TAME PARAMETRISATION THEOREM

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1. Notation

Standard notation, to write on the side board?

$$\begin{aligned}
&F = \text{non-Archimedean local field}\\
&\mathfrak{o}_F = \mathfrak{o} = \text{the discrete valuation ring of } F\\
&\mathfrak{p}_F = \mathfrak{p} = \text{the maximal ideal of } \mathfrak{o}\\
&k_F = k = \mathfrak{o}/\mathfrak{p}\\
&p = \text{the charcteristic of } k\\
&q_F = q = |k|\\
&U_F = \text{the group of units of } \mathfrak{o}\\
&U_F^n = 1 + \mathfrak{p}^n
\end{aligned}$$

Also we have similar for E. We have $G = \mathrm{GL}_2(F)$, and $A = M_2(F)$.
Recall also the definition of a cuspidal type

Definition 1.1. A cuspidal type in G is a triple $(\mathfrak{A}, J, \lambda)$, where $\mathfrak{A}$ is a chain order in $A = M_2(F)$, J is a subgroup of $\mathcal{K}_{\mathfrak{A}}$ and Λ is an irreducible smooth representation of J (of a certain type...).

Introduce later. For ψ character of F, and χ character of $F^\times$, and E/F extension set

$$\begin{aligned}
\psi_E &= \psi \circ \mathrm{Tr}_{E/F}\\
\chi_E &= \psi \circ N_{E/F}\\
\psi_A &= \psi \circ \mathrm{Tr}_A\\
\psi_\alpha = \psi_{A,a} &\colon x \mapsto \psi_A(a(1-x))\,.
\end{aligned}$$

2. Goal

The goal of this talk is to describe the relationship between cuspidal representations of $G = GL_2(F)$, and multiplicative characters of certain quadratic field extensions E/F. We focus in particular on the case of $p := \operatorname{char} k = \operatorname{char}(\mathfrak{o}_F/\mathfrak{p}_F) \neq 2$: the theory can be made to work when $\operatorname{char} k = 2$, but only certain 'unramified' cuspidal representations have such a description. (I will try to indicate what differs later.)

We will describe certain objects called 'admissible pairs' $(E/F, \chi)$ (consisting of a quadratic field extension E/F, and a character χ of $E^\times$), and describe a map $(E/F, \chi) \mapsto \pi_\chi$ which constructs a cuspidal representation π_χ from the 'admissible pair'. With this, we build up to the following theorem.

Theorem 2.1 (Schematic form of Tame parametrisation theorem, Theorem 20.2)**.** *For $p \neq 2$, There is a bijection between*

$$\left\{\begin{array}{l}\textit{The space } \mathbb{P}_2(F) \textit{ of } F\textit{-isomorphism}\\ \textit{classes of 'admissible pairs' } (E/F,\chi)\end{array}\right\} \overset{1:1}{\leftrightarrow} \left\{\begin{array}{l}\textit{The space } \mathcal{A}_2^0(F) \textit{ of equivalence classes of}\\ \textit{irreducible cuspidal representations of } GL_2(F)\end{array}\right\}.$$

3. Admissible pairs

First we deal with some field theoretic constructions.

Definition 3.1 (Ramification)**.** Let E/F be an extension of local fields, of finite degree. Since there is a unique prime ideal $\mathfrak{p}_E$ in $\mathfrak{o}_E$, we have $\mathfrak{p}_E^e = \mathfrak{p}_F\mathfrak{o}_E$, for some integer e. Define the ramification index $e(E/F) \coloneqq e$. Then $[E : F] = efg$, where $e = e(E/F)$, $g = 1$ since there is only one prime, and the inertial degree $f = f(E/F) = [\mathfrak{o}_E/\mathfrak{p}_E\colon \mathfrak{o}_F/\mathfrak{p}_F] = [k_F\colon k_E]$ is the degree of the induced residue field extension.

Definition 3.2 (Tame ramification)**.** The finite extension E/F of local fields is *tamely* ramified if $p \nmid e(E/F)$. If E/F is quadratic, then $2 = [E : F] = ef$, so we see

- $p \neq 2$ means E/F quadratic is always tamely ramified,
- $p = 2$ means E/F quadratic is tamely ramified iff it is unramified

Let's recall a few properties of the norm and trace in tamely ramified extensions. The notation $[x]$ means the greatest integer function, i.e. the floor function.

