The block decomposition of iterated integrals, and cyclic insertion on MZV's

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Abstract. As some background, I will first discuss two (conjectural) families of MZV identities – the cyclic insertion conjecture of Borwein et al, and an identity of a similar flavour, presented by Hoffman. Using the motivic framework due to Goncharov and Brown, I will explain how one can gain some insight into the structure of these identities. I will then present a (conjectural) unification of these identities described using the so-called alternating block decomposition of iterated integrals, and prove a certain symmetrised version always holds for motivic MZV's.

1. Background/motivating identities

Firstly, we state our convention on the definition of MZV's just for clarify

$$\zeta(s_1, \ldots, s_r) = \sum_{0<n_1<\ldots<n_r} \frac{1}{s_1^{n_1} \cdots s_r^{n_r}} \,,$$

so that convergence requires $s_r \geq 2$.

I will start with some identities (proven, or conjectural) which give the background for this talk. The simplest such identity is due to Zagier-Broadhurst, which evaluates a certain MZV as a rational multipel of π^{wt}:

Identity 1 (Broadhurst-Zagier, [BBBL01])**.** *For $n \in \mathbb{Z}_{\geq 0}$, we have*

$$\zeta(\{1,3\}^n) = \frac{1}{2n+1}\zeta(\{2\}^{2n}) = \frac{1}{2n+1}\frac{\pi^{4n}}{(4n+1)!} \,. \qquad \text{(Proven)}$$

This was found numerically by Zagier, on the basis of much computer experimentation. Broadhurst provided a proof using generating functions, hypergeometric series and their differential equations.

A slight generalisation of this leads to a (conjectural) 2-parameter family of evaluable MZV's

Identity 2 ([BBB97])**.** *For $n, m \in \mathbb{Z}_{\geq 0}$, we have*

$$\begin{aligned}\zeta(\{\{2\}^m, 1, \{2\}^m, 3\}^n, \{2\}^m) &\overset{?}{=} \frac{1}{2n+1}\frac{\pi^{(4m+4)n+2m}}{((4m+4)n+2m+1)!} \\ &= \underbrace{\frac{1}{2n+1}\frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}}_{\textit{Write the LHS weight as } \mathrm{wt} \textit{ for simplicity}} \qquad \text{(C: Up to } \mathbb{Q})\end{aligned}$$

Generalising in another direction, leads to a version of Zagier's identity "dressed with 2's".

Identity 3 (Zagier "dressed with 2's" - [BBBL98])**.** *For any $n \in \mathbb{Z}_{\geq 0}$, the following identity holds*

$$\sum_{i=0}^{2n} \zeta(\underset{\uparrow}{1}, \underset{\uparrow}{3}, \underset{\uparrow}{\ldots}, \underset{\uparrow}{1}, \underset{\uparrow}{3}, \underbrace{2}_{\textit{position } i}, \underset{\uparrow}{1}, \underset{\uparrow}{3}, \ldots, \underset{\uparrow}{1}, \underset{\uparrow}{3}) = \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!} \,. \qquad \text{(Proven)}$$

The most general identity of this type seems to be given by the following 'cyclic insertion conjecture'.

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Identity 4 (Cyclic insertion conjecture - [BBBL98])**.** *Let $n \in \mathbb{Z}_{\geq 0}$ and $a_0, \ldots, a_{2n} \in \mathbb{Z}_{\geq 0}$. Then*

$$\sum_{\sigma \in C_n} \zeta(\{2\}^{a_0}, 1, \{2\}^{a_1}, 3, \ldots, 1, \{2\}^{a_{2n-1}}, 3, \{2\}^{a_{2n}}) \overset{?}{=} \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!} \, .$$

"Sum over all cyclic shifts is a known rational multiple of π^{wt}."

