Relating MPL's in weight ≥ 5

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Outline

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1 Introduction and previous work
2 Algebraic tools
3 Relating Weight ≥ 5 MPL's
4 Polylog functional equations from MPL's

1. Firstly, we will set this work in context: why do we want to relate weight 5 MPL's and what is already know.
2. Then we talk about the tools used: the coproduct, and various versions of the symbol (coarser versions, which ignore products or depth 1 terms).
3. Then we present a sample/overview of out weight 5 identities, and how they fit into the broader context of weight 5 MPL's
4. There might only be time to cover one of hte last two sections. Either I will discuss some work that I have currently engaged in: trying to conceptually understand these identities using representation theory. Or I will talk about Goncharov's depth reduction strategy to obtain polylog functonal equations from MPL's

Introduction and previous work

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Multiple polylogarithms

- Classical polylogarithms well-studied
- Too special; conceals algebraic structure
- Introduce multiple-variable version

Definition (MPL)

$$\mathrm{Li}_{s_1,\ldots,s_k}(z_1,\ldots,z_k) := \sum_{0<n_1<\cdots<n_k} \frac{z_1^{n_1}\cdots z_k^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}}$$

- $s_1 + \cdots + s_k$ is the weight
- k is the depth

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- k is the depth

1. Classical polylogarithms are well-studied objects, relating to many area of mathematics: K-theory, hyperbolic geometry, number theory. Also connections with particle-physics, cluster algebras, ...
2. However, classical polylogarithms are somehow *too* special. By restricting to them, we loose much of the useful/interesting stucture
3. Introduce a multiple-variable version, defined by the following series. Generalises the definition of polylogs to a sum over a cone (compare with MZV's, etc).
4. As usual, we call the sum of the indices teh weight of the MPL, and we call the number of indices the depth.

Iterated integrals

- Prefer to work with iterated integrals

Definition (Iterated Integral)

$$I(x_0; x_1, \ldots, x_n; x_{n+1}) := \int_{x_0 < t_1 < \cdots < t_n < x_{n+1}} \frac{\mathrm{d}t_1}{t_1 - x_1} \circ \cdots \circ \frac{\mathrm{d}t_n}{t_n - x_n}$$

$$I_{s_1,\ldots,s_k}(x_1, \ldots, x_k) = I(0; x_1, \{0\}^{s_1-1}, \ldots, x_k, \{0\}^{s_k-1}; 1)$$

- Related to MPL's by a change of variables

$$I_{s_1,\ldots,s_k}(x_1, \ldots, x_k) = (-1)^k \operatorname{Li}_{s_1,\ldots,s_k}\left(\tfrac{1}{z_1 \cdots z_k}, \tfrac{1}{z_2 \cdots z_k}, \ldots, \tfrac{1}{z_k}\right)$$

Iterated integrals
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$I_{s_1,\ldots,s_k}(x_1, \ldots, x_k) = (-1)^k \operatorname{Li}_{s_1,\ldots,s_k}(\tfrac{1}{z_1 \cdots z_k}, \tfrac{1}{z_2 \cdots z_k}, \ldots, \tfrac{1}{z_k})$

1. We prefer to work with iterated integrals, defined by the following integral over a simplex of differential forms $1/(t-x)$. As shorthand, we will collapse the 0's down and write them as indices to the integral.
2. These objects are related to the MPL's on the previous slide by a simple change of variables, so we lose nothing by working here.
3. In fact, in some sense we gain: there are some very explicitly defined algebraic structures on these integrals. We'll return to them momentarily...

Weight ≤ 3

- All weight ≤ 3 MPL's are polylogs

Theorem (Well-known)

$$\begin{aligned} \mathrm{Li}_{1,1,1}(x,y,z) = & -\mathrm{Li}_3(\tfrac{1-xyz}{1-x}) - \mathrm{Li}_3(-\tfrac{(1-x)y}{1-y}) + \mathrm{Li}_3(xy) + \\ & + \mathrm{Li}_3(-\tfrac{(1-x)y(1-z)}{(1-y)(1-xyz)}) - \mathrm{Li}_3(\tfrac{xy(1-z)}{1-xyz}) + \\ & + \mathrm{Li}_3(\tfrac{1}{1-x}) + \mathrm{Li}_3(\tfrac{y}{y-1}) - \mathrm{Li}_3(\tfrac{y(z-1)}{1-y}) + \\ & + \text{products} \end{aligned}$$

Weight ≤ 3
All weight ≤ 3 MPL's are polylogs
Theorem (Well-known)

1. All weight ≤ 3 MPL's are classical polylogs. We can reduce any weight 2, or 3 MPL to classical polylogarithms by explicit formulae.
2. So the first case to consider really is the behaviour at weight 4.

Weight 4 - overview

- New phenomenon at weight 4

Observation

Under 8-fold symmetrisation δ:

$$\mathrm{Li}_4(x) \xrightarrow{\delta} 0$$
$$I_{3,1}(x,y) \xrightarrow{\delta} \mathrm{Li}_2(x) \wedge \mathrm{Li}_2(y)$$

- Cannot write $I_{3,1}$ as Li_4's.
  - Alternatively use $\mathrm{Li}_{3,1}, I_{2,2}, \mathrm{Li}_{2,2}, \ldots$.
- Question: how do weight 4 MPL's relate?

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Alternatively use $\mathrm{Li}_{3,1}, I_{2,2}, \mathrm{Li}_{2,2}, \ldots$.
Question: how do weight 4 MPL's relate?

1. At weight 4 we encounter for the first time a new phenomoenon. The classical polylogarithm Li_4 no longer suffices to write every multiple polylogarithm, modulo products.
2. We can see this using the Lie coalgebra structure. Computing the coboundary map of $I_{3,1}$ gives $\mathrm{Li}_2(x) \wedge \mathrm{Li}_2(y)$, whereas the coboundary of $\mathrm{Li}_4(x)$ is 0 (and so by extension) the coboundary of any combination of Li_4's with any arguments.
3. Since $\mathrm{Li}_2(x) \wedge \mathrm{Li}_2(y) \neq 0$, we have no chance of writing $I_{3,1}$ in terms of Li_4. So it is a genuinely new function.
4. So this naturally leads to the question of how MPL's in weight 4 relate to each other. Which ones can be expressed in terms of others? Do certian combinations of 'new' MPL's reduce to classical polylogarithms? These questions have already been investigated by Gangl...

Weight 4 - Gangl

Geometry of $\mathfrak{M}_{0,n}$ gives 'coupled' cross-ratio arguments

Definition ('Coupled' cross-ratios)

$$\mathrm{cr}(a,b,c,d_1,\ldots,d_{n-3}) = [\mathrm{cr}(a,b,c,d_1),\ldots,\mathrm{cr}(a,b,c,d_{n-3})]$$

Shorthand: $abcd_1\cdots d_{n-3} := \mathrm{cr}(a,b,c,d_1,\ldots,d_{n-3})$

Results include

- Functional equations for $I_{3,1}, I_{2,1,1}, I_{1,1,1,1}\ldots$
- Express $I_{3,1}$, $I_{2,2}$ and $I_{1,3}$ in terms of any others, modulo Li_4.
- Express $I_{2,1,1}$ in terms of $I_{3,1}$

Our goal: extend this to weight ≥ 5.

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Our goal: extend this to weight ≥ 5.

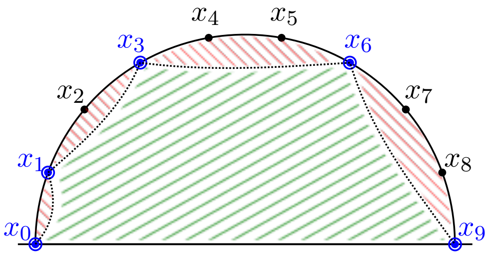
1. Small taste of Gangl's results: to whet the appetite for further investigation in weight ≥ 5.
2. How were these found? Computer assisted multilinear algebra to find null vectors of the symbol map, for this above 'coupled' cross-ratio arguments.

Algebraic tools

Algebraic tools

Iterated integrals - algebraic structure

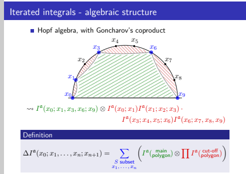
- Hopf algebra, with Goncharov's coproduct



$$\rightsquigarrow I^{\mathfrak{a}}(x_0; x_1, x_3, x_6; x_9) \otimes I^{\mathfrak{a}}(x_0; x_1) I^{\mathfrak{a}}(x_1; x_2; x_3) \cdot I^{\mathfrak{a}}(x_3; x_4, x_5; x_6) I^{\mathfrak{a}}(x_6; x_7, x_8, x_9)$$

Definition

$$\Delta I^{\mathfrak{a}}(x_0; x_1, \ldots, x_n; x_{n+1}) = \sum_{\substack{S \text{ subset} \\ x_1, \ldots, x_n}} \left(I^{\mathfrak{a}}\left(\substack{\text{main} \\ \text{polygon}}\right) \otimes \prod I^{\mathfrak{a}}\left(\substack{\text{cut-off} \\ \text{polygon}}\right) \right)$$



1. (Motivic) iterated integrals have a Hopf algebra structure, given by Gonchrov's coproduct. (Very explicitly defined on the integrals). Arrange the arguments of the integral around a semicircular polygon. For every subset of the points, we can draw in a 'main polygon' and obtain various cut-off polygons. These give the left and right terms in the coproduct.
2. This Hopf algebra structure can be used (as one way) to define the symbol, and the further 'slices' of coproduct will provide refinements to the symbol in later calculations.