Property 3.3. Let E/F be tamely ramified, and $e = e(E/F)$. Then

- For $r \in \mathbb{Z}$, $\mathrm{Tr}_{E/F}(\mathfrak{p}_E^{1+r}) = \mathfrak{p}^{1+[r/e]} = \mathfrak{p}_E^{1+r} \cap F$.
- For $m \geq 1$, $N_{E/F}$ indices isomorphism
$$U_E^{em}/U_E^{em+1} \cong U_F^m/U_F^{m+1}$$
satisfying
$$N_{E/F}(1+x) \equiv 1 + \mathrm{Tr}_{E/F}(x) \pmod{\mathfrak{p}^{m+1}}, x \in \mathfrak{p}_E^{em}$$
- Norm $N_{E/F}$ induces a map $U_E/U_E^1 \to U_F/U_F^1$. Surjective if E/F unramified. Has kernel and cokernel with order $\gcd(e, q-1)$ if E/F is totally ramified, i.e $f = 1$.

Let ψ be a character of F of level 1 (meaning $\mathfrak{p} \subset \ker\psi$, but $\mathfrak{p}^0 = \mathfrak{o} \not\subset \ker\psi$). Lift $\psi = \psi_F$ to a character of E by composing with the trace: $\psi_F \coloneqq \psi \circ \mathrm{Tr}_{E/F}$. Let χ be a character of $F^\times$ of level $m \geq 1$, meaning χ is trivial on U_F^{m+1}, but not on U_F^m. Lift $\chi = \chi_F$ to a character of $E^\times$ by composing with the norm: $\chi_E \coloneqq \chi \circ N_{E/F}$.

Proposition 3.4.
- *The character ψ_E has level 1.*
- *The character χ_E has level em.*
- *If $c \in \mathfrak{p}^{-m}$ satisfies $\chi(1+x) = \psi(cx)$, $x \in \mathfrak{p}^{[m/2]+1}$, then*
$$\chi_E(1+y) = \psi_E(cy), y \in \mathfrak{p}_E^{[em/2]+1}.$$

Aside

Proof.
- From property 1, we have $\psi_E(\mathfrak{p}_E) = \psi(\mathrm{Tr}_{E/F}(\mathfrak{p}_E^{1+0})) = \psi(\mathfrak{p}^{1+[0/e]}) = \psi(\mathfrak{p}^1) = 0$. So $\mathfrak{p}_E \subset \ker\psi_E$. But $\psi_E(\mathfrak{o}_E) = \psi(\mathrm{Tr}_{E/F}(\mathfrak{p}_E^{1-1})) = \psi(\mathfrak{p}^{1+[-1/e]}) = \psi(\mathfrak{p}^0) \neq 0$. So ψ_E has level 1.
- χ_E is defined by composing with $N_{E/F}$. From property 2 above, we see χ is trivial on U_F^{m+1}, but not U_F^m, so the isomorphism implies χ_E is trivial on U_E^{em+1}, but not on U_E^{em}. Thus χ_E has level em.
- Finally, if $x \in \mathfrak{p}_E^{[em/2]+1}$, then
$$N_{E/F}(1+x) \equiv 1 + \mathrm{Tr}_{E/F}(x) \pmod{\mathfrak{p}^{m+1}}.$$
Now we have, for some $\alpha \in \mathfrak{p}^{m+1}$ that
$$\begin{aligned}\chi_E(1+y) &= \chi(N_{E/F}(1+y)) = \chi(1 + \mathrm{Tr}_{E/F}(y) + \alpha)\\ &= \psi(c\,\mathrm{Tr}_{E/F}(y) + c\alpha)\\ &= \psi(c\,\mathrm{Tr}_{E/F}(y))\end{aligned}$$
since $c\alpha \in \mathfrak{p}^{m+1} \subset \mathfrak{p}^1 \subset \ker\psi$. Now $\psi(c\,\mathrm{Tr}_{E/F}(y)) = \psi(\mathrm{Tr}_{E/F}(cy)) = \psi_E(cy)$, since $c \in F$. ∎

Now we are in a position to give the definition of admissible pairs, and the notion of equivalence between admissible pairs.

