Motives and Multiple Zeta Values

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5 April 2017
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Abstract: In this talk I will introduce multiple zeta values (MZV's), a rather mysterious class of real numbers about which many things are conjectured, but relatively little is known.
As we shall see, their analytic definition frequenty causes transcendenality problems and makes understanding the structure of MZV's difficult. To circumvent these problems, we must introduce a purely algebraic lifting – the so-called *motivic* MZV's of Goncharov, and of Brown. Motivic MZV's form a graded Hopf algebra, giving them a much more rigid structure, which we can exploit.

I will aim to discuss some conjectural families of relations on MZV's that I have been able to better understand, and to generalise, thanks to this motivic point of view.

Outline

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1 Definitions and motivations
2 Algebraic structure of MZV's
3 Motivic iterated integrals, and motivic MZV's
4 Alternating block decomposition and cyclic insertion

1. Talk first about definitions and motivations. What are MZV's and why do we care?
2. Then some of the standard results on their algebraic structure. What we do know, and what we don't know. Transcendentality problems!
3. A more algebraic way to study MZV's which deals automatically with the transcendentality problem.
4. Hopefully I do have time to talk about the identities I have understood with this framework, but if not, you can ask me at the end.

Multiple zeta values

Definition (MZV)

Multiple zeta value $\zeta(s_1, s_2, \ldots, s_k)$ is defined by

$$\zeta(s_1, s_2, \ldots, s_k) := \sum_{0<n_1<n_2<\cdots<n_k} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}$$

- 'Interesting' multi-variable version of $\zeta(s)$
- Want to restrict to $s_i \in \mathbb{Z}_{>0}$
- For convergence need $s_k \geq 2$

Also define

- Weight: sum of $s_1 + \cdots + s_k$ of arguments
- Depth: number k of arguments

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Weight: sum of $s_1 + \cdots + s_k$ of arguments
Depth: number k of arguments

1. What do I mean by interesting? If instead of $n_1 < n_2 < \ldots$, we just summed over independent variables $n_1, n_2, \ldots$, then this would reduce to the product $\zeta(s_1)\zeta(s_2)\ldots$, which is nothing new. Forcing $n_1 < n_2 < \ldots$ gives us a genuinely new function.
2. This is perhaps some what arbitrary. Certainly it is possible to study this as a meromorphic function of multiple complex variables. But the structure of the poles is much more complicated than for the Riemann zeta function. We are more motivated by things like Euler's evlauation $\zeta(2) = \frac{\pi^2}{6}$, etc, and want to restriction to integer arguments.
3. Allowing $s_i \in \mathbb{C}$ leads to $s_k > 1$ like for the Riemann zeta function, so we get $s_k \geq 2$ if it is an integer.
4. These are useful auxilliary notions. Pay attention later, when I give some examples of MZV relations. See if you can spot something curious.

Reasons for interest

- Arise naturally in physics calculations
- Have surprising amount of structure
  - At weight 8, $2^{8-2} = 64$ MZV's
  - Spanned by $\{ \zeta(8), \zeta(5,3), \zeta(3,5), \zeta(3,3,2) \}$
  - Generally: suggests lots of $\mathbb{Q}$-linear relations!
- Leads to *difficult* open questions
  - Euler: $\zeta(2) = \frac{\pi^2}{6}, \zeta(4) = \frac{\pi^4}{90}$, generally $\zeta(2k) \in \pi^{2k}\mathbb{Q}$
  - What about $\zeta(3)$? Or $\zeta(5)$?
  - Understand all $\mathbb{Q}$-linear relations.

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1. They occur in the computation of scattering amplitudes. If we understand their structure well, perhaps various calculations can be simplified? It also gives us pure mathematicians something to write on grant applications...
2. These objects have a surprising amount of structure. What do I mean by this? At weight 8, say, their are $2^6 = 64$ different possible MZV's. It turns out that they are all a $\mathbb{Q}$-linear combination of the following 4 $\zeta(2,2,2,2) \sim \zeta(8), \zeta(3,3,2), \zeta(3,2,3), \zeta(2,3,3)$ (Other choices are possible) This implies that MZV's satisfy a lot of $\mathbb{Q}$-linear relations.
3. We all know that $\zeta(2) = \frac{|pi^2}{6}$. More generally Euler evaluated even zetas; $\zeta(2k)$ is an explicit rational multiple of π^{2k}. What do we know about $\zeta(3)$? In 1978 Apèry showed it is irrational, but that's all we know. We don't even know this about $\zeta(5)$, and we certianly don't have an evaluation for $\zeta(2k+1)$.

