(Motivic) multiple zeta values, the block decomposition, and cyclic insertion

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20 October 2015

The title is quite ambitious. I'm probably going to spend a lot of time talking first about the background, giving an introduction to MZVs and some of the standard results.

1 Introduction / recap of Multiple Zeta Values

1.1 Definition/Motivation

Firstly the definition

Definition 1.1. For $s_i \in \mathbb{Z}_{>0}$, the *multiple zeta value* is defined as

$$\zeta(s_1, s_2, \ldots, s_k) \coloneqq \sum_{0<n_1<n_2<\ldots<n_k} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}$$

For this sum to converge, we require $s_k > 1$. (Some conventions use $n_1 > n_2 > \cdots > n_k > 0$, but this just reverses the order of the arguments.)

We call k the *depth*, and $\sum_{i=0}^k s_i$ the *weight*.

View this as a multivariable generalisation of the Rieman zeta function. These are the sort of sums one gets if you look at products of RZV, and try to break them up into their basic blocks...

So why is it worth studying these things? We'll firstly, there they have a huge amount of structure hidden behind this definition. For example, at weight $k = 10$ there are a priori $2^{10-2} = 256$ different MZVs. But it turns out that there are, at most, 7 linearly independent ones. This means there is a huge number of relations. For example

$$\zeta(1, 2) = \zeta(3)$$
$$(2n + 1)\zeta(\{1, 3\}^n) = \zeta(\{2\}^{2n}) = \frac{\pi^{2n}}{(2n + 1)!}$$
$$28\zeta(3, 9) + 150\zeta(5, 7) + 168\zeta(7, 5) = \frac{5197}{691}\zeta(12)$$

First is Euler, Second was a conjecture of Zagier which Broadhusrt proved, and the last is Gangl-Kaneko-Zagier with a connection to non-trivial cusp forms of weight $2k$.

So how to find and understand these?

Secondly, studying them could be motivated by what we don't know about them, despite the apparent simplicity of the definition. Easy-sounding questions about MZVs can be incredibly difficult. Recall, Euler showed that

$$\zeta(2k) = \frac{(-1)^{k+1} B_{2k} (2\pi)^{2k}}{2(2k)!},$$

in particularly all $\zeta(2k)$ are irrational (transcendental), and linearly independent. But the analogous equation for $\zeta(2k+1)$ is firmly unanswered. Arèry showed that $\zeta(3)$ is irrational, and it has been prove there are infinitely irrational $\zeta(odd)$, but we still don't know if $\zeta(5)$ is one of them. Similarly the three relations above are homogeneous in the weight. Are all relations weight graded? No one can prove this yet.

Difficult transcendentality questions:

- What are all relations on MZVs?
- Are all relations homogeneous (graded by weight)?
- Is a particular class of relations (so called double shuffle) sufficient?
- Is $\zeta(5)/\zeta(3)$ irrational?
- Any case where $\dim \text{weight } k$ MZV's > 1?

Part of my work has been in trying to understand and generalise a particular (conjectural) family of relations on MZV's. The first instance of which is:

Conjecture 1.2 (Borwein, Bradley, Broadhurst, Lisoněk cyclic insertion)**.**

$$\sum_{cycle a_i} \zeta(2^{a_0}, 1, 2^{a_1}, 3, \ldots, 1, 2^{a_{2n-1}}, 3, 2^{a_{2n}}) \stackrel{?}{=} \frac{\pi^{\text{wt}}}{(\text{wt}+1)!}$$

We now have

Theorem 1.3 (C)**.**

$$\sum_{permute a_i} \zeta(2^{a_0}, 1, 2^{a_1}, 3, \ldots, 1, 2^{a_{2n-1}}, 3, 2^{a_{2n}}) \in \frac{\pi^{\text{wt}}}{(\text{wt}+1)!}$$

Giving

Corollary 1.4. *The following family is evaluable*

$$\zeta(\{2^n, 1, 2^n, 3\}^m, 2^n) \in \pi^{\text{wt}}\mathbb{Q}$$

Also

Conjecture 1.5 (Hoffman)**.**

$$2\zeta(3, 3, 2^n) - \zeta(3, 2^n, 1, 2) \stackrel{?}{=} -\frac{\pi^{\text{wt}}}{(\text{wt}+1)!}$$

Now

Theorem 1.6 (C)**.**

$$2\zeta(3,3,2^n) - \zeta(3,2^n,1,2) \in \pi^{\mathrm{wt}}\mathbb{Q}$$

Moreover both BBBL cyclic insertion, and Hoffman are part of the same generalised family of identities. (Need some notation to set this up, so will try to give precise statements later.)

