COMPUTING COHOMOLOGY OF ARITHMETIC GROUPS

STEVEN

The goal is to (give an overview of) how to explicitly compute some automorphic forms via arithmetic groups. As a starting point, we review modular symbols and how they can be used compute with holomorphic modular forms.

1. Cohomology and holomorphic modular forms

Let $\Gamma_0(N) = \{ \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \pmod N \} \subset \mathrm{SL}_2(\mathbb{Z})$. This acts on upper half plane $\mathfrak{h} = \{ x + iy \mid y > 0 \}$ by fractional linear transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{ax+b}{cz+d} \,.$$

The quotient $Y_0(N) = \Gamma_0(N) \backslash \mathfrak{h}$ is a smooth algebraic curve defined over $\mathbb{Q}$ called an (open) modular curve. It is not compact.

The action of $\Gamma_0(N)$ extends to the cusps $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \infty$, viewing $\mathbb{Q} \subset \mathbb{C}$ and ∞ being far up the imaginary axis. This forms the boundary of $\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})$. The quotient $\Gamma_0(N)\backslash\mathfrak{h}^*$ is a smooth projective curve, called a modular curve.

Eichler-Haberland-Shimura shows a connection of cohomology of X, Y with modular forms: holomorphic $f \colon \mathfrak{h} \to \mathbb{C}$ satisfying

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)\,, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)\,,$$

(and some growth conditions...) for fixed $k \geq 1$ an integer.

The space of modular forms $M_k(N)$ is a finite-dimensional complex vector space. There is a subspace of cusp forms $S_k(N)$ which decay exponentially as z approaches any cusp. The complement of this is Eisenstein series $\mathrm{Eis}_k(N)$.

We have the following

$$\begin{aligned} H^1(Y_0(N), \mathbb{C}) &\xrightarrow{\cong} S_2(N) \oplus \overline{S_2}(N) \oplus \mathrm{Eis}_2(N) \\ H^1(X_0(N), \mathbb{C}) &\xrightarrow{\cong} S_2(N) \oplus \overline{S_2}(N) \end{aligned}$$

When $N = 11$, we have the explicit facts. $X_0(11)$ has genus 1, i.e. it is a torus so the first (co)homology has 2 generators the meridians. And indeed $S_2(11)$ has dimension 1. Good!

The fundamental domain for $X_0(11)$ has 2 cusps (modulo $\Gamma_0(11)$). So the complement of $Y_0(11)$ in $X_0(11)$ consists of 2 points. So $Y_0(11)$ deformation retracts onto a graph with one point, and 3 loops. Each loop gives a generator of cohomology, so $Y_0(11)$ cohomology is 3 dimensional. Since $M_2(11)$ is 2 dimensional (meaning Eisenstein series is 1 dimensional), the isomorphism above works!

This generalises: we can use any congruence subgroup, $\Gamma_1(N)$, or $\Gamma(N)$. We see higher weigh modular forms by taking cohomology with 'twisted coefficients'. $\mathrm{SL}_2(\mathbb{Z})$ acts on complex vector space P_k of homogeneous polynomials of degree k via

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) P(x,y) = P(ax+cy, bx+dy)\,.$$

This induces a local system $\mathcal{M}_k$ on $X_0(N)$, $Y_0(N)$. We have

$$H^1(Y_0(N),\mathcal{M}_{k-2}) \xrightarrow{\cong} S_k(N)\oplus\overline{S_k}(N)\oplus \mathrm{Eis}_k(N)$$
$$H^1(X_0(N),\mathcal{M}_{k-2}) \xrightarrow{\cong} S_k(N)\oplus\overline{S_k}(N)\,.$$

Hecke operators preserve the decomposition cusp + Eisenstein, so we get operators acting on cohomology and isomorphisms of Hecke modules above.

We can use topological tools to study modular forms. One can explicitly compute with certian automorphic forms of arithmetic interest by generalising the left hand sides of the above.

2. Modular symbols

We review modular symbols briefly, as they form the basis/inspiration for higher dimensional calculations later.