MZV Relations

- $\zeta(3) = \zeta(1,2)$
- $$\zeta(\{1,3\}^n) = \frac{1}{2n+1}\frac{\pi^{4n}}{(4n+1)!} = \frac{1}{2n+1}\zeta(\overbrace{\{2\}^{2n}}^{\substack{\text{Repeat } 2,2,\ldots,2 \\ \text{total of } 2n \text{ times}}})$$
- $28\zeta(3,9) + 150\zeta(5,7) + 168\zeta(7,5) = \frac{5197}{691}\zeta(12)$
- $$\zeta(\{2\}^m,1,3,3,1,2) + \zeta(3,1,2,1,\{2\}^m,3) - \zeta(1,2,1,\{2\}^m,3,1,2) + \\ + \zeta(1,2,1,3,3,\{2\}^m) - \zeta(3,\{2\}^m,1,3,3) \in \pi^{2m+10}\mathbb{Q}$$

Conjecture (Weight grading)

Any $\mathbb{Q}$-linear relation between MZV's is weight graded.
"There are no relations between MZV's of different weights."

MZV Relations
Repeat 2, 2, . . . , 2 total of 2n times
Conjecture (Weight grading)
Any Q-linear relation between MZV's is weight graded.
"There are no relations between MZV's of different weights."

We already know that MZV's should satisfy a lot of relations. What is perhaps not as clear, is that the relations themselves can be highly structured and pretty.

1. This is perhaps one of the first relations between MZV's after Euler first defined them. Euler hoped that all *double* zeta value could be reducd to values of Riemann zeta. This is an example of duality.
2. The left hand side of this was conjectured by Zagier on the basis of numerical evidence. The proof was givey by Broadhurst using generating series methods, and hypergeometric functions.
3. This one perhaps does't look as pretty, but it is a very important relation. Firstly it has a connection to modular forms. At weight 12, we obtain the first non-trivial cusp form for $\mathrm{SL}_2(\mathbb{Z})$, namely the discriminant Δ. This MZV relation is a consequence. (Gangl, Kaneko, Zagier) Secondly, it gives an exceptional relation between $\zeta(odd, odd)$, which ruins a conjectural candidate for a basis. This is the 'depth defect phenomenon'.

MZV Relations

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  $+\,\zeta(1,2,1,3,3,\{2\}^m) - \zeta(3,\{2\}^m,1,3,3) \in \pi^{2m+10}\mathbb{Q}$

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1. This last relation, I just included for fun. It is related to things I will introduce at the end of the talk. So far, the identity is conjectural, but I can prove it is a rational multiple of the right hand side.
2. Did you notice anything curious about these 3 relations? Maybe it is a coincidence because of the small sample size, maybe not? But every relation has the same weight on the left hand side, and the right hand side. Generally this is conjectured to hold.
   Currently, this conjecture is impossible to resolve: it we knew that all MZV relations were weight graded, then we would know automatically that $\zeta(5)$ is irrational. It has weight 5, but a rational number has weight 0. So currently there is no hope. This is the transcendentality problem which plagues MZV!

Integral representation; shuffle product

Definition (Iterated integral)

$$I(a_0; a_1, \ldots, a_N; a_{N+1}) := \int_{\substack{a_0<t_1<t_2<\cdots \\ <t_N<a_{N+1}}} \frac{\mathrm{d}t_1}{t_1 - a_1} \wedge \cdots \wedge \frac{\mathrm{d}t_N}{t_N - a_N}$$

- Multiplication of iterated integrals gives shuffle product
  - Arrange $a_0 < t_i < a_{N+1}$ and $a_0 < s_j < a_{N+1}$ in all compatible ways $t_i < s_j$ or $t_i > s_j$.
  - $I(a; w_1; b) I(a; w_2; b) = I(a; w_1 ⧢ w_2; b)$ where
  $$(xw_1) ⧢ (yw_2) := x(w_1 ⧢ yw_2) + y(xw_1 ⧢ w_2)$$

Proposition (MZV as iterated integral, Kontsevich)

$$\zeta(s_1, \ldots, s_k) = (-1)^k I(0; 1, \{0\}^{s_1 - 1}, \ldots, 1, \{0\}^{s_k - 1}; 1)$$

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We introduce a new object here, seemingly unmotivated. The way it interacts with MZV's makes it important. Later it is used extensively to create motivic MZV's.

