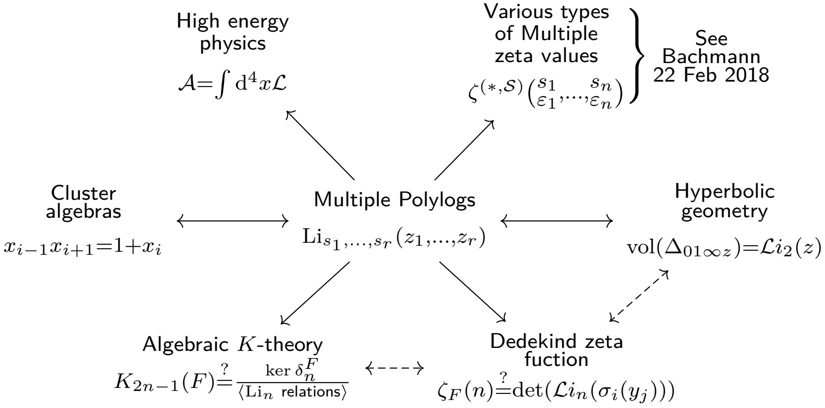
Various aspects of (multiple) polylogs

Steven Charlton

15 March 2018
New guests at the MPIM


High energy physics
$\mathcal{A}=\int \mathrm{d}^4 x \mathcal{L}$
Various types of Multiple zeta values
$\zeta^{(*,\mathcal{S})}\binom{s_1,\ldots,s_n}{\varepsilon_1,\ldots,\varepsilon_n}$
See Bachmann 22 Feb 2018
Cluster algebras
$x_{i-1}x_{i+1}=1+x_i$
Multiple Polylogs
$\mathrm{Li}_{s_1,\ldots,s_r}(z_1,\ldots,z_r)$
Hyperbolic geometry
$\mathrm{vol}(\Delta_{01\infty z})=\mathcal{L}i_2(z)$
Algebraic K-theory
$K_{2n-1}(F)\overset{?}{=}\frac{\ker \delta_n^F}{\langle \mathrm{Li}_n \text{ relations}\rangle}$
Dedekind zeta fuction
$\zeta_F(n)\overset{?}{=}\det(\mathcal{L}i_n(\sigma_i(y_j)))$

Definitions

Definition (Polylogarithm)

For $k \in \mathbb{Z}_{>0}$

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}\,, \quad |z| < 1$$

Analytically continue via

$$\mathrm{Li}_k(z) = \int_0^z \mathrm{Li}_{k-1}(t) \frac{\mathrm{d}t}{t}\,, \quad z \in \mathbb{C} \setminus [1, \infty)$$

Example

$$\mathrm{Li}_1(x) = -\log(1-x)$$

Know $\log(x) + \log(y) = \log(xy)$

$$\rightsquigarrow \mathrm{Li}_1(1-x) + \mathrm{Li}_1(1-y) = \mathrm{Li}_1(1-xy)$$

Does this generalise?

Li_2 Functional equations

Example

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) + \log(z)\log(1-z) = \frac{\pi^2}{6}$$

- Check by differentiation

Theorem (Spence, Abel, . . . - Five term relation)

$$\mathrm{Li}_2\left(\tfrac{x(1-y)}{y(1-x)}\right) - \mathrm{Li}_2\left(\tfrac{y}{1-x}\right) - \mathrm{Li}_2\left(\tfrac{x}{1-y}\right) + \mathrm{Li}_2(x) + \mathrm{Li}_2(y) = -\log(1-x)\log(1-y)$$

- Expected to be the basic/'generic' functional equation for Li_2

Higher function equations

Example ('Trivial' fuctional equations)

$$\mathrm{Li}_k(z^2) = 2^{k-1}(\mathrm{Li}_k(-z) + \mathrm{Li}_k(z))$$
$$\mathrm{Li}_k(z^{-1}) = (-1)^{k-1}\,\mathrm{Li}_k(z) + \text{elementary}$$

Non-trivial:

- $k = 3$: Spence 1809, Kummer 1840,
  Generic: Goncharov 1990 $\rightsquigarrow$ 22-terms
- $k = 4, 5$: Kummer 1840, Wechsung 1965, Lewin 1986,
- $k = 4$: Generic$^?$: Gangl 2012-2017 $\rightsquigarrow$ 931-terms
- $k = 6, 7$: Gangl 1990-91
- $k \geq 8$: Unknown

Goal

Understand Li_k functional equations.

Dedekind zeta functions

Definition (Dedekind zeta function)

For $F/\mathbb{Q}$ number field

$$\zeta_F(s) \coloneqq \sum_{I \neq (0) \subset \mathcal{O}_F} \frac{1}{N_{F/\mathbb{Q}}(I)^s}, \quad \mathrm{Re}(s) > 1$$

Theorem (Class number formula)

$$\mathrm{Res}_{s=1}\, \zeta_F(s) = \frac{2^{r_1} (2\pi)^{r_2} h_F}{w_F \sqrt{|\Delta_F|}} \, \mathrm{Reg}_F$$

$\mathrm{Reg}_F = (r_1 + r_2 - 1)$-determinant of $\log$'s of units

- Generalisation with polylogs?

