Relating MPL's in weight ≥ 5

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Outline

Introduction and previous work

Multiple polylogarithms

- Classical polylogarithms well-studied
- Too special; conceals algebraic structure
- Introduce multiple-variable version

Definition (MPL)

$$\mathrm{Li}_{s_1,\ldots,s_k}(z_1,\ldots,z_k) := \sum_{0<n_1<\cdots<n_k} \frac{z_1^{n_1}\cdots z_k^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}}$$

- $s_1 + \cdots + s_k$ is the weight
- k is the depth

Iterated integrals

- Prefer to work with iterated integrals

Definition (Iterated Integral)

$$I(x_0; x_1, \ldots, x_n; x_{n+1}) \coloneqq \int_{x_0 < t_1 < \cdots < t_n < x_{n+1}} \frac{\mathrm{d}t_1}{t_1 - x_1} \circ \cdots \circ \frac{\mathrm{d}t_n}{t_n - x_n}$$

$$I_{s_1,\ldots,s_k}(x_1, \ldots, x_k) = I(0; x_1, \{0\}^{s_1-1}, \ldots, x_k, \{0\}^{s_k-1}; 1)$$

- Related to MPL's by a change of variables

$$I_{s_1,\ldots,s_k}(x_1, \ldots, x_k) = (-1)^k \operatorname{Li}_{s_1,\ldots,s_k}(\tfrac{1}{z_1 \cdots z_k}, \tfrac{1}{z_2 \cdots z_k}, \ldots, \tfrac{1}{z_k})$$

Weight ≤ 3

- All weight ≤ 3 MPL's are polylogs

Theorem (Well-known)

$$\begin{aligned}\mathrm{Li}_{1,1,1}(x,y,z) = & -\mathrm{Li}_3\left(\tfrac{1-xyz}{1-x}\right) - \mathrm{Li}_3\left(-\tfrac{(1-x)y}{1-y}\right) + \mathrm{Li}_3(xy) + \\ & + \mathrm{Li}_3\left(-\tfrac{(1-x)y(1-z)}{(1-y)(1-xyz)}\right) - \mathrm{Li}_3\left(\tfrac{xy(1-z)}{1-xyz}\right) + \\ & + \mathrm{Li}_3\left(\tfrac{1}{1-x}\right) + \mathrm{Li}_3\left(\tfrac{y}{y-1}\right) - \mathrm{Li}_3\left(\tfrac{y(z-1)}{1-y}\right) + \\ & + \text{products}\end{aligned}$$

Weight 4 - overview

- New phenomenon at weight 4

Observation

Under 8-fold symmetrisation δ:

$$\mathrm{Li}_4(x) \xrightarrow{\delta} 0$$

$$I_{3,1}(x,y) \xrightarrow{\delta} \mathrm{Li}_2(x) \wedge \mathrm{Li}_2(y)$$

- Cannot write $I_{3,1}$ as Li_4's.
  - Alternatively use $\mathrm{Li}_{3,1}, I_{2,2}, \mathrm{Li}_{2,2}, \ldots$.
- Question: how do weight 4 MPL's relate?

Weight 4 - Gangl

Geometry of $\mathfrak{M}_{0,n}$ gives 'coupled' cross-ratio arguments

Definition ('Coupled' cross-ratios)

$$\mathrm{cr}(a, b, c, d_1, \ldots, d_{n-3}) = [\mathrm{cr}(a, b, c, d_1), \ldots, \mathrm{cr}(a, b, c, d_{n-3})]$$

Shorthand: $abcd_1 \cdots d_{n-3} \coloneqq \mathrm{cr}(a, b, c, d_1, \ldots, d_{n-3})$

Results include

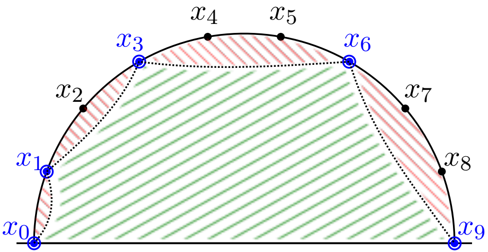
- Functional equations for $I_{3,1}, I_{2,1,1}, I_{1,1,1,1} \ldots$
- Express $I_{3,1}$, $I_{2,2}$ and $I_{1,3}$ in terms of any others, modulo Li_4.
- Express $I_{2,1,1}$ in terms of $I_{3,1}$

Our goal: extend this to weight ≥ 5.

Algebraic tools

Iterated integrals - algebraic structure

- Hopf algebra, with Goncharov's coproduct



$$\rightsquigarrow I^{\mathfrak{a}}(x_0; x_1, x_3, x_6; x_9) \otimes I^{\mathfrak{a}}(x_0; x_1) I^{\mathfrak{a}}(x_1; x_2; x_3) \cdot I^{\mathfrak{a}}(x_3; x_4, x_5; x_6) I^{\mathfrak{a}}(x_6; x_7, x_8, x_9)$$

Definition

$$\Delta I^{\mathfrak{a}}(x_0; x_1, \ldots, x_n; x_{n+1}) = \sum_{\substack{S \text{ subset} \\ x_1, \ldots, x_n}} \left(I^{\mathfrak{a}}\left(\substack{\text{main} \\ \text{polygon}}\right) \otimes \prod I^{\mathfrak{a}}\left(\substack{\text{cut-off} \\ \text{polygon}}\right) \right)$$