Let $\Gamma\subset \mathrm{SL}_2(\mathbb{Z})$ be a torsionfree subgorup, e.g $\Gamma(N)$, $N\geq 3$. Put $Y_\Gamma=\Gamma\backslash\mathfrak{h}$, and $X_\Gamma=\Gamma\backslash\mathfrak{h}^*$. We want to study the cohomology $H^1(Y_\Gamma,\mathbb{C})$ and $H^1(X_\Gamma,\mathbb{C})$.

By Lefschetz duality we have

$$H^1(Y_\Gamma,\mathbb{C})\xrightarrow{\cong} H_1(X_\Gamma,\partial X_\Gamma,\mathbb{C})\,,$$

where the right hand side is homology of X_Γ relative to the cusps. We can compute this homology using standard techniques from algebraic topology: take a triangulation of X_Γ with vertices at the cusps. THis gives a chain complex $C_*(X_\Gamma)$ with subcomplex $C_*(\partial X_\Gamma)$. The relative homology groups are those of the quotient $C_*(X_\Gamma)/C_*(\partial X_\Gamma)$.

Strategy to give triangulations: use the Farey tessellation of $\mathfrak{h}^*$. This triangulation is given by $\mathrm{SL}_2(\mathbb{Z})$ translates of the triangle $\Delta = 01\infty$. This triangulation is explicitly given by view cusp $\alpha=a/b$ in lowest terms. Join two cusps $\alpha=a/b, \beta=c/d$ by an edge iff

$$\det\begin{pmatrix} a & c\\ b & d\end{pmatrix} = \pm 1\,.$$

(Δ is a union of 3 fundamental domains for $\mathrm{SL}_2(\mathbb{Z})$. The fundamental domain for any torsionfree Γ is a union of finitely many copies of Δ, Why?)

We can do the computation before for $Y_0(11)$ explicitly with this point of view…

Generally, in order to use isomorphism $\xrightarrow{\cong} H_1(X_\Gamma,\partial X_\Gamma,\mathbb{C})$ we need to understand generators and relations for the relative homology group. The modular symbols help us to do this.

To define a modular symbol we consider the set Δ_0 of all degree 0 divisors on $\mathbb{P}^1(\mathbb{Q})$. Think of the divisor $\{s\}-\{r\}$ as the path from r to s.

(Sticking to weight 2 for simplicity) We have a map $\psi_f\colon \Delta_0\to\mathbb{C}$ via the period integral

$$\{s\}-\{r\}\to 2\pi i\int_r^s f(z)\,\mathrm{d}z\,.$$

Here f is a cusp form, of weight 2 for some congruence subgroup. Since Δ_0 is generated by elements of the form $\{s\}-\{r\}$, this defines ψ_f generally.

Modularity of f gives the following relation on periods

$$\int_{\gamma r}^{\gamma s} f(z)\mathrm{d}z = \int_r^s f(z)\mathrm{d}z\,.$$

To capture this symmetry of period integrals, endow Δ_0 with the structure of a left $\mathrm{SL}_2(\mathbb{Z})$-module via linear fractional transformations. Then

$$\psi_f\in\mathrm{Hom}_\Gamma(\Delta_0,\mathbb{C})\,,$$

meaning $\psi(\gamma D) = \psi(D)$ for all $\gamma in \Gamma$ and $D \in \Delta_0$. Then $\mathrm{Hom}_\Gamma(\Delta_0, \mathbb{C})$ is the space of $\mathbb{C}$-valued modular symbols of level Γ.

Then Δ_0 is generated as a $\mathbb{Z}$-module via paths $\{s\} - \{r\}$ between cusps in $\mathbb{P}^1(\mathbb{Q})$. But by Manin's contunied fraction trick the set $\{a/b\} - \{c/d\}$ with $\begin{smallmatrix} a & c \\ b & d \end{smallmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ suffices. But these are just the curves in the Farey tessellation!

Taking right coset representatives $\alpha_1, \dots, \alpha_d$ of $\Gamma \backslash \mathrm{SL}_2(\mathbb{Z})$ is sufficient to give a list of generators of Δ_0 as a $\mathbb{Z}[\Gamma]$-module because $\gamma \cdot [x] = [\gamma \cdot x]$ and we view $[\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)] = \{a/b\} - \{c/d\}$.