Indeed, it implies all of the previous ones by appropriately specialising. Namely $a_0 = \cdots = a_{2n} = 0$ gives Zagier-Broadhurst. $a_0 = 1, a_1 = \cdots = a_{2n} = 0$ gives Zagier dressed with 2's. $a_0 = \cdots = a_{2n} = m$ gives the evaluable family.

What is known in this direction, so far?

Theorem 5 (Bowman-Bradley [BB02])**.** *Fix $n, m \in \mathbb{Z}_{\geq 0}$ and then*

$$\sum_{\sigma \in C_{2n+!1}} \zeta(\{2\}^{a_0}, 1, \{2\}^{a_1}, 3, \ldots, 1, \{2\}^{a_{2n-1}}, 3, \{2\}^{a_{2n}}) = \frac{1}{2n+1} \binom{m+2n}{m} \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!} \, .$$

"The sum over all compositions of m into $2n+1$ parts is a known rational multiple of π^{wt}."

Idea. This is a (somewhat) intricate inductive combinatorial argument, using the Zagier-Broadhurst as the base case. □

This is certainly compatible with cyclic insertion, since cyclic permutations of a composition are still a composition. There are $\binom{m+2n}{m}$ compositions of m into $2n+1$ parts. Each term in the sum above should contribute $\frac{1}{2n+1} \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$. So the coefficient matches.

I proved, using the motivic framework of Brown that only permutations of some fixed a_i are necessary to obtain a *rational multiple* of π^{wt}. Though the motivic framework is not sufficient to fix the ratoinal

Theorem 6 (C, [Cha15])**.** *Let $n \in \mathbb{Z}_{\geq 0}$ and $a_0, \ldots, a_{2n} \in \mathbb{Z}_{\geq 0}$. Then*

$$\sum_{\sigma \in C_{2n+1}} \zeta(\{2\}^{a_0}, 1, \{2\}^{a_1}, 3, \ldots, 1, \{2\}^{a_{2n-1}}, 3, \{2\}^{a_{2n}}) \overset{\mathbb{Q}}{=} \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!} \, . \qquad \text{(C: Up to } \mathbb{Q})$$

"Sum over all permutations is an (unknown) rational multiple of π^{wt}."

Idea. The goal of this talk is really to explain/generalise this proof, so I won't say much now. But basically one knows that if certain derivation operators D_r all vanish on this combination, by Brown's characterisation of $\ker D_{<N}$ it must be a rational multiple of $\zeta^{\mathfrak{m}}(N)$. One shows that D_r vanishes by setting up a very explicit cancellation. □

In particular, taking $a_0 = \cdots = a_{2n} = m$ shows that the 2-parameter family above holds up to $\mathbb{Q}$. But beyond this, no further progress has been made (publicly).

And now for something completely different. An identity with a similar flavour (but a priori unrelated) to the cyclic insertion conjecture is given by Hoffman on his MZV info page. (It is not widely known, and is only otherwise mentioned in a set of lecture notes for an MZV course given at University of Newcastle, Australia.)

Identity 7 (Hoffman, [BZ])**.** *For $n \in \mathbb{Z}_{>0}$, we have*

$$2\zeta(3, 3, \{2\}^n) - \zeta(3, \{2\}^n, 1, 2) \overset{?}{=} -\frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!} \, . \qquad \text{(C: Up to } \mathbb{Q}\text{. HS: exact)}$$

Recently this identity was proven by Hirose-san and Sato-san, using their work with Tasaka and Iwaki [HIST17] on iterated integrals over $\mathbb{P}^1 \setminus \{\infty, 0, 1, z\}$.

In fact they proved my conjectural generalisation of this, namely

Identity 8 (C, [HS17])**.** *For $a, b, c \in \mathbb{Z}_{>0}$, we have*

$$\zeta(\{2\}^a, 3, \{2\}^b, 3, \{2\}^c) - \zeta(3, \{2\}^b, \{2\}^c, 1, 2, \{2\}^a) + \\ + \zeta(\{2\}^c, 1, 2, \{2\}^a, 1, 2, \{2\}^b) \overset{?}{=} -\frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!} \qquad \text{(C: Up to } \mathbb{Q}\text{. HS: exact)}$$

We recover Hoffman's identity by taking $a = b = 0$, $c = n$. Then by duality $\zeta(\{2\}^n, 1, 2, 1, 2) = \zeta(3, 3, \{2\}^n)$, giving the coefficient 2 above.