Iterated integrals - symbol

- Algebraic invariant which captures the differential properties

Definition

$$\mathcal{S}\left(\int_w \mathrm{d}F_1 \circ \cdots \circ \mathrm{d}F_n\right) := F_1 \otimes \cdots \otimes F_n$$

- Obtained by maximally iterated coproduct

$$\mathcal{S}(I(x_0; x_1, \ldots, x_n; x_{n+1})) \leftrightarrow \Delta^{[n]} I^{\mathfrak{a}}(x_0; x_1, \ldots, x_n; x_{n+1})$$

$$\begin{aligned}\Delta^{[m]} : \mathcal{A}_m &\xrightarrow{\Delta} \mathcal{A}_{m-1} \otimes \mathcal{A}_1 \\ &\xrightarrow{\Delta \otimes \mathrm{id}} \mathcal{A}_{m-2} \otimes \mathcal{A}_1 \otimes \mathcal{A}_1 \\ &\cdots \xrightarrow{\Delta \otimes \mathrm{id}^{\otimes (m-1)}} \bigotimes_m \mathcal{A}_1\end{aligned}$$

Symbol of $I_{3,1}$

Example

$$\begin{aligned}\mathcal{S}(I_{3,1}(x,y)) = (1-\tfrac{1}{x}) \otimes x \otimes x \otimes (1-\tfrac{1}{y}) + \\ + (1-\tfrac{1}{x}) \otimes x \otimes (1-\tfrac{1}{y}) \otimes x + \\ - (1-\tfrac{1}{x}) \otimes x \otimes (1-\tfrac{1}{y}) \otimes y + \\ + (1-\tfrac{1}{x}) \otimes \tfrac{1-y}{x-y} \otimes \tfrac{x}{y} \otimes x + \\ - (1-\tfrac{1}{x}) \otimes \tfrac{1-y}{x-y} \otimes \tfrac{x}{y} \otimes y + \\ + (1-\tfrac{1}{y}) \otimes (1-\tfrac{y}{x}) \otimes \tfrac{x}{y} \otimes x + \\ + (1-\tfrac{1}{y}) \otimes (1-\tfrac{y}{x}) \otimes \tfrac{y}{x} \otimes y \\ \in \mathbb{Q}(x,y)^{\otimes 4}\end{aligned}$$

- Very concrete/explicit object.
- Computation reduced to multilinear algebra

Symbol modulo products, modulo δ

- Can work with iterated integrals modulo products
  - Pass to Lie coalgebra of irreducibles $\mathcal{L}_\bullet = \mathcal{A}_\bullet / \mathcal{A}^2_{>0}$.
  - Obtain: cobracket $\delta = (\pi \otimes \pi) \circ (\Delta - \Delta^{\mathrm{op}})$ killing Li_n
- Analogue of these construction on the symbol

Strategy to study relations

1. Work modulo δ, to find 'top' slice (depth ≥ 2 part): $\overset{\delta}{=}$
2. Find Li_n terms, to get identity modulo products: $\overset{\sqcup\!\sqcup}{=}$

For more precision

3. Find product terms to get symbol level identity: $\overset{\mathcal{S}}{=}$
4. Compute slices of Δ to find constant $\times$ lower-weight corrections

Relating Weight ≥ 5 MPL's

Weight 5, depth 2 - symmetries

- Depth 2 integrals: $I_{4,1}, I_{3,2}$ (and also $I_{2,3}, I_{1,4}$)

Theorem

Integrals $I_{4,1}$, $I_{3,2}$ satisfy the following anti-symmetry

$$I_{4,1}(x,y) + I_{4,1}(\tfrac{1}{x},\tfrac{1}{y}) \overset{\sqcup\!\sqcup}{=} \mathrm{Li}_5(-[x]-[y]-4[\tfrac{x}{y}])$$

$$I_{3,2}(x,y) + I_{3,2}(\tfrac{1}{x},\tfrac{1}{y}) \overset{\sqcup\!\sqcup}{=} \mathrm{Li}_5(-[x]+4[y]+6[\tfrac{x}{y}])$$

- Compare with

Theorem (Gangl)

$$I_{3,1}(x,y) - I_{3,1}(\tfrac{1}{x},\tfrac{1}{y}) \overset{\sqcup\!\sqcup}{=} \mathrm{Li}_4([x]-[y]+3[\tfrac{x}{y}])$$

Weight n, depth 2 - symmetries

- These (anti-)symmetries generalise to weight n

Theorem

For $a, b \in \mathbb{Z}_{>1}$, the following holds modulo products

$$I_{a,b}(x,y) - (-1)^{a+b} I_{a,b}(\tfrac{1}{x}, \tfrac{1}{y}) \overset{\sqcup\!\sqcup}{=} (-1)^{a+b} \operatorname{Li}_{a+b}(x) + \\ + (-1)^b \binom{a+b-1}{a} \operatorname{Li}_{a+b}(y) - (-1)^a \binom{a+b-1}{b} \operatorname{Li}_{a+b}(\tfrac{x}{y})$$