We can get these coset representatives from a tessellation of the fundamental domain of Γ via the idea triangle 01∞. (This comes from the Farey triangulation!) Only the boundary divisors are relevant since internal edges are homologous to some boundary segment. Moreover, gluing some edges together reduces generators too!

Over $\mathbb{Z}[\Gamma_0(11)]$ the edges $\infty \to 0$, $0 \to 1/3$ $1/3 \to 1/2$ of the fundamental domain generate Δ_0. Also they are independent, which means the space of $\mathbb{C}$-valued modular symbols is 3 dimensional. This matches up with an explicit computation of the (co)homology being $\mathbb{C}^3$.

We see this explicitly by writing down the relevant chain complex, and working out

$$H_1(X_\Gamma, \partial X_\Gamma, \mathbb{C}) = \frac{\langle 0 \to 2/3, 2/3 \to 1/2, \infty \to 0, 0 \to 1/2, 1/2 \to 1, 1 \to 0 \rangle}{\left\langle \begin{array}{l} 1 \to 0, 0 \to 1/2 + 1/2 \to 1 + 1 \to 0, 0 \to 1/2 - 1/2 \to 1/3 \\ - 1/3 \to 0, 1/2 \to 1 - 1 \to 2/3 - 2/3 \to 1/2 \end{array} \right\rangle} = \mathbb{C}^3$$

We can express the setup more algebriacally, as follows.

Let's recall from earlier the modular symbols, and how they give a model for $H_1(X_\Gamma, \partial X_\Gamma, \mathbb{C})$. We have objects $[a, b] = \{a\} - \{b\}$, to be thought of as geodesics from a to b. This gives some natural relations

$$\begin{aligned} [a, b] &= -[b, a] \\ [a, a] &= 0 \\ [a, b] + [b, c] + [c, a] &= 0 \end{aligned}$$

The space U of all such symbols, modulo the given relations is the space of modular symbols.

We quotient out further by $u - \gamma u$, going to the coinvariants U_Γ. Makes sense since Γ acts on the cusps.

We get

$$U_\Gamma \cong H_1(X_\Gamma, \partial X_\Gamma, \mathbb{C}) \,.$$

(There is also the Hecke action.)

In fact, using Manin's continued fraction trick, we reduce to looking at the unimodular symbols U', those with det ± 1. (These are equivalent to the edges of the FArey tessellation). Then U'_Γ is finite(ly) generated, making computations easier.

Upshot: we can compute $H_1(X_\Gamma, \partial X_\Gamma, \mathbb{C})$ (and by extension $H_1(Y_\Gamma, \mathbb{C})$) as the $\mathbb{C}$-vector space of pairs of cusps modulo Γ (modular symbols-ish!). (With some relations coming from finite subgroups of $\mathrm{SL}_2(\mathbb{Z})$...)

Let's try to generalise this!

3. Setting: Algebraic groups and symmetric spaces, Arithmetic groups and cohomology

To begin this generalisation, we need to see how the to get analogues of the upper half plane from group theory. If $G = \mathrm{SL}_2(\mathbb{R})$ Lie group, and $K = SO(2)$ is the maximal compact. Then G

acts on $\mathfrak{h}$. The stabiliser of i is K. So we obtain

$$G/K \xrightarrow{\cong} \mathfrak{h}\,.$$

This exhibits $\mathfrak{h}$ as a Riemannian globally symmetric space.

So the analogue in our setting is going to be the locally symmetric space $\Gamma\backslash G/K$. This is the replacement for $Y_0(N)$, and the compactification is the replacement for $X_0(N)$.

We focus on general linear agroups defined over number fields.

First we briefly talk about reductive or even semi-simple algebraic groups. Particularly $\mathbf{G} = \mathrm{GL}_n$, or $= \mathrm{SL}_n$.

For semisimpmle groups, the space we want to look at is G/K, where $G = \mathbf{G}(\mathbb{R})$, and $K \subset G$ is the maximal compact. If $\mathbf{G}$ is reductive only, then we need to divide by A_G, the connected component of the identity of the group of real points of the maximal $\mathbb{Q}$-split torus in the centre of $\mathbf{G}$.