Iterated integrals - symbol

- Algebraic invariant which captures the differential properties

Definition

$$\mathcal{S}\left(\int_w \mathrm{d}F_1 \circ \cdots \circ \mathrm{d}F_n\right) := F_1 \otimes \cdots \otimes F_n$$

- Obtained by maximally iterated coproduct

$$\mathcal{S}(I(x_0; x_1, \ldots, x_n; x_{n+1})) \leftrightarrow \Delta^{[n]} I^{\mathfrak{a}}(x_0; x_1, \ldots, x_n; x_{n+1})$$

$$\begin{aligned} \Delta^{[m]} \colon \mathcal{A}_m &\xrightarrow{\Delta} \mathcal{A}_{m-1} \otimes \mathcal{A}_1 \\ &\xrightarrow{\Delta \otimes \mathrm{id}} \mathcal{A}_{m-2} \otimes \mathcal{A}_1 \otimes \mathcal{A}_1 \\ &\cdots \xrightarrow{\Delta \otimes \mathrm{id}^{\otimes (m-1)}} \bigotimes_m \mathcal{A}_1 \end{aligned}$$

Iterated integrals - symbol
Algebraic invariant which captures the differential properties
Definition
$\mathcal{S}\left(\int_w \mathrm{d}F_1 \circ \cdots \circ \mathrm{d}F_n\right) := F_1 \otimes \cdots \otimes F_n$
Obtained by maximally iterated coproduct

1. The symbol is an algebraic invariant which captures the differential structure of multiple polylogarithms.
2. The first definition of the symbol given by Goncharov as the $\otimes^m$ invariant was descdribed in terms of certian rooted binary trees constucted from the integral. An alternative description comes from writing the iterated integral as a total derivative. Each term in the integrand then gives a corresponding term in the symbol.
3. Alternatively, it can be described as maximally iterated version of the previous coproduct. Some kind of 'top slice' of the coproduct, containing the most important information.
4. Expect that every relation between MPL's lies in the kernel of the symbol map. Conversely, the symbol should capture the 'main terms' of all relations. That is, modulo constant $\times$ lower-depth.
5. Can supplement this with 'slices' of the full coproduct (related to the total derivative), to obtain identities up to a final numerical constant

Symbol of $I_{3,1}$

Example

$$\begin{aligned}
\mathcal{S}(I_{3,1}(x,y)) = &(1-\tfrac{1}{x})\otimes x\otimes x\otimes(1-\tfrac{1}{y})+\\
&+(1-\tfrac{1}{x})\otimes x\otimes(1-\tfrac{1}{y})\otimes x+\\
&-(1-\tfrac{1}{x})\otimes x\otimes(1-\tfrac{1}{y})\otimes y+\\
&+(1-\tfrac{1}{x})\otimes \tfrac{1-y}{x-y}\otimes \tfrac{x}{y}\otimes x+\\
&-(1-\tfrac{1}{x})\otimes \tfrac{1-y}{x-y}\otimes \tfrac{x}{y}\otimes y+\\
&+(1-\tfrac{1}{y})\otimes(1-\tfrac{y}{x})\otimes \tfrac{x}{y}\otimes x+\\
&+(1-\tfrac{1}{y})\otimes(1-\tfrac{y}{x})\otimes \tfrac{y}{x}\otimes y\\
&\in \mathbb{Q}(x,y)^{\otimes 4}
\end{aligned}$$

- Very concrete/explicit object.
- Computation reduced to multilinear algebra

Symbol of $I_{3,1}$
Example
Very concrete/explicit object.
Computation reduced to multilinear algebra

1. One can compute the symbol of $I_{3,1}$ to be the following. The exact form of it is rather unimportant at the moment. The point is that this is a very concrete object living in some tensor algebra.
2. Can easily do explicit computations; they are reduced to multilinear algebra, which can be implemented with computer programs.

Symbol modulo products, modulo δ

- Can work with iterated integrals modulo products
  - Pass to Lie coalgebra of irreducibles $\mathcal{L}_\bullet = \mathcal{A}_\bullet / \mathcal{A}^2_{>0}$.
  - Obtain: cobracket $\delta = (\pi \otimes \pi) \circ (\Delta - \Delta^{\mathrm{op}})$ killing Li_n
- Analogue of these construction on the symbol

Strategy to study relations

1. Work modulo δ, to find 'top' slice (depth ≥ 2 part): $\overset{\delta}{=}$
2. Find Li_n terms, to get identity modulo products: $\overset{\sqcup\!\sqcup}{=}$

For more precision

3. Find product terms to get symbol level identity: $\overset{\mathcal{S}}{=}$
4. Compute slices of Δ to find constant $\times$ lower-weight corrections

1. We can explict the Hopf algebra structure to simplify/guide the search for identities. We can kill products to obtain a Lie coalgebra, with a cobracket/coboundary δ.
2. This coboundary kills depth 1 terms: Li_n, so acting with it gives us only the the depth ≥ 2 slice of identities. We can use this to fin the 'main terms' in identities, and then try to supplement them with Li_n and product terms to get more 'accurate identities'
3. I said that these constructions were available on the Hopf algebra level, but they have an analogue on the level of the symbol, via explicit formulae.
4. This gives us the following strategy to study relations.

Relating Weight ≥ 5 MPL's

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Weight 5, depth 2 - symmetries

- Depth 2 integrals: $I_{4,1}, I_{3,2}$ (and also $I_{2,3}, I_{1,4}$)

Theorem

Integrals $I_{4,1}$, $I_{3,2}$ satisfy the following anti-symmetry

$$I_{4,1}(x,y) + I_{4,1}(\tfrac{1}{x}, \tfrac{1}{y}) \overset{\sqcup\!\sqcup}{=} \mathrm{Li}_5(-[x] - [y] - 4[\tfrac{x}{y}])$$
$$I_{3,2}(x,y) + I_{3,2}(\tfrac{1}{x}, \tfrac{1}{y}) \overset{\sqcup\!\sqcup}{=} \mathrm{Li}_5(-[x] + 4[y] + 6[\tfrac{x}{y}])$$

- Compare with

Theorem (Gangl)

$$I_{3,1}(x,y) - I_{3,1}(\tfrac{1}{x}, \tfrac{1}{y}) \overset{\sqcup\!\sqcup}{=} \mathrm{Li}_4([x] - [y] + 3[\tfrac{x}{y}])$$

Weight 5, depth 2 - symmetries
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1. The depth 2 integrals all satisfy an antisymmetry under $a \leftrightarrow b$. In terms of $x, y)$ variables, this is $(x,y) \mapsto (\frac{1}{x}, \frac{1}{y})$. The antisymmetry says that the sum vanishes under the coboundary map, i.e. can be written as depth 1 terms. I write $\overset{\delta}{=}$ for this.
2. In both of these cases, we can find the Li_5 terms to get versions of the identity modulo products. (I write $\overset{\sqcup\!\sqcup}{=}$ for this).
3. The $I_{4,1}$ identity should be compared with Gangl's identity in weight 4. One can see similarities in the coefficients. The signs perhaps depend on the weight? Indeed can give a general result.

Weight n, depth 2 - symmetries

- These (anti-)symmetries generalise to weight n

Theorem

For $a, b \in \mathbb{Z}_{>1}$, the following holds modulo products

$$I_{a,b}(x,y) - (-1)^{a+b} I_{a,b}(\tfrac{1}{x}, \tfrac{1}{y}) \overset{\sqcup\!\sqcup}{=} (-1)^{a+b} \operatorname{Li}_{a+b}(x) + \\ + (-1)^b \binom{a+b-1}{a} \operatorname{Li}_{a+b}(y) - (-1)^a \binom{a+b-1}{b} \operatorname{Li}_{a+b}(\tfrac{x}{y})$$

Example

Can give explicit product term for the above identities. Example:

$$I_{4,1}(x,y) + I_{4,1}(\tfrac{1}{x}, \tfrac{1}{y}) \overset{\mathcal{S}}{=} \operatorname{Li}_5(-[x] - [y] - 4[\tfrac{x}{y}]) + \\ \operatorname{Li}_4(y)\log(x) + \operatorname{Li}_4(\tfrac{x}{y})\log(\tfrac{x}{y}) + \tfrac{1}{5!}(\log^5(\tfrac{x}{y}) - \log^5(x)) + \\ - \tfrac{1}{2!}\operatorname{Li}_3(y)\log^2(x) + \tfrac{1}{3!}\operatorname{Li}_2(y)\log^3(x) - \tfrac{1}{4!}\operatorname{Li}_1(y)\log^4(x)$$

Weight n, depth 2 - symmetries
These (anti-)symmetries generalise to weight n
Theorem
For $a, b \in \mathbb{Z}_{>1}$, the following holds modulo products
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Can give explicit product term for the above identities. Example:

1. On the level of the symbol modulo products, we can always find these Li_n terms for the $(x,y) \mapsto (1/x, 1/y)$ (anti-)symmetry.
2. In fact we can go one better, and find explicit product terms for this (anti-)symemtry. For $I_{4,1}$, we have the following product terms, hich is structurally similar to the expression Gangl finds for $I_{3,1}$ in weight 4. Comparing with this was helpfl to find the generalisation to weight n.
3. The proof of this theorem goes via explicit computation of the symbol using Rhode's formula for $I_{a,b}$ using the R-deco polygon algebra.