Example

Can give explicit product term for the above identities. Example:

$$I_{4,1}(x,y) + I_{4,1}(\tfrac{1}{x}, \tfrac{1}{y}) \overset{\mathcal{S}}{=} \operatorname{Li}_5(-[x] - [y] - 4[\tfrac{x}{y}]) + \\ \operatorname{Li}_4(y)\log(x) + \operatorname{Li}_4(\tfrac{x}{y})\log(\tfrac{x}{y}) + \tfrac{1}{5!}(\log^5(\tfrac{x}{y}) - \log^5(x)) + \\ - \tfrac{1}{2!}\operatorname{Li}_3(y)\log^2(x) + \tfrac{1}{3!}\operatorname{Li}_2(y)\log^3(x) - \tfrac{1}{4!}\operatorname{Li}_1(y)\log^4(x)$$

Numerically testable identity

- By computing 'slices' $\Delta_{1,\ldots,1,n}$, can find constant $\times$ lower-weight terms.
- Get numerically testable identity (for $I_{n,1}$)
- Already generalised via analytic techniques to the parity theorem for MPL's

Theorem (MPL Parity Theorem – Panzer, 2015)

$$\mathrm{Li}_{s_1,\ldots,s_k}(x_1,\ldots,x_k) - (-1)^{x_1+\cdots+x_k}\,\mathrm{Li}_{s_1,\ldots,s_k}(\tfrac{1}{x_1},\ldots,\tfrac{1}{x_k})$$
$=$ *explicit lower depth and products*

$I_{4,1}$ symmetry

Proposition

$$I_{4,1}(x,y) - I_{4,1}(y,x) \stackrel{\delta}{=} 0$$

Instance of following exact identity

Theorem

$$I_{n,1}(x,y) - (-1)^n I_{n,1}(y,x) = \sum_{i=1}^{n} (-1)^{n-i} I_i(x) I_{n+1-i}(y)$$

Corollary (Gangl)

$$I_{3,1}(x,y) + I_{3,1}(x,y) \stackrel{\sqcup\!\sqcup}{=} 0$$

$I_{4,1}$ 3-term relation

Proposition

$$I_{4,1}(x,y) + I_{4,1}(\tfrac{1}{1-x}, \tfrac{1}{1-y}) + I_{4,1}(1-\tfrac{1}{x}, 1-\tfrac{1}{y}) \overset{\delta}{=} 0$$

- New phenomenon: Nielsen polylogarithms

Definition

$$S_{p,q}(x) \coloneqq (-1)^p I(0; \{1\}^p, \{0\}^q; x)$$

Nielsen vanishes under coboundary δ.

- Goncharov's 'reduction' conjecture $\leftrightarrow$ Nielsen equals classical
- Not clear how to write $S_{3,2}(x)$ as Li_5

$I_{4,1}$ 3-term - Li_5 and Nielsen terms

Theorem

$$\begin{aligned} I_{4,1}(x,y) + I_{4,1}\left(\tfrac{1}{1-x}, \tfrac{1}{1-y}\right) + I_{4,1}\left(1-\tfrac{1}{x}, 1-\tfrac{1}{y}\right) &\overset{\sqcup\!\sqcup}{=} \\ -2\,\mathrm{Li}_5\left(\tfrac{x}{y}\right) - 2\,\mathrm{Li}_5\left(\tfrac{1-y}{1-x}\right) - 2\,\mathrm{Li}_5\left(\tfrac{y(1-x)}{x(1-y)}\right) &+ \\ -2\,\mathrm{Li}_5(x) - \mathrm{Li}_5\left(1-\tfrac{1}{x}\right) + S_{3,2}(x) &+ \\ -2\,\mathrm{Li}_5(y) - \mathrm{Li}_5\left(1-\tfrac{1}{y}\right) + S_{3,2}(y) & \end{aligned}$$

Remark

- Symmetry broken on RHS, to reduce number of Nielsen's

$$S_{3,2}\left([x] + \left[\tfrac{1}{1-x}\right] + \left[1-\tfrac{1}{x}\right]\right) \overset{\sqcup\!\sqcup}{=} 3S_{3,2}(x) - 3\,\mathrm{Li}_5\left(\left[\tfrac{1}{1-x}\right] + [x]\right)$$

- Can find explicit product terms, to get symbol level identity

$I_{3,2}$ relations

Relations are more complicated

- Simplest is 4-term relation

Proposition

$$\mathrm{Alt}_{d,e}\,\mathrm{Cyc}_{c,d}\,I_{3,2}(ab(\mathbf{c}\;(\mathbf{d})\mathbf{e})) \stackrel{\delta}{=} 0$$

'Anti-symmetrisation' of the 2-term $I_{4,1}$ identity swapping $x \leftrightarrow y$.

- Next is 6-term relation

Proposition

$$\mathrm{Alt}_{d,e}\,\mathrm{Cyc}_{a,b,c}\,I_{3,2}((\mathbf{abc})(\mathbf{de})) \stackrel{\delta}{=} 0$$

'Anti-symmetrisation' of the 3-term $I_{4,1}$ identity

'Exceptional' $I_{3,2}$ relation

- 2-, 4-, 6-term describe 90 out of 91 relations
- Last relation has 30-terms

Proposition

$$\mathrm{Cyc}_{a,b,c,d,e}\,\mathrm{Cyc}_{a,b,c}\,I_{3,2}(abcde) \overset{\delta}{=} \mathrm{Cyc}_{a,c,e,b,d}\,\mathrm{Cyc}_{a,c,e}\,I_{3,2}(acebd)$$

Remark

Better description with 60-terms:

$$\sum_{\sigma \in A_5} I_{3,2}(\sigma \cdot abcde) \overset{\delta}{=} 0$$

- Conceptually explained with representation theory

Relating $I_{3,2}$ and $I_{4,1}$

- Structure of $I_{3,2}$ simplifies, modulo $I_{4,1}$.