I claim that there is a common structure among both this (generalised) Hoffman identity and the cyclic insertion identity, so that one can unify these into a more general identity. I want to explain how to do this. But first, we need to talk about iterated integrals and the motivic viewpoint, so we have all of the tools ready.

2. Iterated integrals and motivic MZV's

Recall that Kontesevich gave the following integral representation for MZV's (modulo conventions!)

$$\zeta(s_1, \ldots, s_r) = (-1)^r \int_{0 \leq t_1 \leq \cdots \leq t_k \leq 1} \omega_1 \wedge \underbrace{(\omega_0 \wedge \cdots \omega_0)}_{s_1 - 1} \wedge \cdots \wedge (\omega_1 \wedge \underbrace{\omega_0 \wedge \cdots \omega_0}_{s_k - 1}) \,,$$

where

$$\omega_i = \frac{\mathrm{d}t}{t - i}$$

We use the following notation for this iterated integral

$$I(a_0; a_1, \ldots, a_n; a_{n+1}) = \int_{a_0 \leq t_1 \leq \cdots \leq t_n \leq a_{n+1}} \omega_{a_1} \wedge \cdots \wedge \omega_{a_n} \,,$$

so that

$$\zeta(s_1, \ldots, s_r) = (-1)^r I(0; 1, \{0\}^{s_1 - 1}, \cdots, 1, \{0\}^{s_r - 1}; 1)$$

These integrals satisfy various standard properties:

- (Unit) $I(a; b) = 1$
- (Equal boundaries) $I(x, a_1, \ldots, a_N; x) = 0$
- (Reversal of paths)
$$I(a_0; a_1, \ldots, a_N; a_{N+1}) = (-1)^N I(a_{N+1}; a_N, \ldots, a_1; a_0)$$
- (Path composition)
$$I(a_0, a_1, \ldots, a_N; a_{N+1}) = \sum_{i=0}^{N} I(a_0, a_1, \ldots, a_i; x) I(x, a_{i+1}, \ldots, a_N; a_{N+1})$$
- (Functoriality, under $t \mapsto \alpha t + \beta$, with $\alpha \neq 0$ and $\beta \in \mathbb{C}$)
$$I(a_0; a_1, \ldots, a_N; a_{N+1}) = I(\alpha a_0 + \beta; \alpha a_1 + \beta, \ldots, \alpha a_N + \beta; \alpha a_{N+1} + \beta)$$
- (MZV Duality)
$$I(0; a_1, \ldots, a_N; 1) = (-1)^N I(0; 1 - a_N, \ldots, 1 - a_1; 1)$$
is implied by functoriality $t \mapsto 1 - t$, and reversal of paths.

Goncharov [Gon05] was able to lift these iterated integrals (for a_i algebraic numbers) to framed Mixed Tate motives over $\overline{\mathbb{Q}}$, endowing them with more rigid structure. (Indeed to mixed Tate motive over F, F a number field.) The actual construction is rather technical, so I will just outline the main steps.