1. When multiplying two such integrals, the domains are independent, so the t_i and s_j do not interact. We are free to decompose into subdomains where $t_i < s_j$, $t_i > s_j$, etc. We don't need to include $t_i = s_j$ since this set has measure 0.
2. This was first observed by Kontsevich. The proof is not complicated, one simply expands out the the fractions $\frac{1}{t_1 - a_1}$ as a geometric series. The result can be recognised as the series definition of the MZV.

Properties of iterated integrals

- (Unit) $I(a;b) = 1$
- (Equal boundaries) $I(x, a_1, \ldots, a_N; x) = 0$
- (Reversal of paths)

$$I(a_0; a_1, \ldots, a_N; a_{N+1}) = (-1)^N I(a_{N+1}; a_N, \ldots, a_1; a_0)$$

- (Path composition)

$$I(a_0, a_1, \ldots, a_N; a_{N+1}) = \sum_{i=0}^{N} I(a_0, a_1, \ldots, a_i; x) I(x, a_{i+1}, \ldots, a_N; a_{N+1})$$

- (Functoriality, under $t \mapsto \alpha t + \beta$, with $\alpha \neq 0$ and $\beta \in \mathbb{C}$)

$$I(a_0; a_1, \ldots, a_N; a_{N+1}) = I(\alpha a_0 + \beta; \alpha a_1 + \beta, \ldots, \alpha a_N + \beta; \alpha a_{N+1} + \beta)$$

- (MZV Duality)

$$I(0; a_1, \ldots, a_N; 1) = (-1)^N I(0; 1 - a_N, \ldots, 1 - a_1; 1)$$

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1. The reversal of paths property, and the functoriality under $t \mapsto 1 - t$ combine to give the duality property of MZV's. Interchange $0 \leftrightarrow 1$ and reverse the integral shows two different MZV's are equal.

Series representation; stuffle product

- Multiply series gives stuffle product $*$
  - Arrange n_i, and m_j in all compatible ways $n_i < m_j$, or $n_i = m_j$ or $n_i > m_j$.
- Simplest case $\zeta(s) * \zeta(t) = \zeta(s,t) + \zeta(t,s) + \zeta(s+t)$.

Example (Comparing $⧢$ and $*$)

$$2\zeta(2,2) + 4\zeta(1,3) \overset{⧢}{=} \zeta(2)\zeta(2) \overset{*}{=} 2\zeta(2,2) + \zeta(4)$$
$$\implies \zeta(1,3) = \frac{1}{4}\zeta(4) = \frac{1}{3}\frac{\pi^4}{5!}$$

Conjecture (Extended double shuffle)

All $\mathbb{Q}$-linear relations on MZV's arise by comparing $⧢ - *$.
(Must allow divergent $\zeta(1)$; formally cancels using regularisation.)

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1. In terms of words in iterated integrals, a recursive definition is more complicated. But the idea is is simple: Shuffle the arguments $s_1, \ldots, s_k$ and $s'_1, \ldots, s'_\ell$. Also *stuff* two into one slot $s_i + s'_j$. This arises by arranging the independent summation indices n_i and m_j in all possible ways. Here we do need to include $n_i = m_j$ since this does not have measure 0.
2. We have two different ways to multiply MZV's now, so we should compare them.
3. This conjecture is even more hopeless than the previous one: it implies all relations are weight graded. The shuffle product of integrals write weight k times weight l as a sum of weight $k+\ell$ integrals. Similarly the series gives weight $k+\ell$, so the linear relation relation which results has weight $k+\ell$. This conjecture does pass extensive numerical testing: any numerically true relation on MZV's can (so far) be written as $⧢ - *$.