[For example, $\mathbf{G} = \mathrm{SL}_n$, leads to $G = \mathrm{SL}_n(\mathbb{R})$, and $K = SO(n)$. This recovers $D = G/K = \mathfrak{h}$, when $n = 2$.]

If $\mathbf{G}$ is a group defined over a number field F, we can restrict the scalars to get a group $R_{F/\mathbb{Q}}\mathbf{G}$ defined over $\mathbb{Q}$. Messy definition, but intuitive familiar idea

[Familiar example of this is $\mathbf{G} = \mathrm{GL}_1$ defined over $\mathbb{C}$. Using the representation

$$a + bi \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

we can see the left hand side as the complex points of $\mathbf{G} = \mathrm{GL}_1$, which is defined over $\mathbb{C}$. The right hand side is defined over $\mathbb{R}$, and this i sthe group of real points of (some) $R_{\mathbb{C}/\mathbb{R}}\mathbf{G}$.]

As an example of this setup, take $F/\mathbb{Q}$ to be real quadratic, and $\mathbf{G} = R_{F/\mathbb{Q}}\,\mathrm{SL}_2$. Then $G = \mathbf{G}(R) \cong \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$, view the two real embeddings $F \hookrightarrow \mathbb{R}$. Then $\mathrm{SL}_2(\mathbb{Q}) \hookrightarrow \mathbf{G}(\mathbb{R})$ view the two two real embeddings The maximal compact K is $\mathrm{SO}(2) \times \mathrm{SO}(2)$, with $A_G = 1$. Then $G/K = \mathfrak{h} \times \mathfrak{h}$, giving the familiar setting for Hilbert modular forms!

In general for $R_{F/\mathbb{Q}}\,\mathrm{SL}_2$ where F has r real, s pairs of complex embeddings, we get

$$\begin{aligned} G &\cong \mathrm{SL}_2(\mathbb{R})^r \times \mathrm{SL}_2(\mathbb{C})^s \\ K &\cong \mathrm{SO}(2)^r \times \mathrm{SU}(2)^s \\ A_G &= 1 \\ G/K &\cong \mathfrak{h}^r \times \mathfrak{h}_3^s \end{aligned}$$

where $\mathfrak{h}_3$ is the 3-dimensional hyperbolic space.

For $R_{F/\mathbb{Q}}\,\mathrm{GL}_2$ we get

$$\begin{aligned} G &\cong \mathrm{GL}_2(\mathbb{R})^r \times \mathrm{GL}_2(\mathbb{C})^s \\ K &\cong O(2)^r \times U(2)^s \\ A_g &\cong \mathbb{R}_{>0} \\ G/(A_G K) &\cong \mathfrak{h}^r \times \mathfrak{h}_3^s \times \mathbb{R}^{r+s-1} \end{aligned}$$

We want to study quotients of $D = G/K$ by arithmetic groups. If $\mathbf{G}$ is a linear algebraic group, a subgroup $\Gamma \subset \mathbf{G}$ is arithmetic if it is commensurable with $\mathbf{G}(\mathbb{Z})$, i.e. the intersection has finite index in both.

For $\mathbf{G} = \mathbb{R}_{F/\mathbb{Q}} \operatorname{GL}_n$, then $\mathbf{G}(\mathbb{Z}) = \operatorname{GL}_n(\mathcal{O}_F)$, where $\mathcal{O}_F$ is the ring of integers. Further examples of arithmetic groups arise by taking quotients. For an ideal I, the kernel of $\operatorname{GL}_n(\mathcal{O}_F) \to \operatorname{GL}_n(\mathcal{O}_F/I$ is a *congruence subgroup.*

Then $Y_\Gamma = \Gamma\backslash(G/A_G K)$ is our analogue for the open modular curve. We ant to study $H^*(Y_\Gamma, \mathbb{C})$. Classes here are anaogues of holoorphic modular forms of weight 2.