2 Standard results about MZVs

So let's now recall some facts about MZVs.

Integral Representation Kontsevich shows/observed that every multiple zeta value can be written as particular Chen iterated integrals. We have

$$\zeta(a_1, \ldots, a_k) = (-1)^k \int_0^1 \frac{\mathrm{d}t}{t-1} \left(\frac{\mathrm{d}t}{t-0}\right)^{a_1-1} \cdots \frac{\mathrm{d}t}{t-1} \left(\frac{\mathrm{d}t}{t-0}\right)^{a_k-1} .$$

Where this iterated integral notation

$$\int_0^1 \omega_1(t) \circ \omega_2(t) \circ \cdots \circ \omega_n(t) \coloneqq \int_{0<t_1<t_2<\ldots<t_n<1} \omega_1(t_1)\omega_2(t_2)\cdots\omega_n(t_n) .$$

It is convenient to write this iterated integral using the following notation

$$(-1)^k I(0; 1, 0^{a_1-1}, \ldots, 1, 0^{a_k-1}; 1)$$

where 0; and ; 1 are endpoints of the integration, and the middle arguments encode the differential forms appearing in the integral

$$a \leftrightarrow \frac{\mathrm{d}t}{t-a}$$

This gives an association between MZVs of weight k, and binary words of length $k+2$ starting 01, and ending 01. Can also write this as $xy^{a_1-1}\cdots xy^{a_k-1}$, as an argument to ζ.

Duality: It was observed early on in the study of MZVs, that they shows a duality - pairs of unrelated MZVs have the same numerical value. For example, Euler showed $\zeta(3) = \zeta(2,1)$, but we also have things like $\zeta(3,4) = \zeta(1,1,2,1,2)$. The integral representation provides a very convenient way to describe, and prove, the duality of MZVs, which is otherwise very difficult to even formulate.

Change variables in the integral, so $t \mapsto 1-t$. Then $\mathrm{d}t/(t-1) \leftrightarrow \mathrm{d}t/t$, and the end points swap. So

$$I(0; 10^{a_1-1}\cdots 10^{a_k-1}; 1) = \pm I(1; 01^{a_1-1}\cdots 01^{a_k-1}; 0)$$

But then reversing the path of integration gives

$$= I(0; 1^{a_k-1}0 \cdots 1^{a_1-1}0; 1)$$

And this is the integral for another MZV. On the binary words: reverse and interchange $0 \mapsto 1$.

So

$$\zeta(1,2) = I(0;110;1) = I(0;100;1) = \zeta(3)$$

Shuffle product: There is a well known way to multiply Chen iterated integrals. By splitting up the product of simplex over which we integrate, one can show it is to take the shuffle product of the words defining the differential forms.

$$I(a;v;b)I(a;w;b) = I(a; v \sqcup\!\sqcup w; b)$$

Here $w \sqcup\!\sqcup v$ can be defined recursively by

- For any word w, $\sqcup\!\sqcup w = w \sqcup\!\sqcup = w$,
- For words v, w, and letters x, y, $(xv) \sqcup\!\sqcup (yw) = x(v \sqcup\!\sqcup yw) + y(xv \sqcup\!\sqcup w)$.

Idea: riffle shuffle the letters of the two words.

So

$$\zeta(2)\zeta(2) = I(0; 10 \sqcup\!\sqcup 10; 1) = I(0; 4 \cdot 1100 + 2 \cdot 1010; 1) = 4\zeta(1,3) + 2\zeta(2,2)$$

Stuffle product Instead of multiplying the integrals. Let's multiply the series representing the MZV. This leads to the stuffle product of MZVs, $\zeta(v)\zeta(w) = \zeta(v * w)$, where $*$ is defined recursively via:

- For any word w, $1 * w = w * 1 = w$,
- For any word w, and any integer $n \geq 1$:

$$x^n * w = w * x^n = wx^n$$

- For any words w_1, w_2, and integers $p, q \geq 0$:

$$yx^p w_1 * yx^q w_2 = yx^p(w_1 * yx^q w_2) + yx^q(yx^p w_1 * w_2) + yx^{p+q+1}(w_1 * w_2)$$

This has a much better interpretation as shuffling the arguments of the MZVs, and possibly *stuffing* two into one split.