Construction of motivic iterated integrals - Goncharov

Goal: fix transcendentality problems by using algebraic objects

- Category of Mixed Tate Motives $\mathcal{MT}(F)$ over a number field F exists. It is Tannakian; equivalent to some $\mathrm{Rep}_F\,\mathcal{G}^{\mathcal{MT}}$
- Recover pro-algebraic group scheme $\mathcal{G}^{\mathcal{MT}}$ from automorphisms of the fibre functor $\widetilde{\omega}\colon \mathcal{MT}(F) \to \mathcal{V}ect$, $\mathcal{G}^{\mathcal{MT}} \cong \mathbb{G}_m \ltimes \mathcal{U}^{\mathcal{MT}}$
- Ring of regular functions $\mathcal{O}(\mathcal{U}^{\mathcal{MT}})$ on the pro-unipotent part of $\mathcal{G}^{\mathcal{MT}}$ defines the fundamental Hopf algebra $\mathcal{A}_\bullet(F)$ of $\mathcal{MT}(F)$
- Isomorphism $\mathcal{A}_\bullet(F)$ to 'path algebra' and algebra of 'formal iterated integrals'. $\mathcal{A}_\bullet(F)$ contains objects $I^{\mathfrak{a}}(a_0; a_1, \ldots, a_n; a_{n+1})$.
- Admits a coproduct $\Delta\colon \mathcal{A}_\bullet(F) \to \mathcal{A}_\bullet(F) \otimes_{\mathbb{Q}} \mathcal{A}_\bullet(F)$

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The construction of motivic MZV's is quite difficult, and very technical. I cannot do justice to all the details here. I will try to sketch the ideas. Goncharov contructs a formal algebra of paths, and a formal algebra of iterated integrals (which satisfy all of the properties listed earlier). He then relates algebras to each other and to functions on (the pro-unipotent part of) some pro-algebraic group scheme, to extract from it the motivic iterated integrals. Since the formal integrals satsify the various properties, so to do the motivic ones.

Coproduct on $\mathcal{A}_\bullet(F)$

$$\Delta I^{\mathfrak{a}}(a_0; a_1, \ldots, a_n; a_{n+1}) =$$
$$\sum_{\substack{0=i_0<i_1<\cdots \\ <i_k<i_{k+1}=n+1 \\ k=0,1,\ldots,N}} \left(I^{\mathfrak{a}}(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1}) \otimes \prod_{p=0}^{k} I^{\mathfrak{a}}(a_{i_p}; a_{i_p+1}, \ldots, a_{i_{p+1}-1}; a_{i_{p+1}}) \right)$$

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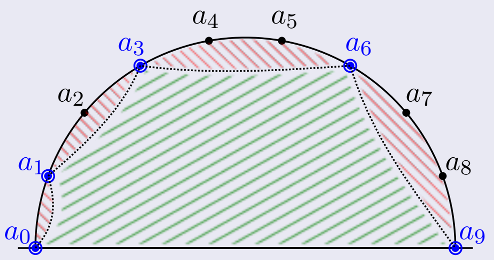
This is a rather complicated and ugly formula. Fortunately I won't display it for very long, I will instead give a mnemonic/pictorial interpretation of the formula.

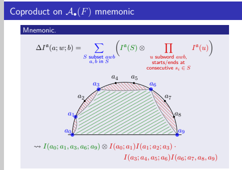
Coproduct on $\mathcal{A}_\bullet(F)$ mnemonic

Mnemonic.

$$\Delta I^{\mathfrak{a}}(a;w;b) = \sum_{\substack{S \text{ subset } awb \\ a,b \text{ in } S}} \left(I^{\mathfrak{a}}(S) \otimes \prod_{\substack{u \text{ subword } awb, \\ \text{starts/ends at} \\ \text{consecutive } s_i \in S}} I^{\mathfrak{a}}(u) \right)$$



$$\rightsquigarrow I(a_0;a_1,a_3,a_6;a_9) \otimes I(a_0;a_1)I(a_1;a_2;a_3) \cdot I(a_3;a_4,a_5;a_6)I(a_6;a_7,a_8,a_9)$$



We view the arguments a_i fo teh integral arranged into a semicircular polygon as follows. The coproduct is then obtained by selecting any subset of vertices. These vertices define a big 'main polygon' coloured green. This gives the left hand factor of the coproduct. In between these vertices we obtian a number of smaller 'cut-off' polygons, coloured red. We take the product of all of these and obtain the right hand term in the coproduct.