Proposition

$$\mathrm{Cyc}_{d,e}\, I_{3,2}(abc(\mathbf{de})) \overset{\sqcup\!\sqcup}{=} -3I_{4,1}(abcde)$$

$$\mathrm{Cyc}_{c,d}\, I_{3,2}(ab(\mathbf{cd})e) \overset{\delta}{=} -\mathrm{Cyc}_{c,d,e}\, I_{4,1}(ab(\mathbf{cde}))$$

Anti-symmetric in $\mathbf{ab}$ *and* $\mathbf{cde}$*, modulo* $I_{4,1}$ *and depth 1.*

- One further 10-term relation

Remark

Expect that index 1 can always be eliminated. Can eliminate $I_{4,1}$, using above.

$I_{3,2}$ in terms of $I_{4,1}$?

Can express $I_{4,1}$ in terms of $I_{3,2}$. Converse?

- 'Coupled' cross-ratios are *not* sufficient
- Modulo δ, see $I_{3,2}$ is $\dim 29$, $I_{4,1}$ is $\dim 20$ subspace.

Observation

$$I_{4,1}(x,y) \xrightarrow{\delta} I_2(x) \wedge I_3(y) - I_3(x) \wedge I_2(y)$$

$$\tfrac{1}{2} I_{4,1}(x, [y] - [\tfrac{1}{y}]) \xrightarrow{\delta} I_3(x) \wedge I_2(y)$$

$$\begin{aligned} I_{3,2}(x,y) \xrightarrow{\delta} & -I_2(x) \wedge I_3(\tfrac{x}{y}) + I_2(y) \wedge I_3(\tfrac{x}{y}) + \\ & - 2I_2(x) \wedge I_3(y) - I_2(y) \wedge I_3(x) \end{aligned}$$

Leads to 'brute force' way to write $I_{3,2}$ as $I_{4,1}$'s

$I_{3,2}$ in terms of $I_{4,1}$

Theorem

$I_{3,2}$ can be expressed in terms of $I_{4,1}$, and explicit Li_5's modulo products

$$I_{3,2}(x,y) \overset{\sqcup\!\sqcup}{=} -\tfrac{1}{2} I_{4,1}\big([x,\tfrac{1}{y}] + [x,\tfrac{y}{x}] + 3[x,y] - [y,\tfrac{x}{y}] - [y,\tfrac{y}{x}]\big) + \\ + \mathrm{Li}_5\big(\cdots + \tfrac{15}{22}[-\tfrac{x(1-y)(x-y)}{(1-x)^2 y}] + \cdots\big)$$

Remark

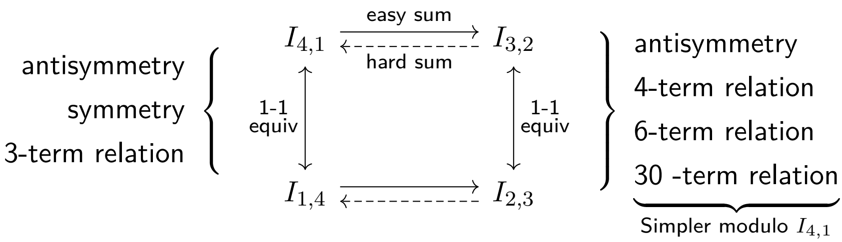
- Involves 141 Li_5 terms
- Found with heavy computer assistance: Radchenko has procedure to find 'good arguments'

Depth 2 summary - modulo products

Observation

Stuffle: $I_{a,b}(x,y) + I_{b,a}(x,\frac{x}{y}) = I_{a+b}(x) + I_b(y) * I_a(\frac{x}{y})$

- No indeed to analyse $I_{1,4}$, $I_{2,3}$



Weight 5, depth 3

- Integrals $I_{3,1,1}, I_{1,3,1}, I_{1,1,3}, I_{2,2,1}, I_{2,1,2}, I_{1,2,2}$.
- Typically relations (modulo δ) are very complicated; (almost) no straight forward symmetries

Proposition

Only symmetry modulo δ is

$$I_{2,2,1}(x,y,z) \overset{ш}{=} I_{2,2,1}(z,y,x)$$

Theorem

$$I_{a,b,1}(x,y,z) + (-1)^{a+b} I_{b,a,1}(z,y,x) =$$
$$\sum_{i=1}^{b} (-1)^i I_i(z) I_{a,b+1-i}(x,y) - (-1)^{a+b} \sum_{i=1}^{a} (-1)^i I_i(x) I_{b,a+1-i}(z,y)$$

Depth 3, modulo depth 2

- Idea: search modulo depth 2
- Only need to search modulo $I_{3,2}$
- Warning: use only 'coupled' cross ratios
  - (Expect: everything in weight 5 is depth ≤ 2.)