- Category of Mixed Tate Motives $\mathcal{MT}(F)$ over a number field F exists. It is Tannakian; equivalent to some $\mathrm{Rep}_F\, \mathcal{G}^{\mathcal{MT}}$
- Recover pro-algebraic group scheme $\mathcal{G}^{\mathcal{MT}}$ from automorphisms of the fibre functor $\widetilde{\omega} \colon \mathcal{MT}(F) \to \mathcal{V}ect$, $\mathcal{G}^{\mathcal{MT}} \cong \mathbb{G}_m \ltimes \mathcal{U}^{\mathcal{MT}}$
- Ring of regular functions $\mathcal{O}(\mathcal{U}^{\mathcal{MT}})$ on the pro-unipotent part of $\mathcal{G}^{\mathcal{MT}}$ defines the *fundamental Hopf algebra* $\mathcal{A}_\bullet(F)$ of $\mathcal{MT}(F)$
- Isomorphism $\mathcal{A}_\bullet(F)$ to 'path algebra' and algebra of 'formal iterated integrals'. $\mathcal{A}_\bullet(F)$ contains objects $I^{\mathfrak{a}}(a_0; a_1, \ldots, a_n; a_{n+1})$.
- Admits a coproduct $\Delta \colon \mathcal{A}_\bullet(F) \to \mathcal{A}_\bullet(F) \otimes_{\mathbb{Q}} \mathcal{A}_\bullet(F)$

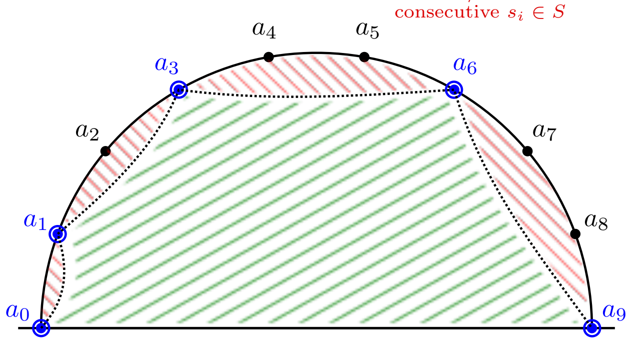
The coproduct on $I^{\mathfrak{a}}(a_0,\dots,a_{n+1})$ is given as follows.

$$\Delta I^{\mathfrak{a}}(a_0;a_1,\dots,a_n;a_{n+1}) =$$
$$\sum_{\substack{0=i_0<i_1<\cdots\\ <i_k<i_{k+1}=n+1\\ k=0,1,\dots,N}} \left(I^{\mathfrak{a}}(a_0;a_{i_1},\dots,a_{i_k};a_{n+1}) \otimes \prod_{p=0}^{k} I^{\mathfrak{a}}(a_{i_p};a_{i_p+1},\dots,a_{i_{p+1}-1};a_{i_{p+1}}) \right)$$

This coproduct lends itself to a nice mnemonic interpretation by means of semicircular polygons.

Mnemonic

$$\Delta I^{\mathfrak{a}}(a;w;b) = \sum_{\substack{S \text{ subset } awb\\ a,b \text{ in } S}} \left(I^{\mathfrak{a}}(S) \otimes \prod_{\substack{u \text{ subword } awb,\\ \text{starts/ends at}\\ \text{consecutive } s_i\in S}} I^{\mathfrak{a}}(u) \right)$$



$$\rightsquigarrow I(a_0;a_1,a_3,a_6;a_9) \otimes I(a_0;a_1)I(a_1;a_2;a_3)\cdot I(a_3;a_4,a_5;a_6)I(a_6;a_7,a_8,a_9)$$

"Draw all possible semicircular polygons connecting these vertices."

We can then define Goncharov's motivic MZV's $\zeta^{\mathfrak{a}}(s_1,\dots,s_k) := (-1)^k I^{\mathfrak{a}}(0;1,\{0\}^{s_1-1},\dots,1,\{0\}^{s_k-1};1)$ by direct analogy with the Kontsevich integral representation.

Goncharov's motivic MZV's already shed much light on hard open MZV questions.