For higher weight, we can take a finite dimensioal complex representation (ρ, M) of G. This gives a representation of Γ in M. For torsionfree Γ, the fundamental gorup of Y_Γ is Γ. The representation $\rho\colon \Gamma \to \operatorname{GL}(M)$ indices a local coefficient system $\mathcal{M}$ on Y_Γ, and we form the cohomology spaces $H^*(Y_\Gamma, \mathcal{M})$.

4. Voroni's reduction theory: rationals and number fields

This gives an alternative way to recovery the Farey tessellation in the case of $\mathbb{Q}$, which can then be generalised to all number fields.

Let $V = \operatorname{Sym}_2(\mathbb{R})$ be the 3-d space of 2×2 real symmetric matrices. Inside V there is a subset C of positive-definition matrices. This C is a convex cone: $x, y \in C$ implies $\rho x, x + y \in \mathbb{C}$ for $\rho \in \mathbb{R}_{>0}$.

V has an inner product $\langle x, y\rangle = \operatorname{Tr}(xy)$, and C is self-adjoint with repect to this. So

$$C = C^* = \{\, y \in V \mid \langle x, y\rangle > 0, \forall x \in C \,\} .$$

The group $G = \operatorname{SL}_2(\mathbb{R})$ acts on V by $(g, x) \mapsto gxg^\top$, this action preserves C. The stabiliser of any point is a conjugate of $K = \operatorname{SO}(2)$. Moding out by scalings (homotheties) lead sto a transitive action, and so we get an idenfitication

$$C/\mathbb{R}_{>0} \xrightarrow{\cong} \mathfrak{h} = G/K .$$

In coordinates $\left(\begin{smallmatrix} x & z \\ z & y \end{smallmatrix}\right)$, the cone is given by $yx - z^2 > 0, x > 0$. We can consider the closure $\overline{C}$ of C, which of certain rank 1 matrices, and the rank 0 matrix. (See diagram!)

The map $\mathbb{Z}^2 \to \overline{C}, x = (a, b)^\top \mapsto xx^\top$ gives a collection of non-zero points ξ on ∂C when restricted to $\mathbb{Z}^2 \setminus 0$. The image is discrete. $\operatorname{SL}_2(\mathbb{Z})$ acts on these points, via V. The points Ξ are (almost) the vertices of the Farey tessellation.

C gives a linear model for $\mathfrak{h}$. From C and Ξ the next step is 'natural' take the convex hull Π of Ξ.

Π is a huge polyhedron, equipped with an $\operatorname{SL}_2(\mathbb{Z})$ action. But Π has nice combinatorial structure. Unfortunately it is not locally finite (vertices meet infinitely many edges), but top dimensional faces are triangles! Modding ou by homotheties means the faces of Π become the vertices, edges, triangles of the Farey tesselation.

Why? If $a, b)^\top$ and $(c, d)^\top$ are primitive vectors giving cups at the ends of an arc. The this arc is the image of the edge between $q(a, b)$ and $q(c, d)$.

From this we see that modulo $\operatorname{SL}_2(\mathbb{Z})$ there are only finiely many vertices, edges and triangles in Π. We also see that every edge meets finitely many triangles (two), but every vertex meets infiintely many edges (at infinity in $\mathfrak{h}$).

This setup generalises to higher rank $\operatorname{SL}_n(\mathbb{R})$. Set $V = \operatorname{Sym}_n(\mathbb{R}$, the real vector space of $n \times n$ symmetric matrices. Let C be the convex cone of positive definite matrices. $G = \operatorname{SL}_n(\mathbb{R})$ acts on C by $(g, x) \mapsto gxg^\top$. The quotient of c by homotheties is isomorphic to $D = \operatorname{SL}_n(\mathbb{R})/SO(n)$, where G acts on D by left translation.

The map $q\colon \mathbb{Z}^n \setminus 0 \to \overline{C}$ determines a point set $\Xi \subset \partial C$. The convex hull of Ξ is the Voronoi polyhedron Π. By construction $\mathrm{SL}_n(\mathbb{Z})$ acts on Π. The the cones on the faces of Π descend to form cells in $\overline{D}$, a compactification of D.