Definition 3.5 (Admissible pairs, F-isomorphic, minimal)**.** (1) Consider the pair $(E/F, \chi)$, where E/F is a tamely ramified quadratic field extension, and χ is a character of $E^\times$. We say $(E/F, \chi)$ is *admissible* if

- χ does not factor through the norm map $N_{E/F}\colon E^\times \to F^\times$ (so that it is not just a trivial lifting of some character of $F^\times$), and
- If $\chi \mid U_E^1$ does factor through $N_{E/F}$, then E/F is unramified

(2) Given $(E/F, \chi)$ and $(E'/F, \chi')$ two admissible pairs. We say they are *F-isomorphic* if there is an F-isomorphism $j\colon E \to E'$ such that $\chi = \chi' \circ j$. For $E = E'$, this means $\chi' = \chi^\sigma$, for some $\sigma \in \mathrm{Gal}(E/F)$. THe set of F-isomorphism classes of admissible pairs is $\mathbb{P}_2(F)$

(3) given $(E/F, \chi)$ admissible, and χ of level n, we say $(E/F, \chi)$ minimal if $\chi \mid U_E^n$ does not factor through $N_{E/F}$.

Some properties of admissible pairs

Property 3.6.

- If $(E/F, \chi)$ admissible, and ϕ a character of $F^\times$, then $(E/F, \chi \otimes \phi_E)$ admissible, where $\phi_E = \phi \circ N_{E/F}$.
- Any admissible $(E/F, \chi)$ is isomorphic to one of the form $(E/F, \chi' \otimes \phi_E)$, for ϕ a character of $F^\times$ and $(E/F, \chi')$ minimal.

Aside

If $(E/F, \chi)$ not minimal, then $\chi = \chi\, |_{F/U_E^n} \otimes (\chi \mid U_E^n) = \chi' \otimes N_{E/F} \circ \phi$, where ϕ a character of F, so $\chi = \chi' \otimes \phi_E$.

4. Construction of representations

How do we attach to $(E/F, \chi)$ an irreducible cuspidal representation of $G = \mathrm{GL}_2(F)$?

4.1. **Special case: level χ is 0.** Assume $(E/F, \chi)$ admissible, and χ character of $E^\times$ has level 0. Then E/F is unramified, by definition as $\chi \mid U_E^1 \equiv 0$ so (trivially) factors through $N_{E/F}$.

Then k_E/k is a quadratic extension. Choose F-embedding of $E \to A$, and let $\mathfrak{A}$ be the unique chain order with $E^\times \subset \mathcal{K}_{\mathfrak{A}}$. We can conjugate by an element of G to assume $\mathfrak{A} = \mathfrak{M} = M_2(\mathfrak{o})$, to embed $\mathfrak{o}$ in $\mathfrak{M}$. So we get a k embedding of k_E in $M_2(k)$.

The character $\chi \mid U_E$ is the inflation of a regular character $\widetilde{\chi}$ of $k_E^\times$. (Recall: inflation $G \twoheadrightarrow G/N \to \mathrm{GL}(V)$ lifts from quotient to whole group, or pulling back under group homomorphism?. Regular means $\theta^\sigma \neq \theta$, where σ is unique automorphism of k_E/k. We have $k_E = \mathfrak{o}_E/\mathfrak{p}_E$, so that $\mathfrak{o}_E \supset U_E \to k_E^\times$.) It was NOT explained earlier in 6.4 how such a $\widetilde{\chi}$ gives rise to an irreducible cuspidal rep $\widetilde{\lambda}$ of $\mathrm{GL}_2(k)$.

Aside

Change of notation! One has: Regular character θ of $l^\times$, l quadratic extension of k. View $l^\times$ as subgroup E of $G = \mathrm{GL}_2(k)$ by writing $l = k \oplus k$ and acting naturally via $l^\times$. Let ψ be a non-trivial character of $N < G$, then define $\theta_\psi)$ of ZN by $a\,\mathrm{Id}_2\, u \mapsto \theta(a)\psi(u)$. Up to equivalence, $\mathrm{Ind}_{ZN}^G \theta_\psi$ is independent of ψ. Then

$$\mathrm{Ind}_{ZN}^G \theta_\psi - \mathrm{Ind}_E^G$$

is an irreducible cuspidal representation of G, of dimension $q-1$. Every irreducible cuspidal rep of G is of this form, and representations are equivalence if and only if $\theta_1 = \theta_2$ or $\theta_1 = \theta_2^q$.