- $\zeta^{\mathfrak{a}}(2k+1)$ are linearly independent
  - $\zeta^{\mathfrak{a}}(2k+1) \neq 0 \in \mathcal{A}_{2k+1}(\mathbb{Q})$
  - So have different gradings
- $\zeta^{\mathfrak{a}}(2k+1)$ are *algebraically* independent
  - Suppose some $\zeta^{\mathfrak{a}}(2k+1)$ satisfy a polynomial
  - Use coproduct Δ to show all coefficients are 0
- $\zeta^{\mathfrak{a}}(3,5)$ is irreducible (i.e. not in $\mathbb{Q}[\zeta(n)]$
  - $(\Delta - \Delta^{\mathrm{op}})\zeta^{\mathfrak{a}}(3,5) = -5\zeta^{\mathfrak{a}}(3) \wedge \zeta^{\mathfrak{a}}(5)$
  - $(\Delta - \Delta^{\mathrm{op}})\zeta^{\mathfrak{a}}(n_1)\cdots\zeta^{\mathfrak{a}}(n_k) = 0$

Unfortunately, Goncharov's motivic MZV's have a small, but significant defect. $\zeta^{\mathfrak{a}}(2) = 0$, since $\zeta(2) \in \pi^2\mathbb{Q}$. This means that there can be no period map down from Goncharov's motivic MZV's back to real numbers (only to the associated graded of some filtered algebra $\mathcal{P}^{\sigma}(F)$, for $\sigma\colon F \hookrightarrow \mathbb{C}$ an embedding.)

Brown [Bro12a] is able to further lift Goncharov's motivic iterated integrals (when defined over $0,1$, in such a way that $\zeta^{\mathfrak{m}}(2) \neq 0$.

- Consider 'motivic torsor' of paths ${}_0\Pi_1$ between 0 and 1 in $\mathbb{P}^1 \setminus \{0,1,\infty\}$. $\mathcal{O}({}_0\Pi_1) \cong \mathbb{Q}\langle e_0, e_1\rangle$.
- Straight line gives function $\mathcal{O}({}_0\Pi_1) \to \mathbb{R}$, evaluating MZV.

- Coalgebra of motivic MZV's is $\mathcal{H} \coloneqq \mathcal{O}({}_0\Pi_1)/J^{\mathcal{MT}}$, $J^{\mathcal{MT}}$ the largest graded ideal in the kernel of above.
- $\mathcal{H} \cong \mathcal{A} \otimes_{\mathbb{Q}} \mathbb{Q}[\zeta^{\mathfrak{m}}(2)]$, $\mathcal{A} \coloneqq \mathcal{A}_\bullet(\mathbb{Z})$
- Period map per: $\mathcal{H} \to \mathbb{R}$, $\zeta^{\mathfrak{m}}(s_1, \ldots, s_k) \mapsto \zeta(s_1, \ldots, s_k)$, ring homomorphism
- Coaction by lifting Gonchrov's coproduct to $\mathcal{H} \to \mathcal{A} \otimes_{\mathbb{Q}} \mathcal{H}$.

The full coproduct on motivic MZV's obviously reflects a lot of the structure and properties of the motivic MZV's. But the number of terms that the coproduct contains (order 2^n) grows rapidly, and make this object difficult to work with in general. Francis Brown introducse an infinitesimal version of the coproduct, and shows it factors through a certian fmaily of operators D_{2r+1}, defined below.

Definition 9 (Derivations D_k)**.** Let $\mathcal{L} \coloneqq \mathcal{A}/(\mathcal{A}_{>0} \cdot \mathcal{A}_{>0})$, which kills products and $\zeta^{\mathfrak{m}}(2)$. For k odd define