Numerically testable identity

- By computing 'slices' $\Delta_{1,\ldots,1,n}$, can find constant $\times$ lower-weight terms.
- Get numerically testable identity (for $I_{n,1}$)
- Already generalised via analytic techniques to the parity theorem for MPL's

Theorem (MPL Parity Theorem – Panzer, 2015)

$$\mathrm{Li}_{s_1,\ldots,s_k}(x_1,\ldots,x_k) - (-1)^{x_1+\cdots+x_k}\,\mathrm{Li}_{s_1,\ldots,s_k}(\tfrac{1}{x_1},\ldots,\tfrac{1}{x_k})$$
$=$ *explicit lower depth and products*

Numerically testable identity
By computing 'slices' $\Delta_{1,\ldots,1,n}$, can find constant $\times$ lower-weight terms.
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Theorem (MPL Parity Theorem – Panzer, 2015)
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= explicit lower depth and products

1. The last step would be to find a numerically testable verison of the identity by adding in the constant times lower-weight terms.
2. One can do this by computing further slices of the coproduct. Taking $\Delta_{1,1,1,2}$ lets us find weight 4 times $i\pi$ terms. Then $\Delta_{1,1,3}$ gives us weight 3 times $\zeta(3)$ terms, and so on...
3. This gives an identity for $I_{4,1}$. By using Gangl/Duhr's weight 4 case, and finding similar results in weight 6 leads to a candidate numerically testable identity in weight n.
4. Already these identities have been proven exactly using analytic techniques, and are contained within the MPL parity theorem of Erik Panzer. The version for iterated integrals is obtained by the usual change of variables.

$I_{4,1}$ symmetry

Proposition

$$I_{4,1}(x,y) - I_{4,1}(y,x) \stackrel{\delta}{=} 0$$

Instance of following exact identity

Theorem

$$I_{n,1}(x,y) - (-1)^n I_{n,1}(y,x) = \sum_{i=1}^{n} (-1)^{n-i} I_i(x) I_{n+1-i}(y)$$

Corollary (Gangl)

$$I_{3,1}(x,y) + I_{3,1}(x,y) \stackrel{ш}{=} 0$$

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Corollary (Gangl)
$I_{3,1}(x,y) + I_{3,1}(x,y) \stackrel{ш}{=} 0$

1. We already have a good understanding of 1 class of symmetries on depth 2 integrals. Are there any others? What other relations does $I_{4,1}$ satisfy?
2. Indeed $I_{4,1}$ satisfies another symmetry, interchanging $x \leftrightarrow y$. $I_{3,2}$ does not satisfy this symmetry.
3. This symmetry is much easier to prove exactly; it holds on any $I_{n,1}$ by expanding out these products using the shuffle product of iterated integrals. So the identity holds exactly.

$I_{4,1}$ 3-term relation

Proposition

$$I_{4,1}(x,y) + I_{4,1}(\tfrac{1}{1-x}, \tfrac{1}{1-y}) + I_{4,1}(1 - \tfrac{1}{x}, 1 - \tfrac{1}{y}) \overset{\delta}{=} 0$$

- New phenomenon: Nielsen polylogarithms

Definition

$$S_{p,q}(x) := (-1)^p I(0; \{1\}^p, \{0\}^q; x)$$

Nielsen vanishes under coboundary δ.

- Goncharov's 'reduction' conjecture $\leftrightarrow$ Nielsen equals classical
- Not clear how to write $S_{3,2}(x)$ as Li_5

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New phenomenon: Nielsen polylogarithms
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Not clear how to write $S_{3,2}(x)$ as Li_5

1. The last type of identity that $I_{4,1}$ satisfies is this 3-term symmetrisation under $(x,y) \mapsto (1/(1-x), 1/(1-y))$.
2. If we try to find Li_5 terms for this identity, we encounter a new phenomenon: some kind of obstruction in the form of Nielsen polylogarithms. The (p,q)-Nielsen polylogarithm is defined by the following iterated integral, so that Li_n is $S_{1,n-1}$.
3. All of these Nielsen polylogarithms vanish under the coboundary. Zagier's polylogarithm conjecture (in some version) says that such objects shoudl be expressible in terms of classical polylogarithms. Unfortunately, it is not clear to me how to do this for $S_{3,2}$. Perhaps one needs rather complicated arguments? Or perhaps it is not possible, and the conjecture needs to be rewritten in weight ≥ 5 to take this into account?

$I_{4,1}$ 3-term - Li_5 and Nielsen terms

Theorem

$$I_{4,1}(x,y) + I_{4,1}(\tfrac{1}{1-x}, \tfrac{1}{1-y}) + I_{4,1}(1-\tfrac{1}{x}, 1-\tfrac{1}{y}) \overset{\sqcup\!\sqcup}{=}$$
$$-2\,\mathrm{Li}_5(\tfrac{x}{y}) - 2\,\mathrm{Li}_5(\tfrac{1-y}{1-x}) - 2\,\mathrm{Li}_5(\tfrac{y(1-x)}{x(1-y)}) +$$
$$-2\,\mathrm{Li}_5(x) - \mathrm{Li}_5(1-\tfrac{1}{x}) + S_{3,2}(x) +$$
$$-2\,\mathrm{Li}_5(y) - \mathrm{Li}_5(1-\tfrac{1}{y}) + S_{3,2}(y)$$

Remark

- Symmetry broken on RHS, to reduce number of Nielsen's

$$S_{3,2}([x] + [\tfrac{1}{1-x}] + [1-\tfrac{1}{x}]) \overset{\sqcup\!\sqcup}{=} 3S_{3,2}(x) - 3\,\mathrm{Li}_5([\tfrac{1}{1-x}] + [x])$$

- Can find explicit product terms, to get symbol level identity

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Relating MPL's in weight ≥ 5
└Relating Weight ≥ 5 MPL's

└$I_{4,1}$ 3-term - Li_5 and Nielsen terms

$I_{4,1}$ 3-term - Li_5 and Nielsen terms
Theorem
Remark
Symmetry broken on RHS, to reduce number of Nielsen's
Can find explicit product terms, to get symbol level identity

1. We can find the following Li_5 and NIelsen terms to get an identity modulo products.
2. There is a clear symmetry on the left hand side. On the right hand side this has been deliberately broken so we can use as few Nielsen polylogas as possible. One could symmetrise under $(x,y) \mapsto (1/(1-x), 1/(1-y))$ to make the symmetry manifest on the right hand side.
3. One can also find explicit product terms to get an identity holding on the level of the symbol, altough they are more complicated than the previous identities.

$I_{3,2}$ relations

Relations are more complicated

- Simplest is 4-term relation

Proposition

$$\mathrm{Alt}_{d,e}\,\mathrm{Cyc}_{c,d}\,I_{3,2}(ab(\mathbf{c}\ (\mathbf{d})\mathbf{e})) \overset{\delta}{=} 0$$

'Anti-symmetrisation' of the 2-term $I_{4,1}$ identity swapping $x \leftrightarrow y$.

- Next is 6-term relation

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$$\mathrm{Alt}_{d,e}\,\mathrm{Cyc}_{a,b,c}\,I_{3,2}((\mathbf{abc})(\mathbf{de})) \overset{\delta}{=} 0$$

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'Anti-symmetrisation' of the 3-term $I_{4,1}$ identity

1. Now we have a good understanding of $I_{4,1}$, so we can start to study other depth 2 integrals like $I_{3,2}$. Immediately w find that the relations for $I_{3,2}$ are more complicated: there is only 1 symmetry.
2. The next simplest functional equation is a 4-term relatoin, which can be viewed somehow as a symmetrisation of the $I_{4,1}$ relation. After that, we have a 6 term relatoin, which again is a kind of symmetrisatoin of the 3-term $I_{4,1}$ relation.