Results from motivic MZV's

- $\zeta^{\mathfrak{a}}(2k+1)$ are linearly independent
  - $\zeta^{\mathfrak{a}}(2k+1) \neq 0 \in \mathcal{A}_{2k+1}(\mathbb{Q})$
  - So have different gradings
- $\zeta^{\mathfrak{a}}(2k+1)$ are *algebraically* independent
  - Suppose some $\zeta^{\mathfrak{a}}(2k+1)$ satisfy a polynomial
  - Use coproduct Δ to show all coefficients are 0
- $\zeta^{\mathfrak{a}}(3,5)$ is irreducible (i.e. not in $\mathbb{Q}[\zeta(n)]$)
  - $(\Delta - \Delta^{\mathrm{op}})\zeta^{\mathfrak{a}}(3,5) = -5\zeta^{\mathfrak{a}}(3) \wedge \zeta^{\mathfrak{a}}(5)$
  - $(\Delta - \Delta^{\mathrm{op}})\zeta^{\mathfrak{a}}(n_1)\cdots\zeta^{\mathfrak{a}}(n_k) = 0$

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1. This follows trivially from the fact that $\mathcal{H}_\bullet$ is an object grade by the weight weight. So each $\zeta^{\mathfrak{m}}(n)$ is in a component of different weight n, therefore must be linearly independent.
2. This also follows reasonably easily. One can take set of such Riemann zeta values, and support they satisfy a only polynomail of some minimal degree. Applying the coproduct allows us to extract a polynomial of lower degree that they satisfy. Hence conclude the lower degree polynomial vanishes, and this tells us about the coefficients of our polynomial. This is enough to establish our starting polynomial also vanishes.
3. The proof of this is rather cute. The reduced coproduct of $\zeta^{\mathfrak{m}}(3,5)$ is (a mulitple of) $\zeta^{\mathfrak{m}}(3) \otimes \zeta^{\mathfrak{m}}(5)$, which gives $\zeta^{\mathfrak{m}}(3) \wedge \zeta^{\mathfrak{m}}(5)$ under antisymmetrisation $\Delta - \Delta^{op}$.
   On the other hand, the reduced coproduct of $\zeta(n)$ is 0. In such a case $\Delta(xy) = x \otimes y + y \otimes x$, so a $\zeta(n)$'s vanishes under antisymmetrisation.

Construction of motivic MZV's - Brown

- Problem: $I^{\mathfrak{a}}(0;1,0;1) \leftrightarrow -\zeta^{\mathfrak{a}}(2)$ vanishes.
- Consider 'motivic torsor' of paths ${}_0\Pi_1$ between 0 and 1 in $\mathbb{P}^1 \setminus \{0,1,\infty\}$. $\mathcal{O}({}_0\Pi_1) \cong \mathbb{Q}\langle e_0, e_1\rangle$.
- Straight line gives function $\mathcal{O}({}_0\Pi_1) \to \mathbb{R}$, evaluating MZV.
- Coalgebra of motivic MZV's is $\mathcal{H} := \mathcal{O}({}_0\Pi_1)/J^{\mathcal{MT}}$, $J^{\mathcal{MT}}$ the largest graded ideal in the kernel of above.
- $\mathcal{H} \cong \mathcal{A} \otimes_{\mathbb{Q}} \mathbb{Q}[\zeta^{\mathfrak{m}}(2)]$, $\mathcal{A} := \mathcal{A}_\bullet(\mathbb{Z})$
- Period map $\mathrm{per}: \mathcal{H} \to \mathbb{R}$, $\zeta^{\mathfrak{m}}(s_1,\ldots,s_k) \mapsto \zeta(s_1,\ldots,s_k)$, ring homomorphism
- Coaction by lifting Gonchrov's coproduct to $\mathcal{H} \to \mathcal{A} \otimes_{\mathbb{Q}} \mathcal{H}$.

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The main problem with Goncharov's motivic iterated integrals is that the $\zeta^{\mathfrak{a}}(2)$ element vanishes. This means there is no period map back down to real numbers, because $\zeta(2) \neq 0$. We do have a map to the associated graded of a certain filtered algebr of periods.

Brown fixes the problem by further lifting Goncharov's iterated integrals with $a_i \in \{0,1\}$, in such a way as $I^{\mathfrak{m}}(0;1,0;1) = -\zeta^{\mathfrak{m}}(2)$ is non-zero. We then do obtain a period map meaning motivic relations descend exactly to real number relations.