$$\zeta(a)\zeta(b) = \zeta(a,b) + \zeta(b,a) + \zeta(a+b)$$
$$\zeta(2)\zeta(2) = 2\zeta(2,2) + \zeta(4)$$

Double Shuffle With the two different ways of multiplying MZVs, we can compare the expressions and get linear relations between MZVs. This even works if we allow the divergent $\zeta(1)$ to appear formally, the divergences cancel out in a way which gives correct results.

$$2\zeta(2,2) + \zeta(4) = 4\zeta(1,3) + 2\zeta(2,2) \implies \zeta(4) = 4\zeta(1,3)$$

Conjecturally regularised doubles shuffle gives all relations, which in turn would imply they are weight graded.

3 Motivic MZVs and the Coproduct / coaction

Many of the difficulties in proving results about MZVs is due to transcendence problems. If there were some way to replace the messy analytic object with some purely algebraic object, things would be easier.

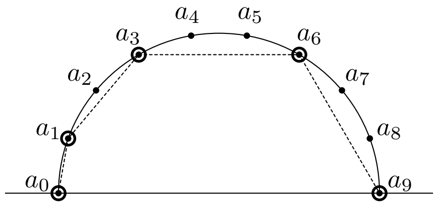
Goncharov's motivic iterated integrals Goncharov (in *Galois symmetries of fundamental groupoids and non-commutative geometry*) showed how the ordinary iterated integrals $I(a_0; a_1, \ldots, a_n; a_{n+1})$ can be upgraded to *framed mixed Tate motives*, so give the motivic iterated integrals $I^{\mathfrak{a}}(a_0; a_1, \ldots; a_n, a_{n+1})$. Unfortunately, since $\zeta^{\mathfrak{a}}(2)$ vanishes (for whatever reason), we don't have a period map back down to $\mathbb{C}$, so we can't compare with real numbers (this is unfortunate).

These form a graded Hopf algebra structure $\mathcal{A}$. (Restrict to $a_i = 0, 1$. Goncharov works more generally, but we're interested in MZVs, so this is fine.) The grading is n. He deduces the following expression for the coproduct:

$$\Delta I^{\mathfrak{a}}(a_0; a_1, \ldots; a_n, a_{n+1}) =$$

$$\sum_{0=i_0<i_1<\cdots<i_k<i_{k+1}=n+1} I^{\mathfrak{a}}(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1}) \otimes \prod_{p=0}^{k} I^{\mathfrak{a}}(a_{i_p}; a_{i_p+1}, \ldots; a_{i_{p+1}-1}, a_{i_{p+1}})$$

The way to remember this using the semicircle polygon pictorial representation



This is the term

$$I^{\mathfrak{a}}(a_0; a_1, a_3, a_6; a_9) \otimes I^{\mathfrak{a}}(a_0; a_1) I^{\mathfrak{a}}(a_1; a_2; a_3) I^{\mathfrak{a}}(a_3; a_4, a_5; a_6) I^{\mathfrak{a}}(a_6; a_7, a_8; a_9)$$

So this for all possible polygons.

By using the Kontsevich integral representation of MZVs, we get their motivic version.

$$\zeta^{\mathfrak{a}}(a_1, \ldots, a_n) = (-1)^n I^{\mathfrak{a}}(0; 10^{a_1} \cdots 10^{a_n}; 1) .$$

For free this gives us some results that have so far been impossible to prove for usual MZVs. The elements $\zeta^{\mathfrak{m}}(2k+1)$ lie in different componenets A_{2k+1}, so must be $\mathbb{Q}$-linearly independent! (In fact, it is almost as easy to prove they are $\mathbb{Q}$-algebraically independent.) Goncharov has proved that any relations between the motivic DZVs follows from the motivic double shuffle relations. Similarly Goncharov has shown that $\zeta^{\mathfrak{a}}(3, 5)$ is irreducible, i.e. not a product of classical motivic ζ's.