'Exceptional' $I_{3,2}$ relation

- 2-, 4-, 6-term describe 90 out of 91 relations
- Last relation has 30-terms

Proposition

$$\mathrm{Cyc}_{a,b,c,d,e}\,\mathrm{Cyc}_{a,b,c}\,I_{3,2}(abcde) \overset{\delta}{=} \mathrm{Cyc}_{a,c,e,b,d}\,\mathrm{Cyc}_{a,c,e}\,I_{3,2}(acebd)$$

Remark

Better description with 60-terms:

$$\sum_{\sigma \in A_5} I_{3,2}(\sigma \cdot abcde) \overset{\delta}{=} 0$$

- Conceptually explained with representation theory

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Better description with 60-terms:
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1. This reduces the number of independent $I_{3,2}(abcde)$ to 30 terms. But it turns out that there is 1 more relation between these terms. The 2-, 4-, and 6- term identities only describe 90 out of 91 relations.
2. When the computer first spat out this relatoin, I tried to describe it using exactly the terms which appeared in it. I was able to give the above describe as some kind of 15-fold symmetrisation of the $I_{4,1}$ 3-term relation. The left and right hand side have the same structure, but are applied to different permutations of $abcde$: viewing them in cycle notation it is $(abcde)$ and $(abcde)^2$.
3. I have since started to revisit some of these identities from a more conceptual point of view using repesentation theory. Perhaps I have time later to explain. But one finds that summing over A_5 gives the extra relation on $I_{3,2}$. This is probaby the more pleasant description, though it involves more terms.

Relating $I_{3,2}$ and $I_{4,1}$

- Structure of $I_{3,2}$ simplifies, modulo $I_{4,1}$.

Proposition

$$\mathrm{Cyc}_{d,e}\, I_{3,2}(abc(\mathbf{de})) \overset{\sqcup\!\sqcup}{=} -3 I_{4,1}(abcde)$$

$$\mathrm{Cyc}_{c,d}\, I_{3,2}(ab(\mathbf{cd})e) \overset{\delta}{=} -\mathrm{Cyc}_{c,d,e}\, I_{4,1}(ab(\mathbf{cde}))$$

Anti-symmetric in $\mathbf{ab}$ *and* $\mathbf{cde}$*, modulo* $I_{4,1}$ *and depth 1.*

- One further 10-term relation

Remark

Expect that index 1 can always be eliminated. Can eliminate $I_{4,1}$, using above.

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Anti-symmetric in ab and cde, modulo $I_{4,1}$ and depth 1.
One further 10-term relation
Remark
Expect that index 1 can always be eliminated. Can eliminate $I_{4,1}$, using above.

1. We see somehow that the structure of $I_{3,2}$ is more complicated, but somehow there is a hint that $I_{4,1}$ is connected. Some of the identities are symmetrisations of $I_{4,1}$ identities.
2. If we try to relate $I_{3,2}$ and $I_{4,1}$, we find the folowing results. The first one shows how to write $I_{4,1}$ in term of $I_{3,2}$ and explains why some of the $I_{3,2}$ identities look like symmetrisations of $I_{4,1}$ identities. They actually are.
3. We see that the structure of $I_{3,2}$ somehow simplifies greatly modulo $I_{4,1}$. We have genunine symmetries now. $I_{3,2}$ is antisymmetric in ab (from the inversion/parity relation), and is antisymmetric in cde) from the above.
4. With the first identity above, we can even eliminate $I_{4,1}$ terms completely. It is expected that index 1 can always be eliminated from MPL's, and I do use this in other work to give an explicit reduction of $I_{1,1,1,1,1}$ to $I_{3,2}$ terms, modulo products and Li_5 on the level of the symbol modulo δ.

$I_{3,2}$ in terms of $I_{4,1}$?

Can express $I_{4,1}$ in terms of $I_{3,2}$. Converse?

- 'Coupled' cross-ratios are *not* sufficient
- Modulo δ, see $I_{3,2}$ is $\dim 29$, $I_{4,1}$ is $\dim 20$ subspace.

Observation

$$I_{4,1}(x,y) \xrightarrow{\delta} I_2(x) \wedge I_3(y) - I_3(x) \wedge I_2(y)$$

$$\tfrac{1}{2} I_{4,1}(x, [y] - [\tfrac{1}{y}]) \xrightarrow{\delta} I_3(x) \wedge I_2(y)$$

$$\begin{aligned} I_{3,2}(x,y) \xrightarrow{\delta} & -I_2(x) \wedge I_3(\tfrac{x}{y}) + I_2(y) \wedge I_3(\tfrac{x}{y}) + \\ & - 2 I_2(x) \wedge I_3(y) - I_2(y) \wedge I_3(x) \end{aligned}$$

Leads to 'brute force' way to write $I_{3,2}$ as $I_{4,1}$'s

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Leads to 'brute force' way to write $I_{3,2}$ as $I_{4,1}$'s

1. We already now that we can express $I_{4,1}$ in terms of $I_{3,2}$. But for completeness, we ask whether it can be done the other way around.
2. Compare this with the weight 4 situation. Gangl was able to express each of $I_{3,1}, I_{2,2}$ and $I_{1,3}$ in terms of the others. Surprisingly this does not work, with cross ratio arguments, at weight 5.
3. To see this is a simple linear algebra problem, really. One knows that $I_{3,2}$ is a 29 dimensional vector space, modulo δ. Whereas $I_{4,1}$ is seen to form a 20 dimensoinal subspace.
4. So we can try to approach this in a more brute force way. It we compute the coboundary of $I_{4,1}$ and $I_{3,2}$, we obtian the following results.
   In particular, this symmetrised version $I_{4,1}(x,y) - I_{4,1}(x,1/y)$ has only a single term as its coboundary. So we can build all of the terms in the coboundary of $I_{3,2}$ by choosing the arguments of $I_{4,1}$ carefully.

$I_{3,2}$ in terms of $I_{4,1}$

Theorem

$I_{3,2}$ can be expressed in terms of $I_{4,1}$, and explicit Li_5's modulo products

$$I_{3,2}(x,y) \overset{\sqcup\!\sqcup}{=} -\tfrac{1}{2} I_{4,1}([x,\tfrac{1}{y}] + [x,\tfrac{y}{x}] + 3[x,y] - [y,\tfrac{x}{y}] - [y,\tfrac{y}{x}]) + \\ + \mathrm{Li}_5(\cdots + \tfrac{15}{22}[-\tfrac{x(1-y)(x-y)}{(1-x)^2 y}] + \cdots)$$

Remark

- Involves 141 Li_5 terms
- Found with heavy computer assistance: Radchenko has procedure to find 'good arguments'

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Remark

- Involves 141 Li_5 terms
- Found with heavy computer assistance: Radchenko has procedure to find 'good arguments'

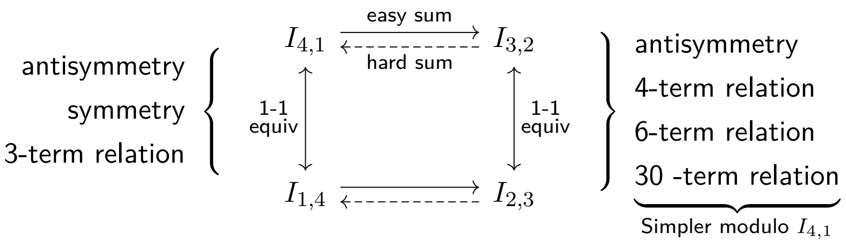
1. This leads to the following expression for $I_{3,2}$ in terms of $I_{4,1}$'s and Li_5's. The Li_5 terms are much more complicated than any we had before: they are not just simple cross ratios. I have an expression involving 141 such terms, and this was found with heavy computer assistance using programs/routines/ideas developed by Danylo Radchenko.

Depth 2 summary - modulo products

Observation

Stuffle: $I_{a,b}(x,y) + I_{b,a}(x,\frac{x}{y}) = I_{a+b}(x) + I_b(y) * I_a(\frac{x}{y})$

- No indeed to analyse $I_{1,4}$, $I_{2,3}$



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Relating MPL's in weight ≥ 5
└Relating Weight ≥ 5 MPL's
└Depth 2 summary - modulo products

Depth 2 summary - modulo products

Weight 5, depth 3

- Integrals $I_{3,1,1}, I_{1,3,1}, I_{1,1,3}, I_{2,2,1}, I_{2,1,2}, I_{1,2,2}$.
- Typically relations (modulo δ) are very complicated; (almost) no straight forward symmetries

Proposition

Only symmetry modulo δ is

$$I_{2,2,1}(x,y,z) \overset{ш}{=} I_{2,2,1}(z,y,x)$$

Theorem

$$I_{a,b,1}(x,y,z) + (-1)^{a+b} I_{b,a,1}(z,y,x) = \sum_{i=1}^{b} (-1)^i I_i(z) I_{a,b+1-i}(x,y) - (-1)^{a+b} \sum_{i=1}^{a} (-1)^i I_i(x) I_{b,a+1-i}(z,y)$$

Weight 5, depth 3

- Integrals $I_{3,1,1}, I_{1,3,1}, I_{1,1,3}, I_{2,2,1}, I_{2,1,2}, I_{1,2,2}$.
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1. So, of course, we now shoudl move on to stud the depth 3 integrals. The situation here, even modulo δ, is significantly more complicated. Typically the integrals do not satisfy any straight forward symmetries. The only such symmetry occurs for $I_{2,2,1}$, and comes about by swapping $x \leftrightarrow z$.
2. This symmetry is actually explained by the following exact identiy whic holds at arbitrary weight. We can switch the indices and arguments to $I_{a,b,1}$ in the following way modulo products. The proof of this propositoin is directly by multiplying the integrals using the shuffle product.