Inflate $\widetilde{\lambda}$ to representation λ of $U_{\mathfrak{M}} = \mathrm{GL}_2(\mathfrak{o})$. (Since $\mathrm{GL}_2(\mathfrak{o}) \to \mathrm{GL}_2(\mathfrak{o}/\mathfrak{p}) = \mathrm{GL}_2(k)$.) Now $\lambda \mid U_F$ is a multiple of $\chi \mid U_F$ (from equation 6.4.1). We then extend λ to a representation Λ of $\mathcal{K}_{\mathfrak{M}} = F^\times U_{\mathfrak{M}}$ by demanding $\Lambda \mid F^\times$ is a multiple of χ.

The triple $(\mathfrak{M}, \mathcal{K}_{\mathfrak{M}}, \Lambda)$ is then a cuspidal type, and we can define

$$\pi_\chi := \text{c-Ind}_{\mathcal{K}_{\mathfrak{M}}}^G \Lambda$$

to get an irreducible cuspidal representation of G, with length $\pi_\chi = 0$.

This construction only depends on the F-isomorphism class of $(E/F, \chi)$ as isomorphic level 0 pairs give rise to conjugate pairs $(k_E^\times, \widetilde{\chi})$ in $\mathrm{GL}_2(k)$.

Write

$$\mathbb{P}_2(F)_0 = \{ \text{ isomorphism classes of } (E/F, \chi), \text{ where } \chi \text{ has level } 0 \}$$

$$\mathcal{A}_2^0(F)_0 = \{ \text{ equivalence classes of irreducible cuspidal representations of } G \text{ with length } 0 \ \}$$

Proposition 4.1. *The map $(E/F, \chi) \to \pi_\chi$ induces a bijection*

$$\mathbb{P}_2(F)_0 \xrightarrow{\cong} \mathcal{A}_2^0(F)_0$$

Proof. The construction of cuspidal representations in $\mathrm{GL}_2(k)$ from Section 6.4 shows that any cuspidal type of level 0 arises from $(E/F, \chi) \in \mathbb{P}_2(F)_0$. By the Exhaustion Theorem 14.5 from Xenia's talk, the map is surjective, then.

Aside

For level 0, there is an equivalence between $(E/F, \chi)$ admissible, and χ a regular character of $k_E^\times$. Regular means $\chi \mid U_E \neq \chi^\sigma \mid U_E$, $\sigma \neq 1 \in \mathrm{Gal}(E/F)$. Equivalent to $\chi \neq \chi^\sigma$ since E/F is unramified and $E^\times = F^\times U_E$. But χ factors through $N_{E/F}$ iff $\chi = \chi^\sigma$. So admissible iff $\chi \neq \chi^\sigma$.

For injectivity, suppose $(E_i/F, \chi_i)$ with π_{χ_i} equivalent. The E_i/F are unramified so are isomorphic, as there is a unique unramified extension of degree 2. So wlog $E_1 = E_2 = E$. The central character relation gives $\chi_1 \mid F^\times = \chi_2 \mid F^\times$, so the (induced?) cuspidal representations $\widetilde{\lambda}_i$ of $\mathrm{GL}_2(k)$ are equivalent. By Theorem 6.4 (2), the characters $\widetilde{\chi}_i$ of $k_E^\times$ are Galois conjugate, and hence so are the characters $\chi_i \mid U_E$. This means the pairs $(E/F, \chi_i)$ are F-isomorphic. □

Property 4.2. This map satisfies some properties

- If ϕ a character of $F^\times$ of level 0, then $\pi_{\chi\phi_E} = \phi\pi_\chi$.
- If $\pi = \pi_\chi$, then $\omega_\pi = \chi \mid F^\times$, (where ω_π is the associated central character).
- The pair $(E/F, \check{\chi})$ is admissible, and $\check{\pi_\chi} = \pi_{\check{\chi}}$.

4.2. **Level of χ is ≥ 1.** How to extend this to characters χ of level ≥ 1?

Fix character $\psi \in \widehat{F}$ of level 1. Let $(E/F, \chi)$ be minimal admissible such that χ has level $n \geq 1$. Set $\psi_E = \psi \circ \mathrm{Tr}_{E/F}$, $\psi_A = \psi \circ \mathrm{Tr}_A$.