Proposition

Obtain many new symmetries

$$I_{3,1,1}((\mathbf{ba})cdef) \overset{I_{3,2}}{=} I_{3,1,1}(abcdef) \overset{I_{3,2}}{=} I_{3,1,1}(ab(\mathbf{fedc}))$$

$$I_{2,1,2}((\mathbf{ba})cdef) \overset{I_{3,2}}{=} I_{2,1,2}(abcdef) \overset{I_{3,2}}{=} I_{2,1,2}(\mathbf{fedc})$$

$$\overset{I_{3,2}}{=}$$

$$I_{2,1,2}(ab(\mathbf{dc})(\mathbf{fe}))$$

Relating depth 3 integrals

Theorem

Weight 5 depth 3 integrals span the same space, modulo $I_{3,2}$.

Example

$$I_{1,3,1}(abc(\mathbf{f})e(\mathbf{d})) \overset{⧢}{=} I_{3,1,1}(abcdef) \overset{I_{3,2}}{=} I_{1,1,3}(ab(\mathbf{dc})(\mathbf{fe}))$$

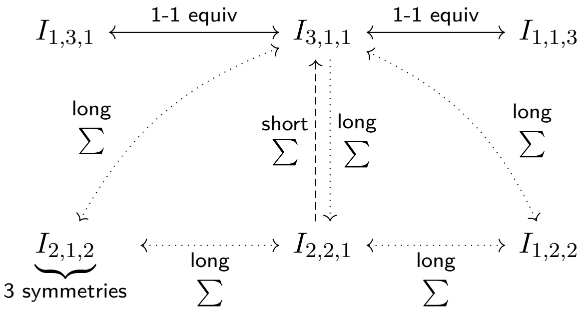
$$I_{2,2,1}(abcdef) \overset{⧢}{=}$$
$$-I_{3,1,1}([abc(\mathbf{def})] + [abc(\mathbf{dfe})] + [abc(\mathbf{fde})] + [abc(\mathbf{fed})])$$

$$I_{3,1,1}(abcdef) \overset{I_{3,2}}{=} \sum 197 I_{2,1,2}\text{'s}$$

$$\vdots$$

Depth 3 summary – modulo $I_{3,2}$

- Each integral has 2 symmetries
- $I_{2,1,2}$ has 3 symmetries



Polylog functional equations from MPL's

Goncharov's 'depth-reduction' idea

Element $\kappa(x,y)$, essentially $I_{3,1}(x,y)$, has coboundary $\mathrm{Li}_2(x) \wedge \mathrm{Li}_2(y)$

- Substituting $x = \mathrm{Li}_2$ functional equations gives coboundary 0
- Expect $I_{3,1}(\mathrm{Li}_2 \text{ FE}, y) \overset{\sqcup\!\sqcup}{=} \sum \mathrm{Li}_4$'s
- Get Li_4 functional equation by expanding in two ways

$$I_{3,1}(\mathrm{Li}_2 \text{ FE}, \mathrm{Li}_2 \text{ FE}) \overset{\sqcup\!\sqcup}{=} \sum \mathrm{Li}_4\text{'s}$$

Similar strategy for $\Phi_5(x,y)$, essentially $I_{4,1}(x, [y] - [\frac{1}{y}])$

- Li_5 FE from $x = \mathrm{Li}_3$ FE, $y = \mathrm{Li}_2$ FE.

Remark

Such functional equations should play a key role in a proof of Zagier's polylogarithm conjecture

Li_4 functional equations

Definition (Algebraic Li_2 FE)

Let $p_i(t)$ be roots of $x^a(1-x)^b = t$, $a \neq b \in \mathbb{Z}_{>0}$. Set $a + b + c = 0$. Then

$$\sum_j \mathrm{Li}_2(p_j(t)) \overset{\sqcup\!\sqcup}{=} 0$$

Theorem (Gangl, 2000)

$$I_{3,1}(\textstyle\sum_j [p_j], y) \overset{\sqcup\!\sqcup}{=} \mathrm{Li}_4(\frac{1}{abc}[\frac{t}{y^a(1-y)^b}] - b[1-\frac{1}{y}] - c[y] +$$
$$- \sum_j \frac{b}{a}[\frac{1-p_j}{1-y}] - \frac{b}{c}[\frac{1-1/y}{1-1/p_j}] - \frac{a}{b}[\frac{y}{p_j}] - \frac{b}{a}[1-p_j])$$

Corollary

2-variable of family of Li_4 functional equations

Li_4 functional equations

- Want to do this for the 5-term equation for Li_2, to obtain 'generic' Li_4 functional equation

Theorem (Gangl, 2016)

$$I_{3,1}(\mathrm{Li}_2 \textit{ five term}, y) = \sum \textit{122 } \mathrm{Li}_4\textit{'s}$$

Corollary

931-term functional equation for Li_4*.*

Remark

Goncharov-Rudenko: announced a proof of Zagier's conjecture for $n = 4$. Geometric interpretation of 122 term relation.