Infinitesimal coproduct

Definition (Derivations D_k)

Let $\mathcal{L} := \mathcal{A}/(\mathcal{A}_{>0} \cdot \mathcal{A}_{>0})$, which kills products and $\zeta^{\mathfrak{m}}(2)$. For k odd define

$$\begin{aligned} D_k : \quad & \mathcal{H} \to \mathcal{L}_k \otimes_{\mathbb{Q}} \mathcal{H} \\ & I^{\mathfrak{m}}(w) \mapsto (\pi \otimes \mathrm{id}) \circ (\Delta - 1 \otimes \mathrm{id})\, I^{\mathfrak{m}}(w) \end{aligned}$$

$$\begin{aligned} & D_k I^{\mathfrak{m}}(a_0; a_1, \ldots, a_N; a_{N+1}) = \\ & \quad \sum_{p=0}^{N-k} I^{\mathfrak{L}}(a_p; a_{p+1}, \ldots, a_{p+k}; a_{p+k+1}) \otimes \\ & \quad\quad I^{\mathfrak{m}}(a_0; a_1, \ldots, a_p, a_{p+k+1}, \ldots, a_N; a_N + 1) \end{aligned}$$

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$D_k I^{\mathfrak{m}}(a_0; a_1, \ldots, a_N; a_{N+1}) = \sum_{p=0}^{N-k} I^{\mathfrak{L}}(a_p; a_{p+1}, \ldots, a_{p+k}; a_{p+k+1}) \otimes I^{\mathfrak{m}}(a_0; a_1, \ldots, a_p, a_{p+k+1}, \ldots, a_N; a_N + 1)$

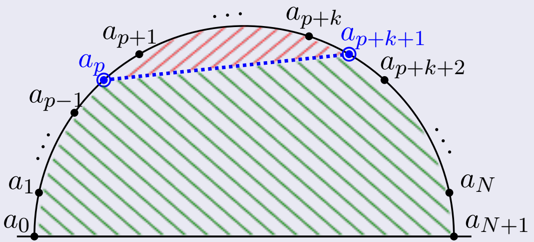
The full coproduct on motivic MZV's obviously reflects a lot of the structure and properties of the motivic MZV's. But the number of terms that the coproduct contains (order 2^n) grows rapidly, and make this object difficult to work with in general.
Francis Brown introducse an infinitesimal version of the coproduct, and shows it factors through a certian fmaily of operators D_{2r+1}. The derivations D_{2r+1} are defined as this composition, and can be explicitly described by the following formula.

Already this is much better, it only has n terms. We have to consider the entire family $D_3, D_5, \ldots$, but still we only get n^2 terms.

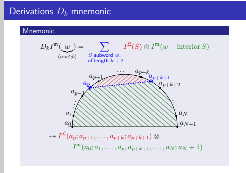
Derivations D_k mnemonic

Mnemonic.

$$D_k I^{\mathfrak{m}}(\underbrace{w}_{(a;w';b)}) = \sum_{\substack{S \text{ subword } w, \\ \text{of length } k+2}} I^{\mathfrak{L}}(S) \otimes I^{\mathfrak{m}}(w - \text{interior } S)$$



$$\rightsquigarrow I^{\mathfrak{L}}(a_p; a_{p+1}, \dots, a_{p+k}; a_{p+k+1}) \otimes$$
$$I^{\mathfrak{m}}(a_0; a_1, \dots, a_p, a_{p+k+1}, \dots, a_N; a_N + 1)$$



A mnemonic for remembering/computing this object is again given by this semicircular polygon. We cut out segments of length $k+2$.

This is a very combinatorial object: at some level we are essentially playing with binary words, and subwords.

Transcendental Galois Theory

Theorem (Brown, 2012)

In weight N, $\ker D_{<N} = \zeta^{\mathfrak{m}}(N)\mathbb{Q}$.

Example

Can show $\zeta^{\mathfrak{m}}(\{2\}^n) \in \zeta^{\mathfrak{m}}(2n)\mathbb{Q}$

As an integral $= \pm I^{\mathfrak{m}}(0; \underbrace{1,0,1,0,\ldots,1,0}_{n \text{ times}}; 1)$

- Alternates 0 and 1
- Odd length subword has same start and end letter
- Integral vanishes because boundaries are equal
- All D_{2r+1} vanish

So $\zeta^{\mathfrak{m}}(\{2\}^n) \in \ker D_{<2n} = \zeta^{\mathfrak{m}}(2n)\mathbb{Q}$.