Choose an element $\alpha \in \mathfrak{p}_E^{-n}$ such that $\chi(1+x) = \psi_E(\alpha x)$, $x \in \mathfrak{p}_E^{[n/2]+1}$. Choose an F-embedding of E into $A = M_2(F)$, and let $\mathfrak{A}$ be the unique chain order in A such that $E^\times \subset \mathcal{K}_{\mathfrak{A}}$. Then $e_{\mathfrak{A}} = e(E/F)$ and $(\mathfrak{A}, n, \alpha)$ is a simple stratum. Attached to this, we have subgroups $J_\alpha, J_\alpha^1, H_\alpha^1$, and we will define an irrep Λ in $C(\psi_\alpha, \mathfrak{A})$ (as in Def 15.3). (Recall the notation ψ_α is shorthand for $\psi_{A,alpha}$, where $\psi_{A,a} \colon x \mapsto \psi_A(a(x-1))$.)

(The representation is much easier to define when the level is odd.)

Case $n = 2m+1$ odd: We define Λ by requiring it is the character of $J_\alpha = E^\times U_{\mathfrak{A}}^{m+1}$ given by

$$\Lambda \mid U_{\mathfrak{A}}^{m+} = \psi_\alpha\,, \quad \Lambda \mid E^\times = \chi\,.$$

These conditions are compatible/consistent since $\mathrm{Tr}_A \mid E = \mathrm{Tr}_{E/F}$ and $E \cup U_{\mathfrak{A}}^{m+1} = U_E^{m+1}$. So the triple $(\mathfrak{A}, J_\alpha, \Lambda)$ is a cuspidal type in G, and

$$\pi_\chi = \text{c-Ind}_{J_\alpha}^G \Lambda$$

is an irreducible cuspidal representation of G containing fundamental stratum $(\mathfrak{A}, n, \alpha)$.

We obtain properties

- $\ell(\pi_\chi) = n/e(E/F)$,
- $\omega_{\pi_\chi} = \chi \mid F^\times$.

Case $n = 2m$ even: Then E/F is unramified (because $\chi \mid U_E^{2m}$ does not factor through $N_{E/F}$...?)

Define a character θ of $H_\alpha^1 = U_E^1 U_{\mathfrak{A}}^{m+1}$ by

$$\theta(ux) = \chi(u)\psi_\alpha(x)$$

where $x \in U_{\mathfrak{A}}^{m+1}$, $u \in U_E^1$. Conditions are consistent. Let ηbe the unique irrep of $J_\alpha^1 = U_E^1 U_{\mathfrak{A}}^1$ which contains θ.

Fact 4.3 (Corollary 19.4)**.** There is a unique irreducible representation Λ of J_α such that

- $\Lambda \mid J_\alpha^1 \cong \eta$
- $\Lambda \mid F^\times$ is a multiple of $\chi \mid F^\times$,
- for every $\zeta \in \mu_E \setminus \mu_F$, we have $\operatorname{Tr}\Lambda(\zeta) = -\chi(\zeta)$.

(Here μ_E are the roots of unity, of order prime to p in E.)

Aside

This requires a technical proposition in the representatoin theory of (finite?) groups.

Proposition 4.4. *There is a unique irrep $\widetilde{\eta}$ of $\mu_E/\mu_F \ltimes J_\alpha^1$, such that $\widetilde{\eta} \mid J_\alpha^1 \cong \eta$, and*

$$\operatorname{Tr}\widetilde{\eta}(\zeta u) = -\theta(u)\,,$$

for $u \in H_\alpha^1$ and $\zeta \neq 1 \in \mu_E/\mu_F$.

The proof of this is long and technical, taking 2 pages.

The representation Λ lies in $C(\psi_\alpha, \mathfrak{A})$, and so we define

$$\pi_\chi = \text{c-Ind}_{J_\alpha}^G \Lambda\,.$$

Then π_χ is an irreducible cuspidal reprsentation of G satisfying the same properties as before. (Have $e(E/F) = 1$, giving $\ell(\pi_\chi) = 1$.)