Li_5 functional equations

Approach in weight 5 uses $I_{4,1}$

Observation

$$I^-_{4,1}(x,y) = I_{4,1}(x, [y] - [\tfrac{1}{y}]) \xrightarrow{\delta} \mathrm{Li}_3(x) \wedge \mathrm{Li}_2(y)$$

- $I^-_{4,1}$ coboundary 0 for $x = \mathrm{Li}_3$ FE or $y = \mathrm{Li}_2$ FE.

Definition (Algebraic Li_3 FE)

$$\sum_j a\,\mathrm{Li}_3(p_j) - b\,\mathrm{Li}_3(1-p_j) \overset{\sqcup\!\sqcup}{=} 0$$

Li_5 functional equations

Theorem

$$I^{+}_{4,1}(\mathrm{Li}_3 \textit{ algebraic}, y) = \sum \mathrm{Li}_5\textit{'s}$$

$$I^{+}_{4,1}(x, \mathrm{Li}_2 \textit{ algebraic}) = \sum \mathrm{Li}_5\textit{'s}$$

$$I^{+}_{4,1}([x] + [\tfrac{1}{1-x}] + [1 - \tfrac{1}{x}], y) = \textit{Nielsen} + \sum \mathrm{Li}_5\textit{'s}$$

Corollary

Two different families of 2-variable Li_5 *functional equations*

Li_5 functional equations

Example

$$
\begin{aligned}
I_{4,1}(x,\textstyle\sum_i[p_i]) - I_{4,1}(x,\textstyle\sum_i[\frac{1}{p_i}]) \overset{\sqcup\!\sqcup}{=} \\
& - c\,\mathrm{Li}_5(x) + 2b\,\mathrm{Li}_5(1-x) + 2b\,\mathrm{Li}_5(1-\tfrac{1}{x}) + \\
& + \tfrac{2}{abc(c-a))}\,\mathrm{Li}_5\left(\left[\tfrac{t}{x^a(1-x)^b}\right] + \left[\tfrac{t}{x^c(x-1)^b}\right]\right) + \\
& + \sum_i \{ -\tfrac{b}{2(c-a)}\mathrm{Li}_5\left(\tfrac{(1-x)^2}{x}\,\tfrac{p_i}{(1-p_i)^2}\right) + \\
& \qquad + \left(\tfrac{c-a}{2b}+2\right)\mathrm{Li}_5\left(xp_i\right) + \left(\tfrac{c-a}{2b}-2\right)\mathrm{Li}_5\left(\tfrac{x}{p_i}\right) + \\
& \qquad + \tfrac{2b}{a}\,\mathrm{Li}_5\left(\left[\tfrac{1}{1-p_i}\right] - \left[\frac{1-x}{1-p_i}\right] - \left[\tfrac{1-1/x}{1-p_i}\right]\right) + \\
& \qquad - \tfrac{2b}{c}\,\mathrm{Li}_5\left(\left[\tfrac{1}{1-1/p_i}\right] - \left[\tfrac{1-x}{1-1/p_i}\right] - \left[\tfrac{1-1/x}{1-1/p_i}\right]\right)\}
\end{aligned}
$$

Li_5 functional equations

Task

Use the 5-term Li_2 relation, and 22-term Li_3 relation to get a 'generic' Li_5 functional equation

- Not much progress so far. Difficult to find enough good arguments to get identities.
- Deadline: sometime in 2032...?

Li_6 functional equations

Have extended a pproach to weight 6 using $I_{5,1}$

Observation

$$I^+_{5,1}(x,y) = I_{5,1}(x, [y] + [\tfrac{1}{y}]) \xrightarrow{\delta} \mathrm{Li}_3(x) \wedge \mathrm{Li}_3(y)$$

$$I^-_{5,1}(x,y) = I_{5,1}(x, [y] - [\tfrac{1}{y}]) \xrightarrow{\delta} -\mathrm{Li}_2(x) \wedge \mathrm{Li}_4(y) - \mathrm{Li}_4(x) \wedge \mathrm{Li}_2(y)$$

- $I^-_{5,1}$: coboundary 0 for Li_3 FE's
- $I^+_{5,1}$: getting coboundary 0 is not so clear

Definition (Algebraic Li_4 FE)

$$\sum_j bc\,\mathrm{Li}_4(p_j) + ac\,\mathrm{Li}_4(\tfrac{1}{1-p_j}) + ab\,\mathrm{Li}_4(1 - \tfrac{1}{p_j}) \overset{\text{ш}}{=} 0$$

- Algebraic Li_4 is a sum of Li_2 FE's $\rightsquigarrow$ coboundary 0

Li_6 functional equations

Theorem

$$I^{+}_{5,1}(\mathrm{Li}_3 \textit{ algebraic}, y) = \sum \mathrm{Li}_6\text{'s}$$

$$I^{-}_{5,1}(\mathrm{Li}_4 \textit{ algebraic}, y) = \sum \mathrm{Li}_6\text{'s}$$

$$I^{+}_{5,1}([x] + [\tfrac{1}{1-x}] + [1 - \tfrac{1}{x}], y) = \textit{Nielsen} + \sum \mathrm{Li}_6\text{'s}$$

Corollary

Three new families of 2-variable Li_6 *functional equations*

Remark

- Partial results for $I_{6,1}$, $I_{7,1}$ in weight 7 and weight 8
- Possible depth 2 functional equations using $I_{4,1,1}$ in weight 6