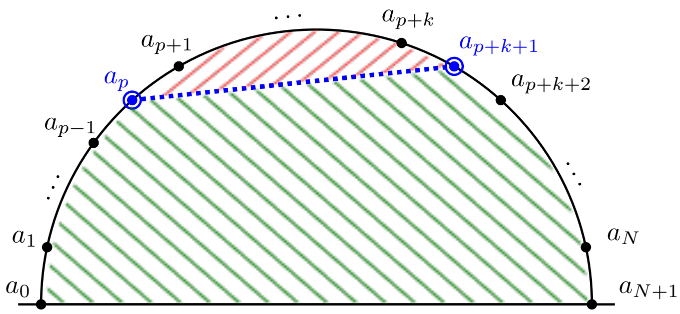
$$\begin{aligned} D_k \colon \quad & \mathcal{H} \to \mathcal{L}_k \otimes_{\mathbb{Q}} \mathcal{H} \\ & I^{\mathfrak{m}}(w) \mapsto (\pi \otimes \mathrm{id}) \circ (\Delta - 1 \otimes \mathrm{id})\, I^{\mathfrak{m}}(w) \end{aligned}$$

$$\begin{aligned} & D_k I^{\mathfrak{m}}(a_0; a_1, \ldots, a_N; a_{N+1}) = \\ & \quad \sum_{p=0}^{N-k} I^{\mathfrak{L}}(a_p; a_{p+1}, \ldots, a_{p+k}; a_{p+k+1}) \otimes \\ & \qquad I^{\mathfrak{m}}(a_0; a_1, \ldots, a_p, a_{p+k+1}, \ldots, a_N; a_N + 1) \end{aligned}$$

Again, a pictorial mnemonic is more helpful.

Mnemonic

$$D_k I^{\mathfrak{m}}(\underbrace{w}_{(a;w';b)}) = \sum_{\substack{S \text{ subword } w, \\ \text{of length } k+2}} I^{\mathfrak{L}}(S) \otimes I^{\mathfrak{m}}(w - \text{interior } S)$$



$$\rightsquigarrow I^{\mathfrak{L}}(a_p; a_{p+1}, \ldots, a_{p+k}; a_{p+k+1}) \otimes$$
$$I^{\mathfrak{m}}(a_0; a_1, \ldots, a_p, a_{p+k+1}, \ldots, a_N; a_N + 1)$$

"Cut out all possible segments of length $k = 2r+1$ odd from the vertices of the semicircular polygon".

These derivations (when taken together) also contain a lot of information about the structure of motivic MZV's. In particular, we have the following important theorem characterising $\ker D_{<N} \coloneqq \bigoplus_{3 \leq 2r+1 \leq N} D_{2r+1}$

Theorem 10 (Brown, [Bro12b])**.** *In weight N,*

$$\ker D_{<N} \coloneqq \bigoplus_{3 \leq 2r+1 \leq N} D_{2r+1} = \zeta^{\mathfrak{m}}(N)\mathbb{Q}\,.$$

"If all D_{2r+1} vanish on a combiantion of weight N motivic MZV's, then it is a rational multiple of $\zeta^{\mathfrak{m}}(N)$".

As a toy example of this, we can see $\zeta(\{2\}^n) \in \mathbb{Q}\zeta(2n) = \mathbb{Q}\pi^{2n}$.

Exmaple 11. We can show $\zeta^{\mathfrak{m}}(\{2\}^n) \in \zeta^{\mathfrak{m}}(2n)\mathbb{Q}$. As an integral

$$\zeta^{\mathfrak{m}}(\{2\}^n) = (-1)^n I^{\mathfrak{m}}(0; \underbrace{1,0,1,0,\ldots,1,0}_{n \text{ times}}; 1)\,.$$

Now observe that the digits alternate $0101\cdots$. Any subword of odd length necessarily starts and ends with the same letter, and so vanishes since the boundaries of integration are equal. This means all D_{2r+1} vanish, giving $\zeta^{\mathfrak{m}}(\{2\}^n) \in \ker D_{<2n} = \zeta^{\mathfrak{m}}(2n)\mathbb{Q}$.

Now we have enough background to investigate the similarities between the above identities, and to give some proofs.

3. Block decomposition and cyclic insertion

The block decomposition used to describe the structural similarities in the cyclic insertion conjecture and Hoffman's identity, so let us do that.