Proposition 4.5. *This construction is independent of choices: π_χ depends only on the isomorphism class of $(E/F, \chi)$, and it is independent of the choice of ψ, α and embedding $E \to A$.*

Moreover if ϕ is a character of $F^\times$ such that $(E/F, \chi\phi_E)$ minimal, then $\pi_{\chi\phi_E} = \phi\pi_\chi$.

Aside

Proof.
- Changing ψ, α does not change the group J_α, or the representation Λ.
- Any two F-embeddings of $E \to A$ are G-conjugate, giving conjugate cuspidal types $(\mathfrak{A}, J_\alpha, \Lambda)$
- F-isomorphic minimal pairs $(E_i/F, \chi_i$ with $j_i \colon E_i \hookrightarrow F$ can be realised by a G-conjugation, taking $j_1(E_1)$ to $j_2(E_2)$ and matching the characters. (Ex 18.2)
- The character $\chi\phi_E$ gives rise to cuspidal type $(\mathfrak{A}, J_\alpha, \Lambda \otimes \phi \circ \det)$.

□

5. Tame parametrisation theorem

We can extend the above construction to all admissible pairs $(E/F, \chi)$, as follows. If $(E/F, \chi)$ is admissible, then there is a character ϕ of $F^\times$ and χ' of $E^\times$ such that $(E/F, \chi')$ is minimal, and $\chi = \chi'\phi_E$. Define

$$\pi_\chi := \phi\pi_{\chi'}$$

This is independent of the decomposition $\chi = \chi'\phi_E$.

Aside

If $\chi''\phi'_E$ is another decomposition, then we get $\phi'\pi_{\chi''}$. But then χ'' can be related to χ' via character ψ_E from character ψ of $F^\times$, so that $\chi''\psi_E = \chi'$. Then we have $\chi = \chi'\phi_E = \chi''\psi_E\phi_E$

and

$$\psi\phi\pi_{\chi''} = \phi\pi_{\chi'} = \pi_\chi$$

using the last property of the above proposition.

We obtain the same properties as before

- $\ell(\pi_\chi) = n/e(E/F)$,
- $\omega_{\pi_\chi} = \chi \mid F^\times$.

From this we obtain the following map from F-isomorphism classes of admissible pairs $(E/F, \chi)$ to equivalence classes of cuspidal representations of $\mathrm{GL}_2(F)$:

$$\mathbb{P}_2(F) \to \mathcal{A}_2^0(F)$$
$$(E/F, \chi) \mapsto \pi_\chi\,,$$

which is independent of all choices.

Aside

We wish to claim now that this map is always a bijection, but this isn't quite true when $p = 2$. We therefore need to introduce the notion of an *unramified* representation: we say π irreducibel cuspidal rep of $G = \mathrm{GL}_2(F)$ is unramified if there is an unramified $\phi \neq 1$ of $F^\times$ such that $\phi\pi \cong \pi$.

Write $\mathcal{A}_2^{nr}(F)$ for the unramified classes in $\mathcal{A}_2^0(F)$.

Then

Theorem 5.1 (Tame parametrisation theorem)**.** *The map $(E/F, \chi) \mapsto \pi_\chi$ induces a bijection*

$$\mathbb{P}_2(F) \xrightarrow{\cong} \begin{cases} \mathcal{A}_2^0(F) & \text{if } p \neq 2, \text{ or} \\ \mathcal{A}_2^{nr}(F) & \text{if } p = 2 \end{cases}$$

This map satisfies the following properties

- *If χ has level $\ell(\chi)$, then $\ell(\pi_\chi) = \ell(\chi)/e(E/F)$,*
- *$\omega_{\pi_\chi} = \chi \mid F^\times$,*
- *if ϕ is a character of $F^\times$, then $\pi_{\chi\phi_E} = \phi\pi_\chi$. Also*
- *the pair $(E/F, \check{\chi})$ is admissible, and $\pi_{\check{\chi}} = \check{\pi_\chi}$.*

Aside

The first 3 properties have been observed above. If $\pi_\chi = \text{c-Ind}_J^G \Lambda$ for cuspidal type $(\mathfrak{A}, J, \Lambda)$ constructed from $(E/F, \chi)$. THen $(E/F, \check{\chi})$ gives rise to type $(\mathfrak{A}, J, \Lambda^\wedge)$. So last property follows from Remark 11.4 (3).