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This upshot of defining these operators comes with an important theorem of Brown, which describes the simultaneously kernel of $D_3, D_5, \ldots$. If all of the D_{odd} vanish on a weight N combination of motivic MZV's, then the result is a rational multiple of $\zeta^{\mathfrak{m}}(N)$.

As an example, we can give a quick proof, up to $\mathbb{Q}$ of the $\zeta(2,2,\ldots,2)$ identity from the start.

Conjectural identities

The following identities appear to hold.

Conjecture (Hoffman)

For $m \in \mathbb{Z}_{\geq 0}$

$$2\zeta(3,3,\{2\}^m) - \zeta(3,\{2\}^m,1,2) \overset{?}{=} -\frac{\pi^{2m+6}}{(2m+7)!} = -\frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$$

Conjecture (Cyclic insertion - Borwein, Bradley, Broadhurst, Lisoněk)

For $n \in \mathbb{Z}_{\geq 0}$*, and* $a_0, \ldots, a_{2n} \in \mathbb{Z}_{\geq 0}$*,*

$$\sum_{\text{cycle } a_i} \zeta(\{2\}^{a_0}, 1, \{2\}^{a_1}, 3, \ldots, 1, \{2\}^{a_{2n-1}}, 3, \{2\}^{a_{2n}}) \overset{?}{=} \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$$

"Insert all cyclic permutations of some blocks $\{2\}^{a_i}$ *of two's."*

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For $n \in \mathbb{Z}_{\geq 0}$, and $a_0, \ldots, a_{2n} \in \mathbb{Z}_{\geq 0}$,
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"Insert all cyclic permutations of some blocks $\{2\}^{a_i}$ of two's."

This conjecture seems to be relatively unknown. It is just listed on Hoffman's website as an example, with little fanfare. This has been checked up to weight 22, where $n = 8$, by using explicit tables of MZV relations. Still unproven in general.
This conjecture is presented by BBBL as a generalisation of the Zagier-Broadhurst identity. When all the $a_i = 0$, we reduce to it. When $n = 0$, we get the $\zeta(2, \ldots, 2)$ evauation. The only case of this conjecture which has been proven is $a_0 = 1$, $a_{>0} = 0$. This direction has been generalised to the Bowman-Bradley theorem where all compositions $a_0 + \ldots + a_{2n} = M$ are taken.
I want to show now how to unify these two conjectures, into a much more general conjecture, by identifying the important underlying structue. With this we can make a little progress towards a proof (via a symmetrisation result). Hopefully with this insight, and more work, an exact proof can be given.

Structure of identities - Hoffman

- Write Hoffman's identity as iterated integrals

$$
\begin{aligned}
& 2\zeta(3,3,\{2\}^n) && -\zeta(3,\{2\}^n,1,2) \\
={}& \zeta(3,3,\{2\}^n) && -\zeta(3,\{2\}^n,1,2) \qquad +\zeta(\{2\}^n,1,2,1,2) \\
\rightsquigarrow{}& I(0100100(10)^n1) + I(0100(10)^n1101) + I(0(10)^n1101101)
\end{aligned}
$$

- Split into 'alternating blocks' at $00 \to 0 \mid 0$ or $11 \to 1 \mid 1$

$$
\begin{aligned}
&= I(010 \mid 010 \mid 0(10)^n1) + I(010 \mid 0(10)^n1 \mid 101) \\
&\qquad + I(0(10)^n1 \mid 101 \mid 101)
\end{aligned}
$$

- Record lengths of the blocks

$$
= I_{\mathrm{bl}}(3,3,2n+2) + I_{\mathrm{bl}}(3,2n+2,3) + I_{\mathrm{bl}}(2n+2,3,3)
$$

- Right hand side is $I_{\mathrm{bl}}(\mathrm{wt}+2)$

Structure of identities - Hoffman
Write Hoffman's identity as iterated integrals
Split into 'alternating blocks' at $00 \to 0 \mid 0$ or $11 \to 1 \mid 1$
Record lengths of the blocks
Right hand side is $I_{\mathrm{bl}}(\mathrm{wt}+2)$

1. First I want to split up the term with coefficient 2 using duality. This is just for convenience. There is also an overall sign coming from the depth that I'm ignoring for simplicity. I only bother with relative differences in depth, which is why the sign of the middle term change swhen goign to the integrals.
   If you work out the overalll sign carefully, everything is okay with the final result on this slide.
2. The right hand side of the identity is $\frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!} = \zeta(\{2\}^{\mathrm{wt}/2})$. Writing this as an iterated integral gives $I(0(10)^{\mathrm{wt}/2}1) = I_{\mathrm{bl}}(\mathrm{wt}+2)$.