[The original reason for defining Π was to understand the reducion theory of positive definite quadratic forms. This essentially boils down to finding a nice fundamental domain of $\mathrm{SL}_n(\mathbb{Z})$ acting on C.

Modulo $\mathrm{SL}_n(\mathbb{Z})$ the polygon Π has finitely many faces, and the facets of Π are in bijection with homothety classes of perfect quadratic forms. IF F is a facet of Π with vertices $\xi_1, \dots, \xi_k$, then $q^{-1)}(\xi_i) \in \mathbb{Z}^n$ are the minimal vectors of the corresponding class.

Moreover, Voroni gave an algorithm that (starting with an initial perfect form) produces a list of perfect forms modulo $\mathrm{SL}_n(\mathbb{Z})$. List given for $n \leq 5$. $n = 8$ is more difficult to understand because of the E_8 root lattice, which gives a fact in Π which contains 2.5×10^{14} maximal faces.

Perfect means the form $Q_A(x) = x^\top Ax$ can be reconstructed from knowledge of its minimum as x ranges over $Z^n \setminus 0$. The minimal vectors are the x for which the minimum occurs. E.g. $x^2 + xy + y^2$ is perfect, but $x^2 + y^2$ is not.

The minimum of $x^2 + y^2$ is 1 at $(\pm 1, 0), (0, \pm 1)$. But this is the same as for $x^2 + 1/2xy + y^2$! Whereas we can recover $x^2 + xy + y^2$ from the minimal vectors and minimum: it must be a form $ax^" + bxy + cy^2$. The minima occur at $(\pm 1, 0), (0, \pm 1)$ and $\pm(1, -1)$, with minimum 1. Plugging these in lets us solve for a, b, c.]

The collection of cones Σ in $\overline{C}$ is obtained by taking the cones on the faces of Π. Modulo $\mathrm{SL}_n(\mathbb{Z})$ there are only finitely many cones in Σ, if a cone meets C then its stabiliser in $\mathrm{SL}_n(\mathbb{Z})$ is finite. The top dimensional cones in Σ are close to fundamental domains of $\mathrm{SL}_n(\mathbb{Z})$. (When $n = 2$, such a Farey triangle is a union of three fundamental domains for $\mathrm{SL}_2(\mathbb{Z})$.) Any point $x \in C$ lies in a unique cone $\sigma(x) \in \Sigma$. Voronoi's algorithm leads to an algorithm to find $\sigma(x)$.

This setup also generalise to number field. If F is a number field with signature (r, s), we can compute the cohomology of $\mathrm{GL}_n(\mathcal{O}_F)$ by building cell decompositions of the corresponding locally symmetric space. I'll sketch the only the differences.

Fix arbitrarily one of each pair of complex embeddings. Identify the infinite places of F with the real and our choice of complex embeddings.

For each real place v, $V_v = \mathrm{Sym}_n(\mathbb{R})$. FOr complex place v, $V_v = Herm_n(\mathbb{C})$. Set C_v to be the corresponding cones of positive definite (Hermitian) forms. Set $V = \prod_v V_v$, $C = \prod_v C_v$.

V has an inner product

$$\langle x, y \rangle = \sum_v c_v \, \mathrm{Tr}(x_v y_v)\,,$$

where $c_v = 1$ for real v, $c_v = 2$ for complex. C is self-adjoint wrt this. View this as the cone of real-valued positive quadratic forms over F in n-variables. More precisely for each $a = (A_v)\imath C$, we get

$$Q_A(x) = \sum c_v x_v^* A_v x_v$$

Note that (A_v) does not necessarily arise from a matrix with entries in F via an embedding $F \to F \otimes \mathbb{R}$. Each A_v is an independent matrix.

The group $G = \mathrm{GL}_n(\mathbb{R})^r \times \mathrm{GL}_n(\mathbb{C})^s$ acts on V by

$$(g \cdot y)_v = \begin{cases} q_V y_V g_v^\top & v \text{ real} \\ q_v y_v \overline{g}_v^\top & v \text{ complex} \end{cases}$$

This action preserves C. We can identify the quotient

$$C/\mathbb{R}_{>0} \xrightarrow{\cong} D = G/(KA_G)$$

where $K = O(n)^r \times U(n)^s$ is the maximal compact.

