Motives and Multiple Zeta Values

Steven Charlton
Tübingen

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BMC, Durham

Outline

Multiple zeta values

Definition (MZV)

Multiple zeta value $\zeta(s_1, s_2, \ldots, s_k)$ is defined by

$$\zeta(s_1, s_2, \ldots, s_k) := \sum_{0<n_1<n_2<\cdots<n_k} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}$$

- 'Interesting' multi-variable version of $\zeta(s)$
- Want to restrict to $s_i \in \mathbb{Z}_{>0}$
- For convergence need $s_k \geq 2$

Also define

- Weight: sum of $s_1 + \cdots + s_k$ of arguments
- Depth: number k of arguments

Reasons for interest

- Arise naturally in physics calculations
- Have surprising amount of structure
  - At weight 8, $2^{8-2} = 64$ MZV's
  - Spanned by $\{ \zeta(8), \zeta(5,3), \zeta(3,5), \zeta(3,3,2) \}$
  - Generally: suggests lots of $\mathbb{Q}$-linear relations!
- Leads to *difficult* open questions
  - Euler: $\zeta(2) = \dfrac{\pi^2}{6}, \zeta(4) = \dfrac{\pi^4}{90}$, generally $\zeta(2k) \in \pi^{2k}\mathbb{Q}$
  - What about $\zeta(3)$? Or $\zeta(5)$?
  - Understand all $\mathbb{Q}$-linear relations.

MZV Relations

- $\zeta(3) = \zeta(1,2)$
- $\zeta(\{1,3\}^n) = \dfrac{1}{2n+1}\dfrac{\pi^{4n}}{(4n+1)!} = \dfrac{1}{2n+1}\zeta(\overbrace{\{2\}^{2n}}^{\text{Repeat } 2,2,\ldots,2 \text{ total of } 2n \text{ times}})$
- $28\zeta(3,9) + 150\zeta(5,7) + 168\zeta(7,5) = \dfrac{5197}{691}\zeta(12)$
- $\zeta(\{2\}^m,1,3,3,1,2) + \zeta(3,1,2,1,\{2\}^m,3) - \zeta(1,2,1,\{2\}^m,3,1,2) +$
  $+\zeta(1,2,1,3,3,\{2\}^m) - \zeta(3,\{2\}^m,1,3,3) \in \pi^{2m+10}\mathbb{Q}$

Conjecture (Weight grading)

Any $\mathbb{Q}$-linear relation between MZV's is weight graded.
"There are no relations between MZV's of different weights."

Integral representation; shuffle product

Definition (Iterated integral)

$$I(a_0; a_1, \ldots, a_N; a_{N+1}) := \int\limits_{\substack{a_0<t_1<t_2<\cdots \\ <t_N<a_{N+1}}} \frac{\mathrm{d}t_1}{t_1 - a_1} \wedge \cdots \wedge \frac{\mathrm{d}t_N}{t_N - a_N}$$

- Multiplication of iterated integrals gives shuffle product
  - Arrange $a_0 < t_i < a_{N+1}$ and $a_0 < s_j < a_{N+1}$ in all compatible ways $t_i < s_j$ or $t_i > s_j$.
  - $I(a; w_1; b) I(a; w_2; b) = I(a; w_1 \sqcup\!\sqcup\, w_2; b)$ where

$$(xw_1) \sqcup\!\sqcup\, (yw_2) := x(w_1 \sqcup\!\sqcup\, yw_2) + y(xw_1 \sqcup\!\sqcup\, w_2)$$

Proposition (MZV as iterated integral, Kontsevich)

$$\zeta(s_1, \ldots, s_k) = (-1)^k I(0; 1, \{0\}^{s_1-1}, \ldots, 1, \{0\}^{s_k-1}; 1)$$