Depth 3, modulo depth 2

- Idea: search modulo depth 2
- Only need to search modulo $I_{3,2}$
- Warning: use only 'coupled' cross ratios
  - (Expect: everything in weight 5 is depth ≤ 2.)

Proposition

Obtain many new symmetries

$$I_{3,1,1}((\mathbf{ba})cdef) \overset{I_{3,2}}{=} I_{3,1,1}(abcdef) \overset{I_{3,2}}{=} I_{3,1,1}(ab(\mathbf{fedc}))$$

$$I_{2,1,2}((\mathbf{ba})cdef) \overset{I_{3,2}}{=} I_{2,1,2}(abcdef) \overset{I_{3,2}}{=} I_{2,1,2}(\mathbf{fedc})$$

$$\overset{I_{3,2}}{\parallel}$$

$$I_{2,1,2}(ab(\mathbf{dc})(\mathbf{fe}))$$

Depth 3, modulo depth 2
Idea: search modulo depth 2
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Proposition
Obtain many new symmetries

1. Since the situation for $I_{3,2}$ simplified rather dramatically moudlo $I_{4,1}$, we might consider doing something similar here. Rather than searching for identities modulo depth 1, search modulo depth 2.
2. Since $I_{4,1}$ can be expressed in terms of $I_{3,2}$, we only need to work modulo $I_{3,2}$. But a word of warning: we do this with a very restricted choice of arguments. It is expected that everything in weight 5 is depth ≤ 2. Indeed I can express $I_{1,1,1,1,1}$ as a sum of $I_{3,2}$'s modulo depth 1, but the arguments are complicated, which verifies this.
3. No, here we only work with cross ratio arguments. Nevertheless, the structure does simplify drastically. We obtain several new symmetries. All of the integrals gain an $(ab) \rightsquigarrow (x, y, z) \leftrightarrow (1/x, 1/y, 1/z)$ symmetry from the parity theorem. But we also have other symmetries.
4. $I_{2,1,2}$ and $I_{3,1,1}$ gain a symmetry by reversnig $cdef$ to $fedc$. But $I_{2,1,2}$ has another symmetry: simultaneously switching $(cd)(ef)$.

Relating depth 3 integrals

Theorem

Weight 5 depth 3 integrals span the same space, modulo $I_{3,2}$.

Example

$$I_{1,3,1}(abc(\mathbf{f})e(\mathbf{d})) \overset{⧢}{=} I_{3,1,1}(abcdef) \overset{I_{3,2}}{=} I_{1,1,3}(ab(\mathbf{dc})(\mathbf{fe}))$$

$$I_{2,2,1}(abcdef) \overset{⧢}{=}$$
$$-I_{3,1,1}([abc(\mathbf{def})] + [abc(\mathbf{dfe})] + [abc(\mathbf{fde})] + [abc(\mathbf{fed})])$$

$$I_{3,1,1}(abcdef) \overset{I_{3,2}}{=} \sum 197 I_{2,1,2}\text{'s}$$

$$\vdots$$

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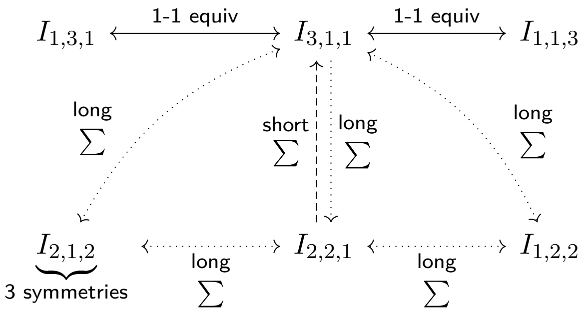
$-I_{3,1,1}([abc(\mathbf{def})] + [abc(\mathbf{dfe})] + [abc(\mathbf{fde})] + [abc(\mathbf{fed})])$
$I_{3,1,1}(abcdef) \overset{I_{3,2}}{=} \sum 197 I_{2,1,2}\text{'s}$
$\vdots$

1. However, all depth 3 integrals do span the same space modulo $I_{3,2}$. Like in the weight 4 case where Gangl could relate $I_{3,1}$, $I_{2,2}$ etc. Here we can write every depth 3 integral as a sum of any other depth 3 integral.
2. Some of the relations are relatively nice: single terms are equal modulo $I_{3,2}$. Or the sum only involves a few terms. (Un)fortunately some of the othe relations seem much longer and more compliated: I can write $I_{3,1,1}$ as a sum of 197 $I_{2,1,2}$ terms modulo $I_{3,2}$.

Depth 3 summary – modulo $I_{3,2}$

- Each integral has 2 symmetries
- $I_{2,1,2}$ has 3 symmetries



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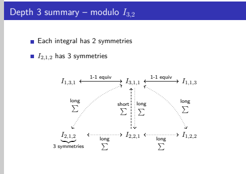
Relating MPL's in weight ≥ 5

└Relating Weight ≥ 5 MPL's

└Depth 3 summary – modulo $I_{3,2}$



Polylog functional equations from MPL's

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Goncharov's 'depth-reduction' idea

Element $\kappa(x, y)$, essentially $I_{3,1}(x, y)$, has coboundary $\mathrm{Li}_2(x) \wedge \mathrm{Li}_2(y)$

- Substituting $x = \mathrm{Li}_2$ functional equations gives coboundary 0
- Expect $I_{3,1}(\mathrm{Li}_2 \text{ FE}, y) \overset{ш}{=} \sum \mathrm{Li}_4$'s
- Get Li_4 functional equation by expanding in two ways

$$I_{3,1}(\mathrm{Li}_2 \text{ FE}, \mathrm{Li}_2 \text{ FE}) \overset{ш}{=} \sum \mathrm{Li}_4\text{'s}$$

Similar strategy for $\Phi_5(x, y)$, essentially $I_{4,1}(x, [y] - [\frac{1}{y}])$

- Li_5 FE from $x = \mathrm{Li}_3$ FE, $y = \mathrm{Li}_2$ FE.

Remark

Such functional equations should play a key role in a proof of Zagier's polylogarithm conjecture

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Li_5 FE from $x = \mathrm{Li}_3$ FE, $y = \mathrm{Li}_2$ FE.
Remark
Such functional equations should play a key role in a proof of Zagier's polylogarithm conjecture

1. Goncharov's deph reductoin strategy is a way to use our knowlege of MPL's in depth 2 to find non-trivial functional equations for Li_n. Goncharov defined an element $\kappa(x, y)$ in the Hopf algebra of iterated integrals (which is essentially $I_{3,1}(x, y)$. This element is constructed so that the coboundary is very simple: only $\mathrm{Li}_2(x) \wedge \mathrm{Li}_2(y)$.
2. This makes is very simple to find combinations which then have 0 coboundary. e can just plug in any Li_2 functional equation. We should then be able to write the result as a sum of Li_4's.
3. From this we can derive a Li_4 functional equation by expanding out $I_{3,1}(\mathrm{Li}_2, \mathrm{Li}_2)$ in two different ways. Hopefully othe arguments used in the different sets of Li_4 terms are different/independent enough that little cancellation occurs after expanding out. One then have two combinations of Li_4's who diffrence is 0 modulo products. This gives us our Li_4 functional equation.
4. Goncharov also outlines a simlar strategy for weight 5, using an element Φ_5, which is essentially $I_{4,1}(x, y) - I_{4,1}(x, 1/y)$. In this

Li_4 functional equations

Definition (Algebraic Li_2 FE)

Let $p_i(t)$ be roots of $x^a(1-x)^b = t$, $a \neq b \in \mathbb{Z}_{>0}$. Set $a+b+c=0$. Then

$$\sum_j \mathrm{Li}_2(p_j(t)) \overset{\sqcup\!\sqcup}{=} 0$$

Theorem (Gangl, 2000)

$$I_{3,1}(\textstyle\sum_j [p_j], y) \overset{\sqcup\!\sqcup}{=} \mathrm{Li}_4\big(\frac{1}{abc}[\frac{t}{y^a(1-y)^b}] - b[1-\frac{1}{y}] - c[y] + \\ -\sum_j \frac{b}{a}[\frac{1-p_j}{1-y}] - \frac{b}{c}[\frac{1-1/y}{1-1/p_j}] - \frac{a}{b}[\frac{y}{p_j}] - \frac{b}{a}[1-p_j]\big)$$

Corollary

2-variable of family of Li_4 functional equations

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Relating MPL's in weight ≥ 5
└ Polylog functional equations from MPL's
└ Li_4 functional equations

1. So what has been done with this already? Well in weight 4, we have had for many years a result for a certain infinite family of Li_2 functional equations. This is the so-called algebraic Li_2 equation, defined by taking the roots of the polynomial $x^a(1-x)^b = t$, as a function of t.
2. In 2000, Gangl was able to find the Li_4 combintaoin, when this was plugged into $I_{3,1}$, and from there derive an 2-variable infinite family of Li_4 Functional equations.

Li_4 functional equations

- Want to do this for the 5-term equation for Li_2, to obtain 'generic' Li_4 functional equation

Theorem (Gangl, 2016)

$$I_{3,1}(\mathrm{Li}_2 \textit{ five term}, y) = \sum \textit{122 } \mathrm{Li}_4\textit{'s}$$

Corollary

931-term functional equation for Li_4.