$\zeta_F(2)$ and hyperbolic geometry

$F = \mathbb{Q}(\sqrt{-a})$

- Humbert: $\mathrm{vol}(\mathfrak{H}_3 / \mathrm{SL}_2(\mathcal{O}_F)) = \zeta_F(2) \frac{|d|^{3/2}}{4\pi^2}$
- Ideal tetrahedra: $\mathrm{vol}(\Delta_{01\infty z}) = \mathcal{L}i_2(z)$
  $\mathcal{L}i_2(z) := \mathrm{Im}(\mathrm{Li}_2(z) + \log(1-z) \log|z|)$ single-valued dilog
- Get:

$$\zeta_2(F) = \frac{\pi^2}{3|d|^{3/2}} \sum_{\nu} n_\nu \mathcal{L}i_2(z_\nu) \,, \quad n_\nu \in \mathbb{Z}, z_\nu \in F$$

Example

$$\zeta_{\mathbb{Q}(\sqrt{-7})}(2) = \frac{4\pi^2}{21\sqrt{7}} \left(2\mathcal{L}i_2(\tfrac{1+\sqrt{-7}}{2}) + \mathcal{L}i_2(\tfrac{-1+\sqrt{-7}}{2}) \right)$$

Zagier: generalisation to all F, and conjecturally to higher $\zeta_F(n)$.

Zagier's polylogarithm conjecture

Conjecture (Zagier, Rough version)

$F/\mathbb{Q}$ number field, with complex embeddings $\sigma_i = \overline{\sigma_{i+r_1+r_2}}$, and real embeddings σ_{i+r_2}. Set

$$d_n = \begin{cases} r_1 + r_2 & n \text{ odd} \\ r_2 & n \text{ even} \end{cases}$$

Then

$$\zeta_F(n) = \text{rational} \cdot \frac{\pi^{(r_1+2r_2-d_n)n}}{|\Delta_F|} \det(\mathcal{L}i_n(\sigma_i(y_j)))_{1 \le i,j \le d_n}$$

for some $y_j \in \mathbb{Z}[F \setminus \{0,1\}]$.

- Single-valued

$$\mathcal{L}i_k(z) = \begin{cases} \mathrm{Re} & k \text{ odd} \\ \mathrm{Im} & k \text{ even} \end{cases} \sum_{j=0}^{k} \frac{2^j B_j}{j!} \log^j|z| \, \mathrm{Li}_{m-j}(z)$$

Progress

- $n = 2$ (Weak version) Zagier
  - $n = 2$ Bloch/Suslin
- $n = 3$ Goncharov
  - Via generic 22-term Li_3 functional equation
  - Defines map in a polylogarithm complex
- $n = 4$ Goncharov/Rudenko
  - Announced May 2017
  - Input from Gangl's 122-term multiple polylog relation
  - ($\rightsquigarrow$ Gangl's 931-term Li_4 functional equation)

Multiple polylogarithms

Definition (Multiple polylogarithm)

For $k_1, \dots, k_r \in \mathbb{Z}_{>0}$

$$\mathrm{Li}_{k_1,\dots,k_r}(z_1,\dots,z_r) = \sum_{0<n_1<\cdots<n_r} \frac{z_1^{n_1}\cdots z_r^{n_r}}{n_1^{k_1}\cdots n_r^{k_r}}, \quad |z_i|<1$$

Change of variables $\rightsquigarrow I_{k_1,\dots,k_r}(z'_1,\dots,z'_r)$ iterated integral.

- Goncharov expects arguments f_i:

$$I_{3,1}(\mathrm{Li}_2 \text{ five-term}, z) = \sum_i \alpha_i \, \mathrm{Li}_4(f_i)$$

Expression found by Gangl $\rightsquigarrow$ input for Zagier $n=4$.

MPL's in weight 5

- Goncharov also expects:

$$I^+_{4,1}(z, \mathrm{Li}_2 \text{ five-term}) = \sum\nolimits_i \beta_i \, \mathrm{Li}_5(g_i)$$
$$I^+_{4,1}(\mathrm{Li}_3 \text{ 22-term}, z) = \sum\nolimits_i \gamma_i \, \mathrm{Li}_5(h_i)$$

- Much more difficult to find; new phenomena appear

Theorem (C, 2016 – Schematic)

$$I^+_{4,1}(\textit{one-variable } \mathrm{Li}_3 \textit{ family}, y) = \sum \mathrm{Li}_5\textit{'s}$$
$$I^+_{4,1}(x, \textit{one-variable } \mathrm{Li}_2 \textit{ family}) = \sum \mathrm{Li}_5\textit{'s}$$
$$I^+_{4,1}([x] + [\tfrac{1}{1-x}] + [1 - \tfrac{1}{x}], y) = \textit{Nielsen} + \sum \mathrm{Li}_5\textit{'s}$$

Work in progress to understand weight 5 (with Gangl/Radchenko/Rudenko).