Exmaple 12 (Hoffman's identity)**.** Write Hoffman's identity as iterated integrals

$$\zeta(\{2\}^a, 3, \{2\}^b, 3, \{2\}^c) - \zeta(3, \{2\}^b, \{2\}^c, 1, 2, \{2\}^a) +$$
$$+ \zeta(\{2\}^c, 1, 2, \{2\}^a, 1, 2, \{2\}^b) = -\frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$$
$$I(0; (10)^a 100 (10)^b 100 (10)^c; 1) + I(0; (10)^b 100 (10)^c 110 (10)^a; 1) +$$
$$+ I(0; (10)^c 110 (10)^a 110 (10)^c; 1) = I(0; (10)^{(\mathrm{wt}/2)})$$

Now split these integrals into 'blocks of alternating 0's and 1's'. I.e. break at $00 \to 0|0$ or at $11 \to 1 \mid 1$, to obtain the *block decomposition.*

$$I(0; (10)^a 10 \mid 0(10)^b 10 \mid 0(10)^c; 1) + I(0; (10)^b 10 \mid 0(10)^c 1 \mid 10(10)^a; 1) +$$
$$I(0; (10)^c 1 \mid 10(10)^a 1 \mid 10(10)^c; 1) = I(0; (10)^{(\mathrm{wt}/2)}; 1)\,.$$

We record the length of the blocks to obtain a *block* integral (to clearly distinguish it notationally from the regular iterated integral).

$$I_{\mathrm{bl}}(2a+3, 2b+3, 2c+2) + I_{\mathrm{bl}}(2b+3, 2c+2, 2a+3) + I_{\mathrm{bl}}(2c+2, 2a+3, 2b+3) = I(\mathrm{wt}+2)$$

Notice that the lengths on the left hand side are just cyclically permuted.

Doing the same for the cyclic insertion identity leads to

Exmaple 13 (Cyclic insertion)**.** Write cyclic insertion as iterated integrals

$$\sum_{\sigma \in C_{2n+1}} \zeta(\{2\}^{a_0}, 1, \{2\}^{a_1}, 3, \ldots, 1, \{2\}^{a_{2n-1}}, 3, \{2\}^{a_{2n}}) \stackrel{?}{=} \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$$
$$\sum_{\sigma \in C_{2n+1}} I(0; (10)^{a_0} 1 \mid (10)^{a_1} 10 \mid 0 \cdots 10 \mid 0(10)^{a_{2n+1}}; 1) \stackrel{?}{=} I(0; (10)^{(\mathrm{wt}/2)})$$
$$\sum_{\sigma \in \mathbb{C}_{2n+1}} I_{\mathrm{bl}}(2a_0+2, 2a_1+2, \cdots, 2a_{2n}+2) \stackrel{?}{=} I(\mathrm{wt}+2)$$

Again the lengths are just cyclically permuted.

Both identities have the form

$$\sum_{\text{cycle } \ell_i} I_{\mathrm{bl}}(\ell_1, \ldots, \ell_n) \stackrel{?}{=} I_{\mathrm{bl}}(\mathrm{wt}+2)\,.$$

Perhaps it is not too much to hope for that we can choose any lengths? Indeed, after some numerical experimentation with various other choices of lengths, I am lead to the following conjecture. (I still call it cyclic insertion because of the cyclic shifting involved in the block integrals.)

Conjecture 14 (Generalised cyclic insertion, C, arXiv 1703.03784)**.** *For any* $[\ell_1, \ldots, \ell_n]$ *with all* $\ell_i > 1$,

$$\sum_{\textit{cycle } \ell_i} I_{\mathrm{bl}}(\ell_1, \ldots, \ell_n) \overset{?}{=} I_{\mathrm{bl}}(\mathrm{wt} + 2)$$