Remark

Goncharov-Rudenko: announced a proof of Zagier's conjecture for $n = 4$. Geometric interpretation of 122 term relation.

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931-term functional equation for Li_4.
Remark
Goncharov-Rudenko: announced a proof of Zagier's conjecture for $n = 4$. Geometric interpretation of 122 term relation.

1. The real aim is to do this for the 5-term Li_2 functional equatoin, which is though to be the basic functional equatoin for Li_2, from which all others can be derived.
2. After the advent of the symbol, Gangl was able to complete this task to write $I_{3,1}$ of the 5-term as a sum of 122 Li_4's, and hence derive a 931-term functional equatoin for Li_4.
3. Recently it was announced by Goncharov-Rudenko a proof of Zagier's conjecture for $n = 4$, which uses this 122-term relation as a key ingredient.

Li_5 functional equations

Approach in weight 5 uses $I_{4,1}$

Observation

$$I^{-}_{4,1}(x,y) = I_{4,1}(x,[y]-[\tfrac{1}{y}]) \xrightarrow{\delta} \mathrm{Li}_3(x) \wedge \mathrm{Li}_2(y)$$

- $I^{-}_{4,1}$ coboundary 0 for $x = \mathrm{Li}_3$ FE or $y = \mathrm{Li}_2$ FE.

Definition (Algebraic Li_3 FE)

$$\sum_j a\,\mathrm{Li}_3(p_j) - b\,\mathrm{Li}_3(1-p_j) \overset{\amalg}{=} 0$$

Li_5 functional equations
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$I^{-}_{4,1}$ coboundary 0 for $x = \mathrm{Li}_3$ FE or $y = \mathrm{Li}_2$ FE.
Definition (Algebraic Li_3 FE)
$\sum_j a\,\mathrm{Li}_3(p_j) - b\,\mathrm{Li}_3(1-p_j) \overset{\amalg}{=} 0$

1. The approach in weight 5 is somewhat similar. The symmetrised version of $I_{4,1}$ has very simple coboundary $\mathrm{Li}_3(x) \wedge \mathrm{Li}_2(y)$, so one makes it vanish by subsituting a Li_3 equaton for x or a Li_2 equation for y.
2. The is a version of the algebraic Li_2 equation for Li_3, which involves summing up over another S_3-orbit. So we can try to find Li_5 terms for these.

Li_5 functional equations

Theorem

$$I^+_{4,1}(\mathrm{Li}_3 \textit{ algebraic}, y) = \sum \mathrm{Li}_5\text{'s}$$

$$I^+_{4,1}(x, \mathrm{Li}_2 \textit{ algebraic}) = \sum \mathrm{Li}_5\text{'s}$$

$$I^+_{4,1}([x] + [\tfrac{1}{1-x}] + [1 - \tfrac{1}{x}], y) = \textit{Nielsen} + \sum \mathrm{Li}_5\text{'s}$$

Corollary

Two different families of 2-variable Li_5 functional equations

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Corollary
Two different families of 2-variable Li_5 functional equations

1. Indeed, I was able to find Li_5 terms for both the algebraic Li_2 and Li_3 equations. I can also give Li_5 and Nielsen terms for the 3-term Li_3 equation.
2. This means we can derive 2 different 2-variable infinite families of functional equations for Li_5. (The resulting combination of Nielsen terms necessarily can be writen as Li_5's, and it can be done very simply in this case.)

Li_5 functional equations

Example

$$
\begin{aligned}
I_{4,1}(x, \textstyle\sum_i [p_i]) - I_{4,1}(x, \textstyle\sum_i [\frac{1}{p_i}]) &\overset{\sqcup\!\sqcup}{=} \\
& - c\,\mathrm{Li}_5(x) + 2b\,\mathrm{Li}_5(1-x) + 2b\,\mathrm{Li}_5(1-\tfrac{1}{x}) + \\
& + \frac{2}{abc(c-a))}\,\mathrm{Li}_5\left(\left[\frac{t}{x^a(1-x)^b}\right] + \left[\frac{t}{x^c(x-1)^b}\right]\right) + \\
& + \sum_i \{ -\frac{b}{2(c-a)}\,\mathrm{Li}_5\left(\frac{(1-x)^2}{x}\,\frac{p_i}{(1-p_i)^2}\right) + \\
& \quad + \left(\tfrac{c-a}{2b} + 2\right)\mathrm{Li}_5\left(xp_i\right) + \left(\tfrac{c-a}{2b} - 2\right)\mathrm{Li}_5\left(\frac{x}{p_i}\right) + \\
& \quad + \frac{2b}{a}\,\mathrm{Li}_5\left(\left[\frac{1}{1-p_i}\right] - \left[\frac{1-x}{1-p_i}\right] - \left[\frac{1-1/x}{1-p_i}\right]\right) + \\
& \quad - \frac{2b}{c}\,\mathrm{Li}_5\left(\left[\frac{1}{1-1/p_i}\right] - \left[\frac{1-x}{1-1/p_i}\right] - \left[\frac{1-1/x}{1-1/p_i}\right]\right) \}
\end{aligned}
$$

Li_5 functional equations
Example

1. As an example, we have the following expression for the Li_2 algebraic equation. It is rather more complicated than Gangl's verson.

Li_5 functional equations

Task

Use the 5-term Li_2 relation, and 22-term Li_3 relation to get a 'generic' Li_5 functional equation

- Not much progress so far. Difficult to find enough good arguments to get identities.
- Deadline: sometime in 2032. . . ?

Li_5 functional equations

Task

Use the 5-term Li_2 relation, and 22-term Li_3 relation to get a 'generic' Li_5 functional equation

- Not much progress so far. Difficult to find enough good arguments to get identities.
- Deadline: sometime in 2032. . . ?

1. As in the weight 4 case, we want to do this for the basic functional equations of Li_2 and Li_3 namely the 5-term and the 22-term respectively.
2. Unfortunately, progress so far has been rather limited. I have not been able to find Li_5 or Nielsen terms in any cases beyond what I already listead. It took Gangl 16 years to move from the algebraic to the 5-term, so I have until 2032 to beat him. So far I have not been able to find the right arguments: either I have too few and the calculation returns 0, or I have too many and the calculation crashes. Probably with a better understanding of the structure of weight 5 MPL's, perhaps some rep theory, one is guided to better choices of arguments.

Li_6 functional equations

Have extended a pproach to weight 6 using $I_{5,1}$

Observation

$$I^+_{5,1}(x,y) = I_{5,1}(x,[y]+[\tfrac{1}{y}]) \xrightarrow{\delta} \mathrm{Li}_3(x) \wedge \mathrm{Li}_3(y)$$

$$I^-_{5,1}(x,y) = I_{5,1}(x,[y]-[\tfrac{1}{y}]) \xrightarrow{\delta} -\mathrm{Li}_2(x) \wedge \mathrm{Li}_4(y) - \mathrm{Li}_4(x) \wedge \mathrm{Li}_2(y)$$

- $I^-_{5,1}$: coboundary 0 for Li_3 FE's
- $I^+_{5,1}$: getting coboundary 0 is not so clear

Definition (Algebraic Li_4 FE)

$$\sum_j bc\,\mathrm{Li}_4(p_j) + ac\,\mathrm{Li}_4(\tfrac{1}{1-p_j}) + ab\,\mathrm{Li}_4(1-\tfrac{1}{p_j}) \overset{ш}{=} 0$$

- Algebraic Li_4 is a sum of Li_2 FE's $\rightsquigarrow$ coboundary 0

2017-11-14

Relating MPL's in weight ≥ 5

└Polylog functional equations from MPL's

└Li_6 functional equations

Li_6 functional equations
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Algebraic Li_4 is a sum of Li_2 FE's $\rightsquigarrow$ coboundary 0

1. As yet, we can't push to the 5-term or 22-term in weight 5. But we can push to higher weight, and find similar results using the algebraic equations.
2. The idea at weight 6 is to use the function $I_{5,1}$, or the symmetrised versions which have simpler coboundary. With plus, the coboundary is $\mathrm{Li}_3(x) \wedge \mathrm{Li}_3(y)$, so Li_3 functional equations will suffice.
3. With the minus symmetrisation, the coboundary has a weight 2 and weight 4 component, so we have to be clever. Fortunately the algebraic Li_4 equation is already a Li_2 equation, so plugging this in does still kill the coboundary.