Structure of identities - BBBL

- Write the BBBL identity as iterated integrals

$$\sum_{\text{cycle } a_i} \zeta(\{2\}^{a_0}, 1, \{2\}^{a_1}, 3, \ldots, 1, \{2\}^{a_{2n-1}}, 3, \{2\}^{a_{2n}})$$

$$\rightsquigarrow \sum_{\text{cycle } a_i} I(0(10)^{a_0}1(10)^{a_1}100\cdots01(10)^{a_{2n-1}}100(10)^{a_{2n}}1)$$

- Split into 'alternating blocks' at $00 \to 0 \mid 0$ or $11 \to 1 \mid 1$

$$= \sum_{\text{cycle } a_i} I(0(10)^{a_0}1 \mid (10)^{a_1}10 \mid 0\cdots01 \mid (10)^{a_{2n-1}}10 \mid 0(10)^{a_{2n}}1)$$

- Record lengths of the blocks

$$= \sum_{\text{cycle } a_i} I_{\text{bl}}(2a_0+2, 2a_1+2, \ldots, 2a_{2n}+2)$$

- Right hand side is $I_{\text{bl}}(\text{wt}+2)$.

1. I'm again ignoring a sign coming from the depth, but the final result is right if you work out this sign carefully.

Common sructure

Both conjectures have the form

$$\sum_{\text{cycle } \ell_i} I_{\text{bl}}(\ell_1, \ldots, \ell_n) \overset{?}{=} I_{\text{bl}}(\text{wt} + 2)$$

Generally:

Conjecture (Generalised cyclic insertion, C., arXiv 1703.03784)

For any $[\ell_1, \ldots, \ell_n]$ *with all* $\ell_i > 1$,

$$\sum_{\text{cycle } \ell_i} I_{\text{bl}}(\ell_1, \ldots, \ell_n) \overset{?}{=} I_{\text{bl}}(\text{wt} + 2)$$

(Extra correction terms needed if some $\ell_i = 1$*.)*

- Numerically tested all cases weight ≤ 18, to 500 decimal places
- Can prove a symmetrised version, up to $\mathbb{Q}$
- Can prove *some* special cases, up to $\mathbb{Q}$

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- Numerically tested all cases weight ≤ 18, to 500 decimal places
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- Can prove some special cases, up to $\mathbb{Q}$

The correction terms have the from $\zeta(\text{wt} - 2k)$ times block integrals starting with $[\{1\}^k, \ldots]$ where the divergent $\{1\}^k$ string is removed.

1. The fourth identity on an earlier slide is an example of such a motivically provable special case. It arises from $I(0(10)^m 1 \mid 10 \mid 010 \mid 01 \mid 101) = I_{\text{bl}}(2m + 2, 2, 3, 2, 3)$.

Progress

Theorem (Symmetric insertion, C., arXiv 1703.03784)

For any $[\ell_1, \ldots, \ell_n]$, *with even weight,*

$$\sum_{\text{permute } \ell_i} I_{\mathrm{bl}}(\ell_1, \ldots, \ell_n) \in I_{\mathrm{bl}}(\mathrm{wt} + 2)\mathbb{Q}$$

Proof (Sketch).

- Lift to motivic version $I^{\mathfrak{m}}$.
- Define a reflection $\mathcal{R}$ on subsequences
- Set up a pairwise cancellation in $D_{<N}$.
- Conclude $\in \zeta^{\mathfrak{m}}(\mathrm{wt})\mathbb{Q} = I^{\mathfrak{m}}_{\mathrm{bl}}(\mathrm{wt}+2)\mathbb{Q}$ using Brown. $\square$

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Since the period map preserves motivic relations, it is sufficient ot show this on hte level of motivic MZV's