6. Proof of tame parametrisation theorem

In full, the proof of this theorem is quite long. The main step is to show injectivity, but this requires using a technical lemma. The proof of this lemma requires a further 4 lemmas, so I will only mention the main steps.

6.1. **Injectivity.** We need to show injectivity.

The map $(E/F, \chi) \mapsto \pi_\chi$ is injective on isomorphism classes of minimal pairs.

Proof. Given minimal pairs $(E_i/F, \chi_i)$ such that $\pi = \pi_{\chi_1} \cong \pi_{\chi_2}$, either π is unramified and hence so are E_i/F (This is the result of a Lemma 20.3).

Otherwise, find some ramified character $\phi \neq 1$ of $F^\times$ such that $\phi\pi \cong \pi$. Then $N_{E_i/F}(E_i^\times) = \ker\phi$. (This is the result of Proposition 20.3.)

In both cases $E_1 \cong E_2$ (by Exercise 18.2 (2)), so wlog we have $E_1 = E_2 = E$, and the level of χ_i is $n = e(E/F)\,\ell(\pi)$.

Choose an F-embedding of E in A and let $\mathfrak{A}$ be the chain order normalised by (the image of) $E^\times$. Then form $J = E^\times U_{\mathfrak{A}}^{[(n+1)/2]}$. The pair $(E/F, \chi_i)$ gives a cuspidal type $(\mathfrak{A}, J, \Lambda_i)$, and restricting Λ_i to $U_{\mathfrak{A}}^{[n/2]+1]}$ is a multiple of ψ_{α_i}, some $\alpha_i \in \mathfrak{p}_E^{-n}$. These two cuspidal types intertwine in G, meaning the characters ψ_{α_i} of $U_{\mathfrak{A}}^{[n/2]+1}$ are $U_{\mathfrak{A}}$ conjugate (using 15.2).

A lemma shows that there exists $u \in U_{\mathfrak{A}}$, such hat $u\alpha_2 u^{-1} \in E$ and $u\alpha_2 u^{-1} \equiv \alpha_1 \pmod{\mathfrak{p}_E^{-[n/2]}}$.

So we can assume, after conjugating, that both α_i lie in E , and the characters ψ_{α_1} and ψ_{α_2} are the same. THe representations Λ_i intertwine in G, so are equivalent (using 15.6).

By the definitions of Λ_i, we can recover χ_i from $\Lambda_i \mid E^\times$ if n is odd, or from a formula given in Corollary 19.4 (Fact above!). Whence $\chi_1 = \chi_2$, which shows injectivity. □

6.2. **Surjectivity.** If $\pi \in \mathcal{A}_2^0(F)$ satisfies $0 < \ell(\pi) \leq \ell(\phi\pi)$ for all characters ϕ of $F^\times$ (and π is unramified, if $p = 2$). Then there exits a minimal pair $E/F, \chi) \in \mathbb{P}_2(F)$ such that $\pi_\chi \cong \pi$.

Proof. The representation π contains a cuspidal type $(\mathfrak{A}, J, \Lambda)$ atached to a simple stratum $(\mathfrak{A}, n, \alpha)$. We have $J = J_\alpha$, and $\Lambda \mid U_{\mathfrak{A}}^{[n/2]+1}$ is a multiple of ψ_α. TAke $E = F[\alpha]$, and get $J_\alpha = E^\times U_{\mathfrak{A}}^{[(n+1)/2]}$.

If n is odd, put $\chi = \Lambda \mid E^\times$, to get minimal admissible pair $(E/F, \chi)$.

If n is even, let θ be the unique character of H_α^1 occuring in $\Lambda \mid H_\alpha^1$.

Then there is a unique character χ of $E^\times$ such that $\chi \mid U_E^1$, $\chi \mid F^\times = \omega_\pi$, and $\chi(\zeta) = -\operatorname{Tr}\Lambda(\zeta)$, for every $\zeta \in \mu_E \setminus \mu_F$. (This is Lemma 21.3)

The pair $(E/F, \chi)$ is a minimal admissible pair.

This proves the map is surjecive. □

7. Final comments

The constructions here can be generalised. One can define admissible pairs of degree n, and obtain irreducible cuspidal representations π_χ of $\mathrm{GL}_n(F)$. This map also turns out to be bijective (assuming $n \not\equiv 0 \pmod{p}$ for simplicity).