Li_6 functional equations

Theorem

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Corollary

Three new families of 2-variable Li_6 *functional equations*

Remark

- Partial results for $I_{6,1}$, $I_{7,1}$ in weight 7 and weight 8
- Possible depth 2 functional equations using $I_{4,1,1}$ in weight 6

Li_6 functional equations
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Corollary
Three new families of 2-variable Li_6 functional equations
Remark
Partial results for $I_{6,1}$, $I_{7,1}$ in weight 7 and weight 8
Possible depth 2 functional equations using $I_{4,1,1}$ in weight 6

1. In all of these cases, I can find the Li_6 terms. I can also find them for the 3-term Li_3 equation.
   This time we get 3 different infinite families of 2-variable Li_6 functional equations.
2. Probably the natural question now is whether this can be pushed to weight 7 (where we only know 3 individual functional equations), or to weight 8 where we no none. I am investigating some ideas, but the difficulty now becomes finding good functional equations in weight 4 or 5 to allow us to make the coboundary of $I_{6,1}$ vanish.
3. I also have some ideas on how to produce a depth 2 version, by tryign to write certain depth 3 integrals as a sum of depth 2 stuff. But this is very much work in progress, with no results yet.

Summary

- Relations between weight 5 MPL's
  - Depth 2: symmetries and functional equations and relations modulo δ and modulo products
  - Depth 3: symmetries and relations modulo $I_{3,2}$
- Goncharov's 'depth reduction' strategy
  - Gives polylog functional equations from MPL's
  - Results in weight 5 and 6
  - Ideas for higher weight and depth

Summary
- Relations between weight 5 MPL's
  - Depth 2: symmetries and functional equations and relations modulo δ and modulo products
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- Goncharov's 'depth reduction' strategy
  - Gives polylog functional equations from MPL's
  - Results in weight 5 and 6
  - Ideas for higher weight and depth

1. In this talk we have seen how to relate MPL's. We have focused on relating weight 5 MPL's, typically in depth 2 and depth 3 where the calculations are tractable enoguh to be completed.
2. We have also looked at an approach using representation theory to conceptually understand these identities and guide us to new ones.
3. We have also used our knowlege of depth 2 MPL's in weight 5 and 6 to derive some new functional equations for Li_5 and Li_6, using Goncharov's depth reduction strategy.

Representation theory approach

Representation theory approach

Integrals as $\mathfrak{S}_n$ representation

- $\mathfrak{S}_n$ acts on $\mathfrak{M}_{0,n}$
- Descends to $\mathrm{cr}(a, b, c, d_1, \ldots, d_{n-3})$
- So $\mathfrak{S}_n$ acts on weight k iterated integrals

Remark

Some earlier investigations by Brown, unfinished/unpublished draft

Goal

Reduce the amount of brute force computation, conceptually understand previous identities

Integrals as $\mathfrak{S}_n$ representation
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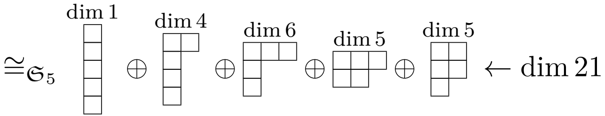
Goal
Reduce the amount of brute force computation, conceptually understand previous identities

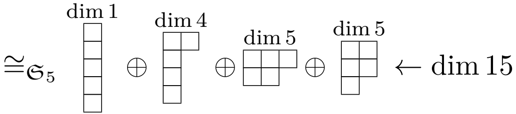
1. The previous identities were all found via brute force computer calculation, and lots of staring, tryign to identify patterns in output. Little use was made of the underlying symmetry of the problem, namely the S_n action on the marked points in $\mathfrak{M}_{0,n}$
2. This gives an action of $\mathfrak{S}_n$ cross ratios, and thus on iterated integrals. We should therefore study these spaces as $\mathfrak{S}_n$ representations. This is motivated by an unpublished/unfinished draft of Francis Brown. The goal is to get a better, more conceptual, understanding of these identities. Try to reduce the amount of brute force computation, and somehow guide ourselves to the nice/correct identities in each case.

Rep theory in weight 4

- 2-variable, weight 4 integrals, modulo products



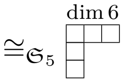
- 2-variable, weight 4 Nielsen polylogs, modulo products

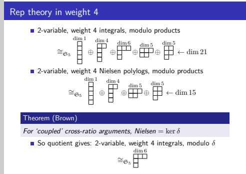


Theorem (Brown)

For 'coupled' cross-ratio arguments, Nielsen $= \ker \delta$

- So quotient gives: 2-variable, weight 4 integrals, modulo δ

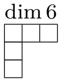




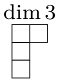
1. Perhaps it is good to start with the weight 4 case, where we already have simple/explicit results from Gangl. We can try to understand/recover these first, before studying the weight 5 case.
2. One can show that 2-variable weight 4 integrals, modulo products, form a 21 dimensional representation of S_5 which decompoases as indicated. This decomposition actually has a conceptual proof, which is explained in Brown's unpublished work. Similarly, the 2-variable weight 4 Nielsen/Li_4 representation is a 15 dimensional subspace of this, decomposing as indicated.
3. In that work Brown shows that with cross ratio arguments, Nielsen is exactly the kernel of the coboundary map. So working with integrals modulo δ is equivalent to taking the quotient of integrals modulo products by the Nielsen subreprsenation.
4. This shows that 2-variable, weight 4 integrals form a 6 dimensional space. This turns out to be an irreducible represenation given by young diagram 311.

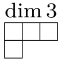
Rep theory identities for $I_{3,1}$

- $I_{3,1}(x,y) \xrightarrow{\delta} I_2(x) \wedge I_2(y)$, non-trivial.
- 2-variable, $I_{3,1}$, modulo δ

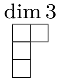
$\cong_{\mathfrak{S}_5}$ 

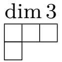
- See a symmetry $a \leftrightarrow b \leftrightarrow c$ and $d \leftrightarrow e$
- At most $\frac{4!}{3!} = 4$ integrals $I_{3,1}((\mathbf{abcd})^\sigma e)$, e fixed
- Restricting to $\mathfrak{S}_4$

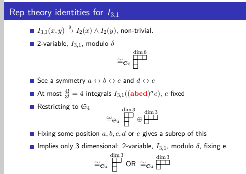
$\cong_{\mathfrak{S}_4}$ 

$\oplus$ 

- Fixing some position a, b, c, d or e gives a subrep of this
- Implies only 3 dimensional: 2-variable, $I_{3,1}$, modulo δ, fixing e

$\cong_{\mathfrak{S}_4}$ 

OR $\cong_{\mathfrak{S}_4}$ 



1. We then observe that any integral modulo δ must be in this space. We see that that $I_{3,1}(x,y)$ has non-trivial coboundary, so gives a non-zero vector in thsi space.
2. In particular, we see that 2-variable $I_{3,1}$ modulo δ must be the whole representation (it is irreducible). So there are 6 linearly independent $I_{3,1}$'s modulo δ. A simlarly argument shows that $I_{2,2}$ and $I_{3,1}$ also span this space, so we must be able to write each in terms of the other.
3. We can spot (using the above expression for $I_{3,1}$ coboundary), that there is a symmetry under abc and de. (Or we refer back to Gangl's results...) Either way, this means that there are at most 4 integrals, when we fix position e.
4. If we restrict the representatoin to $\mathfrak{S}_4$, the branching rule shows it decomposes into 2 3-dimensional reprsentations. Since there are only 4 linearly independent integrals, we do no get the whole space, therefore fixing e must give only one of the irreducible components.

Rep theory identities for $I_{3,1}$

- Must exist a relation

$$\{\, I_{3,1}(abc(\mathbf{d})e), I_{3,1}(abd(\mathbf{c})e), I_{3,1}(acd(\mathbf{b})e), I_{3,1}(bcd(\mathbf{a})e) \,\}$$

- Can show 2-variable, $I_{3,1}$, modulo δ, fixing e $\cong_{\mathfrak{S}_4} \overset{\dim 3}{\yng(3,1)}$
  (Compute trace of $\sigma = (1,2)$.)
- Restrict to C_4: $\operatorname{Res}^{\mathfrak{S}_4}_{C_4} \overset{\dim 3}{\yng(3,1)} \cong_{C_4} \zeta_4 \oplus (-1) \oplus \zeta_4^3$
- Trivial representation doesn't appear, but

$$I_{3,1}((\mathbf{abcd})^{\mathrm{cyc}} e)$$

is a copy of the trivial representation

Theorem (Gangl)

$$I_{3,1}((\mathbf{abcd})^{\mathrm{cyc}} e) \overset{\delta}{=} 0$$

Rep theory identities for $I_{3,1}$
Must exist a relation
$\{ I_{3,1}(abc(\mathbf{d})e), I_{3,1}(abd(\mathbf{c})e), I_{3,1}(acd(\mathbf{b})e), I_{3,1}(bcd(\mathbf{a})e) \}$
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Trivial representation doesn't appear, but
$I_{3,1}((\mathbf{abcd})^{\mathrm{cyc}} e)$
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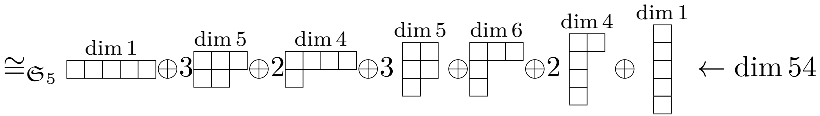
Theorem (Gangl)
$I_{3,1}((\mathbf{abcd})^{\mathrm{cyc}} e) \overset{\delta}{=} 0$