Sketch. If $\Delta'(x) = \Delta'(y) = 0$, then $\Delta'(xy) = x \otimes y + y \otimes x$, so vanishes under antisymmetrisation $\otimes \to \wedge$. However $\Delta'(\zeta(3,5)) = -5\zeta(3) \otimes \zeta(5)$ which does not vanish under antisymmetrisation. □

Brown's motivic MZVs Goncharov's motivic MZVs aren't quite good enough. For him, $\zeta^{\mathfrak{a}}(2) = 0$. Francis Brown (*Mixed Tate motives over* $\mathbb{Z}$ and *On the decomposition of motivic multiple zeta values*) shows how to lift these even further in such a way that $\zeta^{\mathfrak{m}}(2) \neq 0$. This is done with a graded algebra comodule $\mathcal{H} \cong \mathcal{A} \otimes_{\mathbb{Q}} \mathbb{Q}[\zeta^{\mathfrak{m}}(2)]$ over $\mathcal{A}$, and Goncharov's coproduct lifts to a coaction $\Delta \colon \mathcal{H} \to \mathcal{A} \otimes_{\mathbb{Q}} \mathcal{H}$, defined by the same formula as before (up to swapping factors)

$$\Delta I^{\mathfrak{m}}(a_0; a_1, \ldots; a_n, a_{n+1}) =$$

$$\sum_{0=i_0<i_1<\cdots<i_k<i_{k+1}=n+1} \prod_{p=0}^{k} I^{\mathfrak{a}}(a_{i_p}; a_{i_p+1}, \ldots; a_{i_{p+1}-1}, a_{i_{p+1}}) \otimes I^{\mathfrak{m}}(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1})$$

.

This time, we do get a period map back down to $\mathbb{C}$, so all motivic relations hold numerically as well. We can compare with numbers.

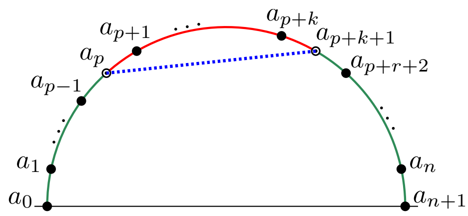
To make this coaction easier to work with, Brown introduces an infinitesimal version of it via the operators he calls D_r, as follows. Take $\mathcal{L} = \mathcal{A}_{>0}/\mathcal{A}_{>0}\mathcal{A}_{>0}$, and π is the projection. Then

$$D_r \colon \mathcal{H}_N \xrightarrow{\Delta_{r,N-r}} \mathcal{A}_r \otimes_{\mathbb{Q}} \mathcal{H}_{N-r} \xrightarrow{\pi\otimes\mathrm{id}} \mathcal{L}_r \otimes_{\mathbb{Q}} \mathcal{H}_{N-r}$$

The action of this on a motivic iterated integral can be explicitly computed as

$$D_r I^{\mathfrak{m}}(a_0; a_1, \ldots, a_n; a_{n+1}) =$$
$$\sum_{p=0}^{n-r} I^{\mathfrak{L}}(a_p; a_{p+1}, \ldots, a_{p+r}; a_{p+r+1}) \otimes I^{\mathfrak{m}}(a_0; a_1, \ldots, a_p, a_{p+1}, \ldots, a_n; a_{n+1})$$

So in the picture from before, we're just cutting off one segment with r interior points each time.



The real upshot of this comes from the following Theorem of Brown

Theorem 3.1. *The kernel of* $D_{<N} \coloneqq \bigoplus_{3\leq 2k+1<N} D_{2k+1}$ *is* $\zeta^{\mathfrak{m}}(N)\mathbb{Q}$ *in weight* N*.*

Brown uses this and the D_r operators to provide an exactly-numerical algorithm to decompose motivic multiple zeta values into a chosen basis. This provides a combinatorial method to find/prove certain identities on the level of real MZVs using the period map.

Some simple examples of this include

Example 3.2.

$$t = \zeta(\underbrace{2, 2, \ldots, 2}_{n}) \in \pi^{2n}\mathbb{Q}$$

because if we compute $D_r I^{\mathfrak{m}}(0; (10)^n; 1)$, then (draw picture) cut off segment always starts and ends with the same symbol. So $D_{<N} t^{\mathfrak{m}} = 0$, which implies $t^{\mathfrak{m}} \in \zeta^{\mathfrak{m}}(2n)\mathbb{Q}$, and gives the above result on taking the period map.

Sadly, Brown's decomposition method cannot find the coefficient exactly in this case, so we'd have to resort to numerical evaluation write an explicit version of the 'almost' identity

$$\zeta(\underbrace{2, 2, \ldots, 2}_{n}) = \alpha\pi^{2n} \, .$$

4 The block decomposition of motivic iterated integrals

In order to prove the various theorems I mentioned in the introduction, an to state the generalised version, I need to set up a framework.