Properties of iterated integrals

- (Unit) $I(a;b) = 1$
- (Equal boundaries) $I(x, a_1, \ldots, a_N; x) = 0$
- (Reversal of paths)

$$I(a_0; a_1, \ldots, a_N; a_{N+1}) = (-1)^N I(a_{N+1}; a_N, \ldots, a_1; a_0)$$

- (Path composition)

$$I(a_0, a_1, \ldots, a_N; a_{N+1}) = \sum_{i=0}^{N} I(a_0, a_1, \ldots, a_i; x) I(x, a_{i+1}, \ldots, a_N; a_{N+1})$$

- (Functoriality, under $t \mapsto \alpha t + \beta$, with $\alpha \neq 0$ and $\beta \in \mathbb{C}$)

$$I(a_0; a_1, \ldots, a_N; a_{N+1}) = I(\alpha a_0 + \beta; \alpha a_1 + \beta, \ldots, \alpha a_N + \beta; \alpha a_{N+1} + \beta)$$

- (MZV Duality)

$$I(0; a_1, \ldots, a_N; 1) = (-1)^N I(0; 1 - a_N, \ldots, 1 - a_1; 1)$$

Series representation; stuffle product

- Multiply series gives stuffle product $*$
  - Arrange n_i, and m_j in all compatible ways $n_i < m_j$, or $n_i = m_j$ or $n_i > m_j$.
- Simplest case $\zeta(s) * \zeta(t) = \zeta(s,t) + \zeta(t,s) + \zeta(s+t)$.

Example (Comparing $ш$ and $*$)

$$2\zeta(2,2) + 4\zeta(1,3) \overset{ш}{=} \zeta(2)\zeta(2) \overset{*}{=} 2\zeta(2,2) + \zeta(4)$$
$$\implies \zeta(1,3) = \frac{1}{4}\zeta(4) = \frac{1}{3}\frac{\pi^4}{5!}$$

Conjecture (Extended double shuffle)

All $\mathbb{Q}$-linear relations on MZV's arise by comparing $ш - *$.
(Must allow divergent $\zeta(1)$; formally cancels using regularisation.)

Construction of motivic iterated integrals - Goncharov

Goal: fix transcendentality problems by using algebraic objects

- Category of Mixed Tate Motives $\mathcal{MT}(F)$ over a number field F exists. It is Tannakian; equivalent to some $\mathrm{Rep}_F\,\mathcal{G}^{\mathcal{MT}}$
- Recover pro-algebraic group scheme $\mathcal{G}^{\mathcal{MT}}$ from automorphisms of the fibre functor $\widetilde{\omega}\colon \mathcal{MT}(F) \to \mathcal{V}ect$, $\mathcal{G}^{\mathcal{MT}} \cong \mathbb{G}_m \ltimes \mathcal{U}^{\mathcal{MT}}$
- Ring of regular functions $\mathcal{O}(\mathcal{U}^{\mathcal{MT}})$ on the pro-unipotent part of $\mathcal{G}^{\mathcal{MT}}$ defines the fundamental Hopf algebra $\mathcal{A}_\bullet(F)$ of $\mathcal{MT}(F)$
- Isomorphism $\mathcal{A}_\bullet(F)$ to 'path algebra' and algebra of 'formal iterated integrals'. $\mathcal{A}_\bullet(F)$ contains objects $I^{\mathfrak{a}}(a_0; a_1, \ldots, a_n; a_{n+1})$.
- Admits a coproduct $\Delta\colon \mathcal{A}_\bullet(F) \to \mathcal{A}_\bullet(F) \otimes_{\mathbb{Q}} \mathcal{A}_\bullet(F)$

Coproduct on $\mathcal{A}_\bullet(F)$

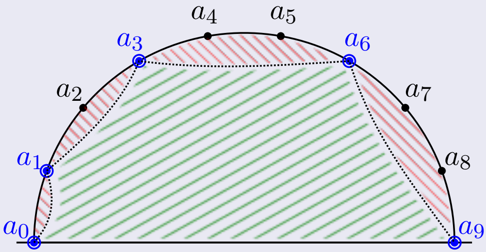
$$\Delta I^{\mathfrak{a}}(a_0; a_1, \ldots, a_n; a_{n+1}) =$$

$$\sum_{\substack{0=i_0<i_1<\cdots \\ <i_k<i_{k+1}=n+1 \\ k=0,1,\ldots,N}} \left(I^{\mathfrak{a}}(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1}) \otimes \prod_{p=0}^{k} I^{\mathfrak{a}}(a_{i_p}; a_{i_p+1}, \ldots, a_{i_{p+1}-1}; a_{i_{p+1}}) \right)$$