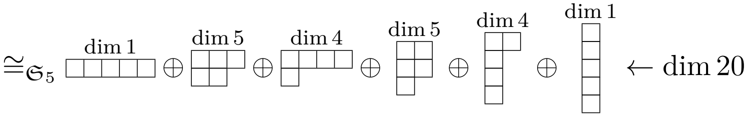
1. This means that there are only 3 linearly independent integrals upon fixing e. So there is a relation between the 4 integrals listed.
2. By computing the trace, one can show that the fixing e representation corresponds to 31. And if we restrict further to C_4 inside S_4, it decopmoses as $\zeta_4 \oplus (-1) \oplus \zeta_4^3$. The trivial reprsentation does not appear.
3. We can use this to get an identity by manufactuing a copy of the trivial rep inside this space. Such a copy is given by cycling the first 4 entries. But then this sum must be 0 as the trivial rep is not present.
4. From this we recover the following theorem from Gangl

Rep theory in weight 5

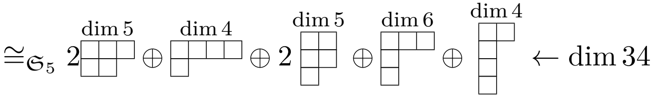
- More complicated!
- 2-variable, weight 5, mod ⧢

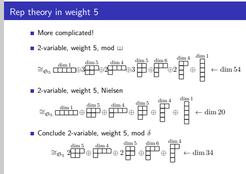


- 2-variable, weight 5, Nielsen



- Conclude 2-variable, weight 5, mod δ

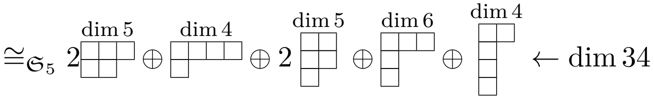




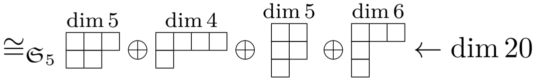
1. We can start to play the same games at weight 5, but of course the situation is more compliated. 2-variable integrals, mod products span a 54 dimensional space. The Nielsens span a 20 dimensional space. So integral modulo δ span a 34 dimensional space. All of these decompose as indicated.

Rep theory in weight 5, Depth 2

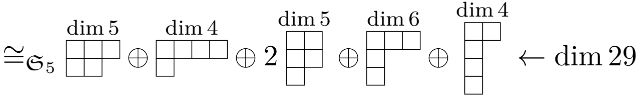
- 2-variable, weight 5, mod δ



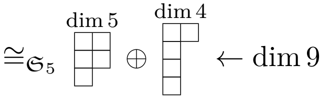
- 2-variable, $I_{4,1}$, δ

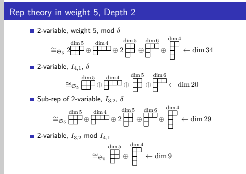


- Sub-rep of 2-variable, $I_{3,2}$, δ



- 2-variable, $I_{3,2}$ mod $I_{4,1}$





1. Then we can consider the specific depth 2 integrals living inside this space. We know that $I_{4,1}$ spans a 20 dimensional subspace. One can copute it gives the following reprsentation. (I don't really have yet a conceptual explanation.)
2. Similarly $I_{3,2}$ spans the 29 dimensional rep, which decomposes as indicated.
3. From here, one sees that perhaps $I_{4,1}$ is a subrep of $I_{3,2}$. (Perhaps one has to be careful about the 2 copies of 32 appearing in integrals modulo δ... We could get two different copies of 32 in the $I_{4,1}$ and $I_{3,2}$ reps?)
4. But from our earlier results, we know that indeed $I_{4,1}$ can be expressed in terms of $I_{3,2}$, so it must be a subrep.
5. Taking the quotient gives us a way to study integrals $I_{3,2}$ modulo integrals $I_{4,1}$. We see this decomposes as indicated, meaning it is a 9-dimensional space (as we already know!)

Rep theory for $I_{3,2}$ mod $I_{4,1}$

Proposition

There is a relation between the following 10 elements which span $I_{3,2}$ modulo $I_{4,1}$

$$\{\, I_{3,2}((\mathbf{a_1 a_2})\,(\mathbf{b_1 b_2 b_3}))\,\}$$

- Restrict to $\mathrm{GA}(1,5) = \langle\, (1\,2\,3\,4\,5), (2\,3\,5\,4)\,\rangle < \mathfrak{S}_5$

shape	$[1]$	$[4]$	$[2,2]$	$[5]$	$[4]$
#ccl	1	5	5	4	5
triv	1	1	1	1	1
sgn	1	-1	1	1	-1
χ_i	1	i	-1	1	$-i$
χ_{-i}	1	$-i$	-1	1	i
4d	4	0	0	-1	0

Rep theory for $I_{3,2}$ mod $I_{4,1}$
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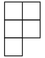
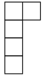
Restrict to $\mathrm{GA}(1,5) = \langle (1\,2\,3\,4\,5), (2\,3\,5\,4) \rangle < \mathfrak{S}_5$

1. Since the dimension of the space is 9, we know there must be a relation betwen the above 10 elements. This might suggest looking for a D_5-symmetric relation. (Since $\mathbb{Z}_10$ is not a subgroup of S_5.)
2. It turns out that D_5 does not describe any non-trivial identities. If one restricts the above character to D_5, then one finds the sign representation is missing, so gets identities like

$$\sum_{g \in D_5} \mathrm{sgn}(g) I_{3,2}(g \circ abcde) \overset{I_{4,1}}{=} 0$$

, but it turns out this follows trivialy from the anti-symmetries above.
3. The smallsest subgroupgroup to look at seems to be the general affine group $\mathrm{GA}(1,5)$ (degree 1 over $\mathbb{F}_5$). The character table of this group is given below.

$\mathrm{GA}(5,1)$-identity for $I_{3,2}$

$\mathrm{Res}^{\mathfrak{S}_5}_{\mathrm{GA}(1,5)}$ dim 5  $\oplus$ dim 4  $\cong_{\mathrm{GA}(1,5)}$ triv $\oplus\, 2 \cdot$ $\overset{\dim 4}{\text{4d}}$

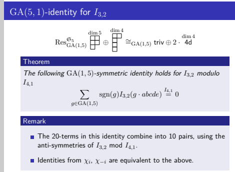
Theorem

The following $\mathrm{GA}(1,5)$*-symmetric identity holds for* $I_{3,2}$ *modulo* $I_{4,1}$

$$\sum_{g \in \mathrm{GA}(1,5)} \mathrm{sgn}(g) I_{3,2}(g \cdot abcde) \overset{I_{4,1}}{=} 0$$

Remark

- The 20-terms in this identity combine into 10 pairs, using the anti-symmetries of $I_{3,2}$ mod $I_{4,1}$.
- Identities from χ_i, χ_{-i} are equivalent to the above.



1. If we restrict the $\mathfrak{S}_5$ charcter to $\mathrm{GA}(1,5)$, we find it decomposes as the trivial rep, and 2 copies of the 4d rep. So we are missing the sign rep, and the $\chi_{\pm i}$ reps.
2. By writing down a combination which is invariant under the sign-rep we guarentee the result vanishes, as no copy of the sign rep appears. This leads to the following theorem describnig an identity on $I_{3,2}$ modulo $I_{4,1}$.
3. A priori this identity consists of 20 terms, but they combine into 10 pairs using the previous antisymmetries. This is exactly the relation between the 10 elements $I_{3,2}(a_1 a_2\, b_1 b_2 b_3)$ previously sought/mentioned.
   Moreover, the identitie from $\chi_{\pm i}$ turn out to just be scalar multiples of this. (There can't be another identity, since the dimension of the space is known to be 9!)
4. Can also be described as $\sum_{A_5}$, where terms combine into 10 6-tuples using symmetries.

GA$(5,1)$-identity for $I_{3,2}$

- Can refine the identity so that there is no duplication of terms

$$G := \mathrm{GA}(5,1) = \underbrace{\mathrm{ccl}_G(e)}_{\text{size } 1} \cup \underbrace{\mathrm{ccl}_G((1\,2\,3\,4\,5))}_{\text{size } 4} \cup \underbrace{\mathrm{ccl}_G((2\,3\,5\,4))}_{\text{size } 5} \cup \underbrace{\mathrm{ccl}_G((1\,2)(3\,5))}_{\text{size } 5} \cup \underbrace{\mathrm{ccl}_G((1\,2\,5\,4))}_{\text{size } 5}$$

Theorem

$$\sum_{\substack{g \in \mathrm{ccl}(\mathrm{id}) \\ \cup\, \mathrm{ccl}((1\,2\,3\,4\,5)) \\ \cup\, \mathrm{ccl}((2\,3\,5\,4))}} \mathrm{sgn}(g) I_{3,2}(g \cdot abcde) \overset{I_{4,1}}{=} 0$$

GA$(5,1)$-identity for $I_{3,2}$
- Can refine the identity so that there is no duplication of terms
Theorem

1. If we decopmose the group into conjugacy classes, one can check (conceptual reason?) that the 10 terms from the ccls in the first row already give the identity. (And so do the 10 terms in the second row)