Given a motivic MZV (more generally an motivic iterated integral), we locate in the argument string positions where 00 or 11 occurs, and break the integral into 'blocks' of alternating 0's and 1's, at these points.

$$\begin{aligned}\zeta(1,4,2,2,1,2) &= I(0,1,1,0,0,0,1,0,1,0,1,1,0,1) \\ &\mapsto I(0,1 \mid 1,0 \mid 0 \mid 0,1,0,1,0,1 \mid 1,0,1) \\ &\mapsto I_{bl}(2,2,1,6,3)\end{aligned}$$

Definition 4.1. The *(alternating) block decomposition* of the iterate integral is formed by recording the lengths of the resulting blocks.

This encodes all of the information about the original integral, since we can assume by reversal of paths that the integral starts with a 0.

Using Brown's D_{2k+1} operators, we can prove the following theorem.

Theorem 4.2 (C)**.**

$$S = \sum_{permute l_i} I_{bl}(l_1, \ldots, l_k) \in \zeta(\mathrm{wt})\mathbb{Q}$$

(Note, this is only interesting in even weight, since in odd weight duality means $I_{bl}(l_1, \ldots, l_k) = -I_{bl}(l_k, \ldots, l_1)$*).*

Main idea of proof. Recall that the D_{2k+1} operation marks out a subsequence of length $2k+3$ on the integral.

$$I(010 \mid 0101\,\underbrace{010 \mid 0101 \mid 10101 \mid 1}\,01) \mapsto I_{bl}(3,7,4,5,3)\,.$$

We show that the D_{2k+1} cancel pairwise when applied to S. Do this by defining a reflection operation on the subsequences and block integrals.

$$I(010 \mid 01\,\underbrace{0 \mid 01010 \mid 0101 \mid 101}\,0101) \mapsto I_{bl}(3,3,5,4,7)\,.$$

We get a subsequence defined on an integral with some permutation of the block lengths. When we work out contributions to D_{2k+1}, we get

$$I^{L}_{bl}(1;3,4,5,1) \otimes I^{m}_{bl}(3,8=7+4+5+3-(2k+1))$$

from the first, and

$$I^{L}_{bl}(0;1,5,4,3) \otimes I^{m}_{bl}(3,8=7+4+5+3-(2k+1))\,.$$

The second factors are equal. The first factors are negatives, since reversal of paths turns first into second with sign $(-1)^{2k+1} = -1$.

Conclusion: all terms in D_{2k+1} cancel pairwise, therefore $S \in \ker D_{<N} = \mathbb{Q}\zeta(\mathrm{wt})$, by Brown. □

Example 4.3 (BBBL)**.** We have

$$\zeta(2^{a_0},1,2^{a_1},3,\ldots,1,2^{a_{2n-1}},3,2^{a_{2n}}) = (-1)^{depth} I_{bl}(2a_0+2, 2a_1+2, \ldots, 2a_{2n}+2)\,,$$

so permuting blocks, means permuting a_i. Hence

$$\sum_{permute a_i} \zeta(2^{a_0},1,2^{a_1},3,\ldots,1,2^{a_{2n-1}},3,2^{a_{2n}}) \in \pi^{\mathrm{wt}}\mathbb{Q}\,.$$

Example 4.4 (Hoffman)**.** We have

$$\zeta(3,3,2^n) = I_{bl}(3,3,2n+1)\,,$$

so summing all permutations leads to

$$\begin{aligned}&2I_{bl}(3,3,2n+1) + 2I_{bl}(3,2n+1,3) + 2I_{bl}(2n+1,3,3)\\ &2\zeta(3,3,2^n) - 2\zeta(3,2^n,1,2) + 2\zeta(2^n,1,2,1,2) \in \pi^{\text{wt}}\mathbb{Q}\,.\end{aligned}$$

Combining term 1 and term 3 by duality gives Hoffman.