Coproduct on $\mathcal{A}_\bullet(F)$ mnemonic

Mnemonic.

$$\Delta I^{\mathfrak{a}}(a; w; b) = \sum_{\substack{S \text{ subset } awb \\ a, b \text{ in } S}} \Bigg(I^{\mathfrak{a}}(S) \otimes \prod_{\substack{u \text{ subword } awb, \\ \text{starts/ends at} \\ \text{consecutive } s_i \in S}} I^{\mathfrak{a}}(u) \Bigg)$$



$$\rightsquigarrow I(a_0; a_1, a_3, a_6; a_9) \otimes I(a_0; a_1) I(a_1; a_2; a_3) \cdot I(a_3; a_4, a_5; a_6) I(a_6; a_7, a_8, a_9)$$

Results from motivic MZV's

- $\zeta^{\mathfrak{a}}(2k+1)$ are linearly independent
  - $\zeta^{\mathfrak{a}}(2k+1) \neq 0 \in \mathcal{A}_{2k+1}(\mathbb{Q})$
  - So have different gradings
- $\zeta^{\mathfrak{a}}(2k+1)$ are *algebraically* independent
  - Suppose some $\zeta^{\mathfrak{a}}(2k+1)$ satisfy a polynomial
  - Use coproduct Δ to show all coefficients are 0
- $\zeta^{\mathfrak{a}}(3,5)$ is irreducible (i.e. not in $\mathbb{Q}[\zeta(n)]$)
  - $(\Delta - \Delta^{\mathrm{op}})\zeta^{\mathfrak{a}}(3,5) = -5\zeta^{\mathfrak{a}}(3) \wedge \zeta^{\mathfrak{a}}(5)$
  - $(\Delta - \Delta^{\mathrm{op}})\zeta^{\mathfrak{a}}(n_1)\cdots\zeta^{\mathfrak{a}}(n_k) = 0$

Construction of motivic MZV's - Brown

- Problem: $I^{\mathfrak{a}}(0;1,0;1) \leftrightarrow -\zeta^{\mathfrak{a}}(2)$ vanishes.
- Consider 'motivic torsor' of paths ${}_0\Pi_1$ between 0 and 1 in $\mathbb{P}^1 \setminus \{0,1,\infty\}$. $\mathcal{O}({}_0\Pi_1) \cong \mathbb{Q}\langle e_0, e_1\rangle$.
- Straight line gives function $\mathcal{O}({}_0\Pi_1) \to \mathbb{R}$, evaluating MZV.
- Coalgebra of motivic MZV's is $\mathcal{H} := \mathcal{O}({}_0\Pi_1)/J^{\mathcal{MT}}$, $J^{\mathcal{MT}}$ the largest graded ideal in the kernel of above.
- $\mathcal{H} \cong \mathcal{A} \otimes_{\mathbb{Q}} \mathbb{Q}[\zeta^{\mathfrak{m}}(2)]$, $\mathcal{A} := \mathcal{A}_\bullet(\mathbb{Z})$
- Period map $\mathrm{per}\colon \mathcal{H} \to \mathbb{R}$, $\zeta^{\mathfrak{m}}(s_1,\ldots,s_k) \mapsto \zeta(s_1,\ldots,s_k)$, ring homomorphism
- Coaction by lifting Gonchrov's coproduct to $\mathcal{H} \to \mathcal{A} \otimes_{\mathbb{Q}} \mathcal{H}$.