Summary

- Relations between weight 5 MPL's
  - Depth 2: symmetries and functional equations and relations modulo δ and modulo products
  - Depth 3: symmetries and relations modulo $I_{3,2}$
- Goncharov's 'depth reduction' strategy
  - Gives polylog functional equations from MPL's
  - Results in weight 5 and 6
  - Ideas for higher weight and depth

Representation theory approach

Integrals as $\mathfrak{S}_n$ representation

- $\mathfrak{S}_n$ acts on $\mathfrak{M}_{0,n}$
- Descends to $\mathrm{cr}(a, b, c, d_1, \ldots, d_{n-3})$
- So $\mathfrak{S}_n$ acts on weight k iterated integrals

Remark

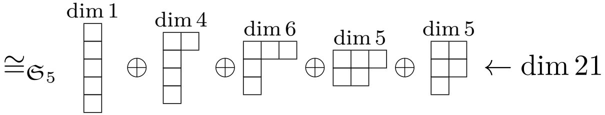
Some earlier investigations by Brown, unfinished/unpublished draft

Goal

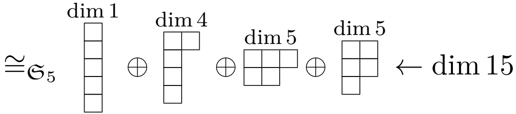
Reduce the amount of brute force computation, conceptually understand previous identities

Rep theory in weight 4

- 2-variable, weight 4 integrals, modulo products



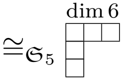
- 2-variable, weight 4 Nielsen polylogs, modulo products



Theorem (Brown)

For 'coupled' cross-ratio arguments, Nielsen $= \ker \delta$

- So quotient gives: 2-variable, weight 4 integrals, modulo δ



Rep theory identities for $I_{3,1}$

- $I_{3,1}(x,y) \xrightarrow{\delta} I_2(x) \wedge I_2(y)$, non-trivial.
- 2-variable, $I_{3,1}$, modulo δ

$$\cong_{\mathfrak{S}_5} \overset{\dim 6}{\ydiagram{3,1,1}}$$

- See a symmetry $a \leftrightarrow b \leftrightarrow c$ and $d \leftrightarrow e$
- At most $\frac{4!}{3!} = 4$ integrals $I_{3,1}((\mathbf{abcd})^{\sigma} e)$, e fixed
- Restricting to $\mathfrak{S}_4$

$$\cong_{\mathfrak{S}_4} \overset{\dim 3}{\ydiagram{2,1,1}} \oplus \overset{\dim 3}{\ydiagram{3,1}}$$

- Fixing some position a, b, c, d or e gives a subrep of this
- Implies only 3 dimensional: 2-variable, $I_{3,1}$, modulo δ, fixing e

$$\cong_{\mathfrak{S}_4} \overset{\dim 3}{\ydiagram{2,1,1}} \text{ OR } \cong_{\mathfrak{S}_4} \overset{\dim 3}{\ydiagram{3,1}}$$

Rep theory identities for $I_{3,1}$

- Must exist a relation

$$\{\, I_{3,1}(abc(\mathbf{d})e), I_{3,1}(abd(\mathbf{c})e), I_{3,1}(acd(\mathbf{b})e), I_{3,1}(bcd(\mathbf{a})e) \,\}$$

- Can show 2-variable, $I_{3,1}$, modulo δ, fixing e $\cong_{\mathfrak{S}_4} \overset{\dim 3}{\yng(3,1)}$
(Compute trace of $\sigma = (1,2)$.)
- Restrict to C_4: $\mathrm{Res}^{\mathfrak{S}_4}_{C_4} \overset{\dim 3}{\yng(3,1)} \cong_{C_4} \zeta_4 \oplus (-1) \oplus \zeta_4^3$
- Trivial representation doesn't appear, but

$$I_{3,1}((\mathbf{abcd})^{\mathrm{cyc}} e)$$

is a copy of the trivial representation

Theorem (Gangl)

$$I_{3,1}((\mathbf{abcd})^{\mathrm{cyc}} e) \overset{\delta}{=} 0$$

Rep theory in weight 5

- More complicated!
- 2-variable, weight 5, mod ⧢

$$\cong_{\mathfrak{S}_5} \overset{\dim 1}{\ydiagram{5}} \oplus 3\overset{\dim 5}{\ydiagram{3,2}} \oplus 2\overset{\dim 4}{\ydiagram{4,1}} \oplus 3\overset{\dim 5}{\ydiagram{2,2,1}} \oplus \overset{\dim 6}{\ydiagram{3,1,1}} \oplus 2\overset{\dim 4}{\ydiagram{2,1,1,1}} \oplus \overset{\dim 1}{\ydiagram{1,1,1,1,1}} \quad \leftarrow \dim 54$$

- 2-variable, weight 5, Nielsen

$$\cong_{\mathfrak{S}_5} \overset{\dim 1}{\ydiagram{5}} \oplus \overset{\dim 5}{\ydiagram{3,2}} \oplus \overset{\dim 4}{\ydiagram{4,1}} \oplus \overset{\dim 5}{\ydiagram{2,2,1}} \oplus \overset{\dim 4}{\ydiagram{2,1,1,1}} \oplus \overset{\dim 1}{\ydiagram{1,1,1,1,1}} \quad \leftarrow \dim 20$$