Generalisation to $I_{bl}(2a+3,2b+3,2c+2)$ gives

$$\begin{aligned}&\zeta(2^a,3,2^b,3,2^c) + \zeta(2^b,3,2^a,3,2^c)\\ &-\zeta(2^b,3,2^c,1,2,2^a) - \zeta(2^a,3,2^c,1,2,2^b)\\ &+\zeta(2^c,1,2,2^a,1,2,2^b) + \zeta(2^c,1,2,2^b,1,2,2^a) \in \pi^{\text{wt}}\mathbb{Q}\,.\end{aligned}$$

Can combine terms by duality to get

$$\zeta(2^a,3,2^b,3,2^c) - \zeta(2^b,3,2^c,1,2,2^a) - \zeta(2^a,3,2^c,1,2,2^b) \in \pi^{\text{wt}}\mathbb{Q}\,.$$

Easy to generate provable new identities now. Also have a few 'odd' cases like

$$\begin{aligned}&\zeta(1,3,3,(1,2)\mid 0,0,0,0,n) + \zeta(3,(1,2),1,3,\mid 0,0,n,0,0) +\\ &\quad -\zeta((1,2),1,3,(1,2)\mid 0,n,0,0,0) + \zeta((1,2),1,3,3\mid 0,0,0,n,0) +\\ &\quad -\zeta(3,1,3,3\mid n,0,0,0,0) \overset{?}{=} \frac{\pi^{\text{wt}}}{(\text{wt}+1)!}\end{aligned}$$

also can be proven motivically, but it doesn't fit into the theorem. (Notice that as blocks, this is just cyclic permutations.)

Lots and lots of numerical expermentations trying to find other relations between these block decompositions suggests that the sum in Theorem over all permutations in fact breaks up in to the sum of cyclic shifts. With a very precise factor.

Conjecture 4.5 (Generalised cyclic insertion, C)**.** Suppose that no pairs of blocks $(l_i, l_{i+1}) = (1,1)$, then

$$\sum_{cycle l_i} I_{bl}(l_1,\ldots,l_k) = \begin{cases} I_{bl}(\sum l_i) & \text{even weight} \\ 0 & \text{odd weight}\end{cases}\,.$$

Notice that this is a direct generalisation of BBBL. If $l_i = 2a_i + 2$, then get the same result since

$$I_{bl}(\sum l_i) = (-1)^{\text{wt}/2=depth}\zeta(2^{\text{wt}/2})\,.$$

It should be possible to partially tackle some version of this generalise cyclic insertion conjecture using Brown's motivic framework. These identities appear somewhat stable under D_{2k+1}, in the sense that computing D_{2k+1} leads to lower weight versions of the identities. We can then cancel using the exact values of the identities, and conclude motivically an up to $\mathbb{Q}$ version at the next weight.

Some problems: we need to know the exactly values, and this can't be done motivically (yet). So proof at each weight would be conditional on lower weight exact proof. Moreover, not entirely yet which lower weight versions arise under D_{2k+1}. Finally, some cases where products in I^L factor have to be killed directly. Not clear how to do this generally.

Would also like to try to generalise to case where $(l_i, l_{i+1}) = (1,1)$ is allowed. Seems like this conjecture is just the leading term in some more general result when we allow $(l_i, l_{i+1}) = (1,1)$. But don't have results yet.

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5 Outline of Goncharov's construction

It is known that the category $\mathcal{M} = \mathcal{MTM}(F)$ of mixed Tate motives over a number field F exists. This is a Tannakian category

- An abelian, k-linear, tensor ridig category, with an exact failthful fibre functor compatible with $\otimes$ structure

with an invertible object $\mathbb{Q}(1)$.

Being a mixed Tate category means that the objects $\mathbb{Q}(n) \coloneqq \mathbb{Q}(1)^{\otimes n}$, with $\mathbb{Q}(-1) = \mathbb{Q}(1)^\vee$, are mutually non-isomorphic. And any simple object in $\mathcal{M}$ is isomorphic to one of them. The extensions $\mathrm{Ext}^1_{\mathcal{M}}(\mathbb{Q}(0), \mathbb{Q}(n)) = 0$ if $n \leq 0$.

Any object M of $\mathcal{M}$ has a weight filtration indexed by $2\mathbb{Z}$. The graded pieces $\mathrm{gr}^W_{2n} M \coloneqq W_{2n}M/W_{2n-2}M$ are a direct sum of copies of $\mathbb{Q}(-n)$. Morphisms are compatible with the weight filtration.

The functor to the category of graded $\mathbb{Q}$-vector spaces

$$\begin{aligned} \omega_{\mathcal{M}} = \omega \colon \mathcal{M} &\to \mathcal{V}ect_\bullet \\ M &\mapsto \bigoplus_n \mathrm{Hom}_{\mathcal{M}}(\mathbb{Q}(-n), \mathrm{gr}^W_{2n} M) \end{aligned}$$

is a fibre functor (faithful $\mathbb{Q}$-linear tensor functor to a category of finite dimensional $\mathbb{Q}$ vector spaces). After forgetting the grading, get $\widetilde{\omega}$.