Infinitesimal coproduct

Definition (Derivations D_k)

Let $\mathcal{L} := \mathcal{A}/(\mathcal{A}_{>0} \cdot \mathcal{A}_{>0})$, which kills products and $\zeta^{\mathfrak{m}}(2)$. For k odd define

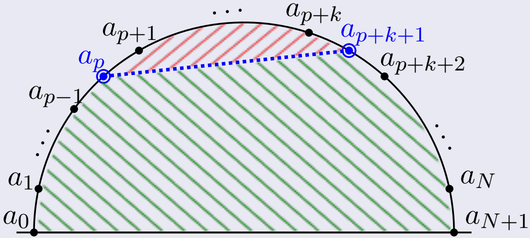
$$\begin{aligned} D_k : \quad & \mathcal{H} \to \mathcal{L}_k \otimes_{\mathbb{Q}} \mathcal{H} \\ & I^{\mathfrak{m}}(w) \mapsto (\pi \otimes \mathrm{id}) \circ (\Delta - 1 \otimes \mathrm{id})\, I^{\mathfrak{m}}(w) \end{aligned}$$

$$\begin{aligned} & D_k I^{\mathfrak{m}}(a_0; a_1, \ldots, a_N; a_{N+1}) = \\ & \quad \sum_{p=0}^{N-k} I^{\mathfrak{L}}(a_p; a_{p+1}, \ldots, a_{p+k}; a_{p+k+1}) \otimes \\ & \qquad I^{\mathfrak{m}}(a_0; a_1, \ldots, a_p, a_{p+k+1}, \ldots, a_N; a_N + 1) \end{aligned}$$

Derivations D_k mnemonic

Mnemonic.

$$D_k I^{\mathfrak{m}}(\underbrace{w}_{(a;w';b)}) = \sum_{\substack{S \text{ subword } w, \\ \text{of length } k+2}} I^{\mathfrak{L}}(S) \otimes I^{\mathfrak{m}}(w - \text{interior } S)$$



$$\rightsquigarrow I^{\mathfrak{L}}(a_p; a_{p+1}, \ldots, a_{p+k}; a_{p+k+1}) \otimes$$

$$I^{\mathfrak{m}}(a_0; a_1, \ldots, a_p, a_{p+k+1}, \ldots, a_N; a_N + 1)$$

Transcendental Galois Theory

Theorem (Brown, 2012)

In weight N, $\ker D_{<N} = \zeta^{\mathfrak{m}}(N)\mathbb{Q}$.

Example

Can show $\zeta^{\mathfrak{m}}(\{2\}^n) \in \zeta^{\mathfrak{m}}(2n)\mathbb{Q}$

As an integral $= \pm I^{\mathfrak{m}}(0; \underbrace{1, 0, 1, 0, \ldots, 1, 0}_{n \text{ times}}; 1)$

- Alternates 0 and 1
- Odd length subword has same start and end letter
- Integral vanishes because boundaries are equal
- All D_{2r+1} vanish

So $\zeta^{\mathfrak{m}}(\{2\}^n) \in \ker D_{<2n} = \zeta^{\mathfrak{m}}(2n)\mathbb{Q}$.

Conjectural identities

The following identities appear to hold.

Conjecture (Hoffman)

For $m \in \mathbb{Z}_{\geq 0}$

$$2\zeta(3,3,\{2\}^m) - \zeta(3,\{2\}^m,1,2) \stackrel{?}{=} -\frac{\pi^{2m+6}}{(2m+7)!} = -\frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$$

Conjecture (Cyclic insertion - Borwein, Bradley, Broadhurst, Lisoněk)

For $n \in \mathbb{Z}_{\geq 0}$*, and* $a_0, \ldots, a_{2n} \in \mathbb{Z}_{\geq 0}$*,*

$$\sum_{\text{cycle } a_i} \zeta(\{2\}^{a_0},1,\{2\}^{a_1},3,\ldots,1,\{2\}^{a_{2n-1}},3,\{2\}^{a_{2n}}) \stackrel{?}{=} \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$$