We construct $\Xi \subset \partial\overline{C}$ using the different embeddings of F. Non-zero vecors in $\mathcal{P}^n \setminus 0$ determine points in V via

$$q\colon x \mapsto (x_v x_c v^*)\,.$$

The image of this defines Ξ. For a form A, we define the minimum $m(A)$ and minimal vectors $M(A)$ by using points in Ξ. A form is perfect if it can be recovered from knowledge of the minimum and minimal vectors.

Given a perfect form, the perfect pyramid

$$\sigma(A) = \left\{ \sum \rho_\xi \xi \;\middle|\; \xi \in M(A), \xi \geq 0 \right\}$$

behaves like Voronoi's perfect cones, having nice combinatorial properties.

Since $\mathrm{GL}_n(\mathcal{O})$ acts on C, and takes Ξ to itself. It acts on the perfect pyramids, and so on the cones in Σ.

There are finitely many $\mathrm{GL}_n(\mathcal{O})$ orbits in Σ. Each $\sigma \in \Sigma$ that meets C has a finite stabiliser. Quotienting by homotheties we get a decomposition of D, analagous to Farey tessellation, which we can use to compute cohomology of finite index subgroups of $\mathrm{GL}_n(\mathcal{O})$.

We can define the cohomological dimension of Γ to be the smallest i such that $H^i(Y_\Gamma, \mathcal{M}) = 0$, for all $\mathcal{M}$. For non-torsionfree Γ, we can extend this to the virtual cohomological dimension $gcd(\Gamma)$ as the cohomological dimension of any finite index torsion free subgroup. By Borel-Serre, we have

$$vcd(\Gamma) = \dim(D) - r_{\mathbb{Q}}(\mathbf{G}/R(\mathbf{G}))\,,$$

where R is the radical.

This leads to stuff on spines, and computing cohomology via certain other isomorphisms...

5. From modular symbols to sharblies

The very explicit construction of modular symbols above gets generalised to the Sharbly complex which $\mathcal{S}_*$ which computes $H^*(Y_\Gamma, \mathbb{C})$. We work with $\mathbf{G} = R_{F/\mathbb{Q}}\,\mathrm{GL}_n$, to compute $H^*(Y_\Gamma, \mathbb{C})$.

Let A_k be he set of formal C-linear ocmbinations of symbols $u = [x_1, \ldots, x_{k+1}]$, each $x_i \imath \mathcal{O}^n \setminus 0$. Let C_k be the submodule generated by

$[x_{\sigma(1)}, \ldots, x_{\sigma(k+n)}] - sgn(\sigma)[x_1, \ldots, x_{k+n}], \sigma \in \mathrm{Sym}_{k+n}$

$[x, x_2, \ldots, x_{k+n}] - [y, y_2, \ldots, y_{k+n}]$ if $x \sim y$

$[x_1, \ldots, x_{k+n}]$ if x_i are contained in a hyerplane, so this is degenerate

Here $x \sim y$ means $q(x) = \lambda q(y)$, $\lambda \in \mathbb{R}_{>0}$, so that x, y determine the ram say in $\overline{C}$.

The quotient $\mathcal{S}_k = A_k/C_k$ is the space of k-sharblies. The boundary map $\partial\colon S_{k+1} \to S_k$ is defined in the usual way

$$\partial[x_1, \ldots, x_{k+n}] = \sum_{i=1}^{k+n} (-1)^i [x_1, \ldots, \widehat{x_i}, \ldots, x_{k+n}]\,.$$

[When $\mathbf{G} = \mathrm{SL}_2/\mathbb{Q}$, this reduces to the modular symbols setup.]

The complex $\mathcal{S}_*$ is homological (boundary decreases degree). The complex has a left Γ-action for any $\Gamma \subset \mathrm{GL}_n(\mathcal{O})$ by putting

$$g \cdot u = [gx_1, \dots, gx_{k+n}]$$

The Γ action commutes with ∂, so form the complex $(\mathcal{S}_*)_\Gamma$ of coinvariants. Then

$$H^{vcd(\Gamma)-k}(Y_\Gamma, \mathbb{C}) \xrightarrow{\cong} H_k((\mathcal{S}_*)_\Gamma)\,.$$

This follows from Borel-Serre duality, as I can try to explain.