Moreover, if some $\ell_i = 1$*, we can give an identity involving lower order correction terms in the form of products. Let*

$$\mathfrak{L}_k = \{ [m_{k+1}, \ldots, m_n] \mid [\underbrace{1, \ldots, 1}_{k \text{ times}}, m_{k+1}, \ldots, m_n] \text{ is a cyclic permutation of } [\ell_1, \ldots, \ell_n] \}$$

"Take all cyclic permutations of $[\ell_1, \ldots, \ell_n]$ *which start with k consecutive 1's. Then drop the ones"*

Then

$$\sum_{\sigma \in C_n} I_{\mathrm{bl}}(\ell_{\sigma(1)}, \ldots, \ell_{\sigma(n)}) \overset{?}{=} I_{\mathrm{bl}}(N + 2) - \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \frac{2(2\pi)^{2k}}{(2k+2)!} \sum_{\vec{m} \in \mathfrak{L}_{2k}} I_{\mathrm{bl}}(\vec{m}) \,.$$

Evidence/outlook

What evidence do we have for this conjecture?

- Numerical validation in all cases with weight ≤ 18, to 500 decimal places.
- Proofs of a symmetrised version, up to $\mathbb{Q}$
- Proofs of some special cases,up to $\mathbb{Q}$
- Proofs of other special cases exactly (HS for Hoffman's identity)

I had at some point better state a theorem, rather than just various conjectures. That theorem is alluded to above, in the proof of a symmetrised version of cyclic insertion

Theorem 15 (Symmetric insertion, C, arXiv 1703.03784)**.** *For any* $[\ell_1, \ldots, \ell_n]$ *with all* $\ell_i > 1$*, and even weight (odd holds trivially!)*

$$\sum_{\textit{permute } \ell_i} I^{(\mathfrak{m})}_{\mathrm{bl}}(\ell_1, \ldots, \ell_n) \in I^{(\mathfrak{m})}_{\mathrm{bl}}(\mathrm{wt} + 2)\mathbb{Q} \,.$$

Proof. The goal of the proof is simple: compute $D_{<N}$ for the LHS and show it vanishes. Then we can conclude by Brown's theorem that the LHS is a rational multiple of $\zeta^{(\mathfrak{m})}(\mathrm{wt})$. (Which is equal to the the above RHS using standard identities.

To show that $D_{<N}$ vanishes, we exploit the block decomposition structure to setup a pairwise cancellation in the terms. Namely given the following substring on some $I_{\mathrm{bl}}(\ell_1, \ldots, \ell_n)$

$$I_{\mathrm{bl}}(\ell_1, \ldots, \underbrace{\overbrace{\ell_s}^{\text{start at position } \alpha} \quad , \ldots, \quad \overbrace{\ell_t}^{\text{end at position } \beta}} , \ldots, \ell_n)$$

we match this with the subsequence

$$I_{\mathrm{bl}}(\ell_1, \ldots, \underbrace{\overbrace{\ell_t}^{\text{start at position } \beta} \quad , \ldots, \quad \overbrace{\ell_s}^{\text{end at position } \alpha}} , \ldots, \ell_n)$$

obtained by reflecting/reversing the blocks containing the subsequence.

The resulting subsequence is the dual or the reverse of the original. Since it has odd length, we pick up a sign $(-1)^{\mathrm{length}} = -1$ from the reversal step. The quotient sequences agree because the blocks match exactly outside of the subsequence. Since reversing the blocks just gives a permutation, the second term will appear in D_{2r+1}, and therefore cancel with the first term as they have opposite signs.

This shows that all terms in D_{2r+1} cancel, and the result follows. (One does need to do some work to set up the notation and make this completely rigorous.) ∎

Hirose-san and Sato-san have done much work towards proving this conjecture fully. We still need to discuss exactly how the work relates. They propose a more general conjecture which implies the first part of my cyclic insertion conjecture. (The case $\ell_i > 1$. As I understand it, they have a proof of this conjecture in certain special cases (or it is now in all cases?) However, they do not yet understand how the $\ell_i = 1$ with products case fits into their framework?

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