"Insert all cyclic permutations of some blocks $\{2\}^{a_i}$ *of two's."*

Structure of identities - Hoffman

- Write Hoffman's identity as iterated integrals

$$\begin{aligned}
&2\zeta(3,3,\{2\}^n) \quad - \zeta(3,\{2\}^n,1,2) \\
&= \zeta(3,3,\{2\}^n) \quad - \zeta(3,\{2\}^n,1,2) \quad + \zeta(\{2\}^n,1,2,1,2) \\
&\rightsquigarrow I(0100100(10)^n 1) + I(0100(10)^n 1101) + I(0(10)^n 1101101)
\end{aligned}$$

- Split into 'alternating blocks' at $00 \to 0 \mid 0$ or $11 \to 1 \mid 1$

$$\begin{aligned}
&= I(010 \mid 010 \mid 0(10)^n 1) + I(010 \mid 0(10)^n 1 \mid 101) \\
&\qquad + I(0(10)^n 1 \mid 101 \mid 101)
\end{aligned}$$

- Record lengths of the blocks

$$= I_{\mathrm{bl}}(3,3,2n+2) + I_{\mathrm{bl}}(3,2n+2,3) + I_{\mathrm{bl}}(2n+2,3,3)$$

- Right hand side is $I_{\mathrm{bl}}(\mathrm{wt}+2)$

Structure of identities - BBBL

- Write the BBBL identity as iterated integrals

$$\sum_{\text{cycle } a_i} \zeta(\{2\}^{a_0}, 1, \{2\}^{a_1}, 3, \ldots, 1, \{2\}^{a_{2n-1}}, 3, \{2\}^{a_{2n}})$$

$$\rightsquigarrow \sum_{\text{cycle } a_i} I(0(10)^{a_0}1(10)^{a_1}100\cdots01(10)^{a_{2n-1}}100(10)^{a_{2n}}1)$$

- Split into 'alternating blocks' at $00 \to 0 \mid 0$ or $11 \to 1 \mid 1$

$$= \sum_{\text{cycle } a_i} I(0(10)^{a_0}1 \mid (10)^{a_1}10 \mid 0\cdots01 \mid (10)^{a_{2n-1}}10 \mid 0(10)^{a_{2n}}1)$$

- Record lengths of the blocks

$$= \sum_{\text{cycle } a_i} I_{\text{bl}}(2a_0 + 2, 2a_1 + 2, \ldots, 2a_{2n} + 2)$$

- Right hand side is $I_{\text{bl}}(\text{wt} + 2)$.

Common sructure

Both conjectures have the form

$$\sum_{\text{cycle } \ell_i} I_{\mathrm{bl}}(\ell_1, \ldots, \ell_n) \overset{?}{=} I_{\mathrm{bl}}(\mathrm{wt} + 2)$$

Generally:

Conjecture (Generalised cyclic insertion, C., arXiv 1703.03784)

For any $[\ell_1, \ldots, \ell_n]$ *with all* $\ell_i > 1$,

$$\sum_{\text{cycle } \ell_i} I_{\mathrm{bl}}(\ell_1, \ldots, \ell_n) \overset{?}{=} I_{\mathrm{bl}}(\mathrm{wt} + 2)$$

(Extra correction terms needed if some $\ell_i = 1$*.)*

- Numerically tested all cases weight ≤ 18, to 500 decimal places
- Can prove a symmetrised version, up to $\mathbb{Q}$
- Can prove *some* special cases, up to $\mathbb{Q}$

Progress

Theorem (Symmetric insertion, C., arXiv 1703.03784)

For any $[\ell_1, \ldots, \ell_n]$*, with even weight,*

$$\sum_{\textit{permute } \ell_i} I_{\mathrm{bl}}(\ell_1, \ldots, \ell_n) \in I_{\mathrm{bl}}(\mathrm{wt} + 2)\mathbb{Q}$$

Proof (Sketch).

- Lift to motivic version $I^{\mathfrak{m}}$.
- Define a reflection $\mathcal{R}$ on subsequences
- Set up a pairwise cancellation in $D_{<N}$.
- Conclude $\in \zeta^{\mathfrak{m}}(\mathrm{wt})\mathbb{Q} = I^{\mathfrak{m}}_{\mathrm{bl}}(\mathrm{wt} + 2)\mathbb{Q}$ using Brown. □