[Just an outline. If $V = F^n$ is an n-d vecto space over F, we build a simplicial complex called the Tits building: vertices are proper non-zero subspaces of V. Subspaces $V_1, \dots, V_{k+1}$ form a k-simplex if they can be arranged into a flag

$$\{0\} \subsetneq V_1 \subsetneq V_2 \cdots \subsetneq V_{k+1} \subsetneq V\,.$$

olomon-Tits theorem shows T has homotopy type of $\vee^{n-2} S^s 1$, so reduced homology groups $\widetilde{H}_*(T)$ are nonzero only in degree $n-2$.

Classes in $\widetilde{H}_*(T)$ can be constructed by taking fudamental classes of apartments: take a basis $E = v_1, \dots, v_n$ of V, and consider all possible flags constructed by taking spans of permutations of subsets. One gets $\langle v_1, \dots, v_n \rangle \in \widetilde{H_{n-2}}(T)$. Such classes are known to span the homology. $\mathbf{G}(\mathbb{Q})$ acts, and this makes $\widetilde{H_{n-2}}(T)$ into the Steinberg module St_n.

Borel-Serre duality states now that for any arithmetic group $\Gamma \subset \mathbf{G}(\mathbb{Q})$, one has

$$H^{vcd(\Gamma)-k}(Y_\Gamma, \mathcal{M}) \xrightarrow{\cong} H_k(\Gamma, St_n \otimes \mathcal{M})$$

We compute this by taking a resolution of the Steinberg module. But the sharbly complex does this:

$$\epsilon : \mathcal{S}_0 \to St_n$$
$$[x_1, \dots, x_n] \mapsto \langle x_1, \dots, x_n \rangle$$

And $\partial \circ \epsilon \colon \mathcal{S}_1 \to St_n$ vanishes, giving a map of complexes $\mathcal{S}_* \to St_n$. (This is a resolution because $\mathcal{S}_*$ is acyclic.)]

So we have a complex $(S_*)_\Gamma$ which computes the cohomology of Y_Γ, like the modular symbols. UNfortunately, each $(S_k)_\Gamma$ is not finite. If $U = [x_1, \dots, x_n]$ is a 0-sharbly, we can compute a 'size' using determinants. Take $x' \sim x$ with $q(x')$ the closest point to the origin in the ray through $q(x)$.

$$size(u) = \left| N_{F/\mathbb{Q}} \det(x'_1, \dots, x'_n) \right| \in \mathbb{Z}_{>0}\,.$$

The size is constant on $\mathrm{GL}_n(\mathcal{P}$ orbits in S_0, but size is unbounded on S_0. SO we want an analogue of the unimodular symbols.

- determinant 1 modular symbols,
- modular symbols whose support is an edge in the Farey tessellation

These are the same for $\mathrm{SL}_2(\mathbb{Z})$, but the second condition is the right one to generalise.

One can try to build a subcomplex using Koecher cones (cf the unidmodular subspace is the subspace of Voronoi 2-cones). But there are technical problems here, since the Keocher cones might not be simplicial. Instead one can look at reduced k-sharblies (those where there is a top-dimensional koecher cone containing rays through $q(x_1), \dots, q(x_{k+n})$).

Reduced sharblies are the analogue to modular symbols. INdeed he complex $\mathcal{R}_*$ is finite modulo Γ since there are only finitely many Koecher cones. Unfortunately $(\mathcal{R}_*)_\Gamma$ doesn't work, since the Koecher fan isn't simplicial and so the complex can be midding some relatoins necessary to capture the cohomology.

Three approaches

(1) Restrict to torsion free Γ, and use subdivisions. But main groups of interest have torsion!

(2) Work with $\mathcal{R}_*$ and manually identify extra relations.
(3) Only compute cohomology where the Koecher fan is simplicial!

The author takes the approach that iii) works well in practice.