Then $\mathrm{Aut}^\otimes \widetilde{\omega} =: \mathcal{G}^{MT}$ is a pro-algebraic group scheme over $\mathbb{Q}$. This is the motivic Galois group of the category $\mathcal{MT}(F)$. It decomposes as $\mathcal{G}^{MT} = \mathbb{G}_m \ltimes \mathcal{U}^{MT}$, where $\mathcal{U}^{MT}$ is a pro-unipotent gorup scheme defined over $\mathbb{Q}$.

By the usual Tannakian formalism the category $\mathcal{MT}(F)$ is equivalent to $\mathrm{Rep}_F \mathcal{G}^{MT}$, and also to $\mathrm{CoMod}\mathcal{O}(\mathcal{G}^{MT})$.

Denote by $\mathfrak{u}$ the completition of the pro-nilpotent grded Lie aglebra of $\mathcal{U}^{MT}$. Then

$$\mathcal{A}_\bullet(\mathcal{M}) = \mathcal{A}^{MT} \coloneqq \mathcal{O}(\mathcal{U}^{MT}) \cong U(\mathfrak{u})^\vee$$

is called the fundamental Hopf algebra of $\mathcal{MT}(F)$.

[Goncharov also shows how to obtain $\mathcal{A}^{MT}$ from the Hopf algebra of framed objects. An object M in $\mathcal{M}$ is n-framed, denoted (M, v_0, f_n) if it is supposed with non-zero morphisms $v_0 \colon \mathbb{Q}(0) \to \mathrm{gr}^W_0 M$, and $f_n \colon \mathrm{gr}^W_{-2n} M \to \mathbb{Q}(n)$. There is a notion of equivalence given by morphisms $M_1 \to M_2$ respecting framings. Taking $\mathcal{A}_n(\mathcal{M})$ to be the set of all equivalence classes of n-framed objects leads to the Hopf algebra $\mathcal{A}_\bullet(\mathcal{F})$.]

Now let S be any subset of $F = \mathbb{A}^1(F)$, possibly $S = F$ with a cannonical choice $v_s = \mathrm{d}t$ of tangent vector at every point $s \in F$. Consider the fundamental groupoid of paths $\mathcal{P}^{\mathcal{M}}(\mathbb{A}^1 - S, S)$. This is a pro-object in $\mathcal{M}$ (This is defined by Deligne-Goncharov for the mixed Tate motives over F case.)

This fundamental groupoid of paths has a number of algebra structures. The space $\mathrm{Hom}(\mathbb{Q}(0), \mathrm{gr}^W_0 \mathcal{P}^{\mathcal{M}}(\mathbb{A}^1 - S, S))$ is one dimensionaln, with a natural generator $p_{a,b}$. There is a composition of paths $*$ leading to $p_{a,s_1,\ldots,s_m,b} = p_{a,s_1} * p_{s_1,s_2} * \cdots * p_{sm,b}$. Goncharov establishes an isomorphism between the object $\mathcal{P}^{\mathcal{M}}(S)$, defined by $\mathcal{P}^{\mathcal{M}}(S)_{a,b} = \mathcal{P}^{\mathcal{M}}(\mathbb{A}^1 - S, a, b)$, and an explicitly constructed *path-algebra* (given by generators, relations, etc).

Goncharov then defines the motivic iterated integral $I^{\mathcal{M}}(a_0; a_1, \ldots, a_m; a_{m+1}) \in \mathcal{A}_m(\mathcal{M})$ as the linear functional on $\mathrm{End}(\omega)$ given by the matrix element

$$\begin{aligned} \mathrm{End}(\omega) &\to \mathrm{End}(\omega) \\ F &\mapsto \langle F(p_{a_0,a_{m+1}}), p_{a_0,a_1,\ldots,a_{m+1}} \rangle \,. \end{aligned}$$

[And offers a description in terms of the framed object

$$(\mathcal{P}^{\mathcal{M}}(\mathbb{A}^1 - S, a_, a_{m+1}), p_{a_0,a_{m+1}}, p^*_{a_0,a_1,\ldots,a_{m+1}})$$

]

Goncharov relates this Hopf algebra, and the fundamental groupoid to his more explicitly defined Hopf algebra of (formal) iterated integrals, and his path algebra. This is by relating the automorphism group of the path algebra and the Galois group of $\mathcal{M}$. From here he extracts all that $I^{\mathcal{M}}(a_0, \ldots, a_{m+1})$ satisfies the various properties of iterated integral, and establishes the coproduct formula.