- Conclude 2-variable, weight 5, mod δ

$$\cong_{\mathfrak{S}_5} 2\overset{\dim 5}{\ydiagram{3,2}} \oplus \overset{\dim 4}{\ydiagram{4,1}} \oplus 2\overset{\dim 5}{\ydiagram{2,2,1}} \oplus \overset{\dim 6}{\ydiagram{3,1,1}} \oplus \overset{\dim 4}{\ydiagram{2,1,1,1}} \quad \leftarrow \dim 34$$

Rep theory in weight 5, Depth 2

- 2-variable, weight 5, mod δ

$$\cong_{\mathfrak{S}_5} 2\overset{\dim 5}{\ydiagram{3,2}} \oplus \overset{\dim 4}{\ydiagram{4,1}} \oplus 2\overset{\dim 5}{\ydiagram{2,2,1}} \oplus \overset{\dim 6}{\ydiagram{3,1,1}} \oplus \overset{\dim 4}{\ydiagram{2,1,1,1}} \leftarrow \dim 34$$

- 2-variable, $I_{4,1}$, δ

$$\cong_{\mathfrak{S}_5} \overset{\dim 5}{\ydiagram{3,2}} \oplus \overset{\dim 4}{\ydiagram{4,1}} \oplus \overset{\dim 5}{\ydiagram{2,2,1}} \oplus \overset{\dim 6}{\ydiagram{3,1,1}} \leftarrow \dim 20$$

- Sub-rep of 2-variable, $I_{3,2}$, δ

$$\cong_{\mathfrak{S}_5} \overset{\dim 5}{\ydiagram{3,2}} \oplus \overset{\dim 4}{\ydiagram{4,1}} \oplus 2\overset{\dim 5}{\ydiagram{2,2,1}} \oplus \overset{\dim 6}{\ydiagram{3,1,1}} \oplus \overset{\dim 4}{\ydiagram{2,1,1,1}} \leftarrow \dim 29$$

- 2-variable, $I_{3,2}$ mod $I_{4,1}$

$$\cong_{\mathfrak{S}_5} \overset{\dim 5}{\ydiagram{2,2,1}} \oplus \overset{\dim 4}{\ydiagram{2,1,1,1}} \leftarrow \dim 9$$

Rep theory for $I_{3,2}$ mod $I_{4,1}$

Proposition

There is a relation between the following 10 elements which span $I_{3,2}$ modulo $I_{4,1}$

$$\{\, I_{3,2}((\mathbf{a_1 a_2})\,(\mathbf{b_1 b_2 b_3}))\,\}$$

- Restrict to $\mathrm{GA}(1,5) = \langle\,(1\,2\,3\,4\,5),(2\,3\,5\,4)\,\rangle < \mathfrak{S}_5$

shape	$[1]$	$[4]$	$[2,2]$	$[5]$	$[4]$
#ccl	1	5	5	4	5
triv	1	1	1	1	1
sgn	1	-1	1	1	-1
χ_i	1	i	-1	1	$-i$
χ_{-i}	1	$-i$	-1	1	i
4d	4	0	0	-1	0

$\mathrm{GA}(5,1)$-identity for $I_{3,2}$

$$\mathrm{Res}^{\mathfrak{S}_5}_{\mathrm{GA}(1,5)} \overset{\dim 5}{\ydiagram{2,2,1}} \oplus \overset{\dim 4}{\ydiagram{2,1,1,1}} \cong_{\mathrm{GA}(1,5)} \text{triv} \oplus 2 \cdot \overset{\dim 4}{\text{4d}}$$

Theorem

The following $\mathrm{GA}(1,5)$*-symmetric identity holds for* $I_{3,2}$ *modulo* $I_{4,1}$

$$\sum_{g \in \mathrm{GA}(1,5)} \mathrm{sgn}(g) I_{3,2}(g \cdot abcde) \overset{I_{4,1}}{=} 0$$

Remark

- The 20-terms in this identity combine into 10 pairs, using the anti-symmetries of $I_{3,2}$ mod $I_{4,1}$.
- Identities from χ_i, χ_{-i} are equivalent to the above.

$\mathrm{GA}(5,1)$-identity for $I_{3,2}$

- Can refine the identity so that there is no duplication of terms

$$G := \mathrm{GA}(5,1) = \overbrace{\mathrm{ccl}_G(e)}^{\text{size 1}} \cup \overbrace{\mathrm{ccl}_G((1\,2\,3\,4\,5))}^{\text{size 4}} \cup \overbrace{\mathrm{ccl}_G((2\,3\,5\,4))}^{\text{size 5}} \cup \underbrace{\mathrm{ccl}_G((1\,2)(3\,5))}_{\text{size 5}} \cup \underbrace{\mathrm{ccl}_G((1\,2\,5\,4))}_{\text{size 5}}$$

Theorem

$$\sum_{\substack{g \in \mathrm{ccl}(\mathrm{id}) \\ \cup\, \mathrm{ccl}((1\,2\,3\,4\,5)) \\ \cup\, \mathrm{ccl}((2\,3\,5\,4))}} \mathrm{sgn}(g) I_{3,2}(g \cdot abcde) \overset{I_{4,1}}{=} 0$$