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6 Brown's motivic iterated integrals

Goncharov's motivic MZV's are not good enough since they lack a period map, $\zeta^{\mathfrak{M}}(2) = 0$. Brown refines the construction to produce motivic MZV's in which $\zeta^{\mathfrak{m}}(2) \neq 0$.

Working over the category of mixed Tate motives over $\mathbb{Z}$.

Consider the motivic torsor of paths ${}_0\Pi_1$ of paths on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ between 0 and 1 with tangent vectors $1, -1$. (Is this just some version of $\mathcal{P}^{\mathcal{M}}(\mathbb{A}^1 - \{0,1\}, 0, 1)$?)

Its ring of affine functions over $\mathbb{Q}$ is

$$\mathcal{O}({}_0\Pi_1) \cong \mathbb{Q}\langle e^0, e^1 \rangle \,.$$

The image of the straight line path $0 \to 1$ defines $dch \in_0 \Pi_1(\mathbb{R})$ (a formal power series with MZV coefficients, the Drinfel'd associator). It defines a function

$$dch \colon \mathcal{O}({}_0\Pi_1) \to \mathbb{R} \,.$$

The full Tannakian subcategory $\mathcal{MT}'(\mathbb{Z})$ generated by the motivic fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ has motivic Galois group $\mathcal{G}^{MT'} = \mathcal{G}^{U'} \ltimes \mathbb{G}_m$. Take $\mathcal{A} = \mathcal{O}(\mathcal{G}^{U'})$. This would be the fundamental Hopf algebra of $\mathcal{M}(\mathbb{Z})$ as considered by Gonchaov.

The graded coalgebra of motivic multiple zeta values is

$$\mathcal{H} = \mathcal{O}(0\Pi1)/J^{MT} \,,$$

where J^{MT} is the largest graded ideal contained in the kernel of dch. The image of a word $w \in \mathbb{Q}\langle e^0, e^1 \rangle$ gives the motivic iterated integral $I^{\mathfrak{m}}(0; w; 1) \in \mathcal{H}$. The coaction $\mathcal{O}(0\Pi1) \to \mathcal{A} \otimes \mathcal{O}(0\Pi1)$ indices a coaction $\mathcal{H} \to \mathcal{A} \otimes \mathcal{H}$. The map dch factors through $\mathcal{H}$ to give a period map $per \colon \mathcal{H} \to \mathbb{R}$.

Brown shows that the isomorphism

$$\mathcal{G}^{U'} \times \mathbb{A}^1 \cong \overline{\mathcal{G}_{MT} dch}$$

gives rise to an isomorphism

$$\mathcal{H} = \mathcal{A} \otimes_{\mathbb{Q}} \mathbb{Q}[\zeta^{\mathfrak{m}}(2)] \,.$$

With the graded ring of affine functions on $\mathcal{G}_U$ denoted $\mathcal{A}^{MT}$, and $\mathcal{H}^{MT_+} = \mathcal{A}^{MT} \otimes \mathbb{Q}[f_2]$, we get an injective morphism $\mathcal{H} \to \mathcal{H}^{MT+}$ dual to the quotient $\mathcal{G}_U \to \mathcal{G}_{U'}$. It is then shown that (non-cannonically)

$$\mathcal{H}^{MT_+} \cong \mathbb{Q}\langle f_3, f_5, \ldots \rangle \otimes \mathbb{Q}[f_2]\,,$$

where $\mathbb{Q}\langle f_3, f_5, \ldots \rangle$ is the graded dual of the universal envoloping algebra of the lie algebra of $\mathcal{G}_U$ and is non-caononicaly isomorphic to $\mathcal{A}^{MT}$.

It is in $\mathcal{U}$ first that Brown establishes a result on $\ker D_{<N} = \mathbb{Q} f_N$, where the elements can be explicitly dealt with. Via the isomorphism $\mathcal{H}^{MT_+} \to \mathcal{U}$ normalised so that $\zeta^{\mathfrak{m}}(2n+1) \in \mathcal{H} \subset \mathcal{H}^{MT_+}$ goes to f_{2n+1}, the same result is for $\ker D_{<N} = \mathbb{Q}\zeta^{\mathfrak{m}}(N)$ is established on the motivic MZV's.