Cyclic insertion on MZV's and the alternating block decomposition

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Outline

1 Introduction to MZV's

2 Algebraic structure of MZV's

3 Cyclic insertion conjecture

4 Tools from motivic MZV's

5 The alternating block decomposition

Introduction to MZV's

Mutiple zeta values

Definition (MZV)

Multiple zeta value $\zeta(s_1, s_2, \ldots, s_k)$ is defined by

$$\zeta(s_1, s_2, \ldots, s_k) \coloneqq \sum_{0<n_1<n_2<\cdots<n_k} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}$$

"*Interesting* multi-variable version of $\zeta(s)$"

- Where $s_i \geq 1 \in \mathbb{Z}$
- For convergence $s_k \geq 2$

Also define

- Weight is sum $s_1 + \cdots + s_k$ of arguments
- Depth is number k of arguments

Reasons for interest

- Arise naturally in physics calculations
- Connection to modular forms
- Periods of $\mathfrak{M}_{0,n}$ are in $\mathbb{Q}[\frac{1}{2\pi i}, \text{MZV's}]$

Reasons for interest

More concretely:

- Have a surprising amount of structure
  - At weight 8, expect $2^{8-2} = 64$ MZV's
  - At most 4 $\mathbb{Q}$-linearly independent ones, e.g. $\{\zeta(8), \zeta(3,5), \zeta(5,3), \zeta(3,3,2)\}$
  - Implies lots of linear relations $\zeta(2,1,1,4) = -\frac{51}{40}\zeta(3,5) - \frac{3}{4}\zeta(5,3) + \frac{1}{4}\zeta(3,3,2) + \frac{539}{2880}\zeta(8)$
- Leads to difficult open questions
  - Euler $\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(2k) \in \pi^{2k}\mathbb{Q}$
  - Apéry (1970): $\zeta(3) \notin \mathbb{Q}$. What about $\zeta(5) \overset{?}{\in} \mathbb{Q}$?
  - Linear independence $\zeta(3)/\pi^3 \overset{?}{\in} \mathbb{Q}$?

Goal: Understand all $\mathbb{Q}$-linear relations

MZV Relations

- $\zeta(3) = \zeta(1,2)$
- $\zeta(\{1,3\}^n) = \frac{1}{2n+1}\frac{\pi^{4n}}{(4n+1)!} = \frac{1}{2n+1}\zeta(\overbrace{\{2\}^{2n}}^{\text{Repeat } 2,2,\ldots,2 \text{ total of } 2n \text{ times}})$
- $28\zeta(3,9) + 150\zeta(5,7) + 168\zeta(7,5) = \frac{5197}{691}\zeta(12)$
- $\zeta(\{3\}^n, 4) = \zeta(1,3,\{3\}^n) + \zeta(2,\{3\}^n,2)$

Conjecture (Weight grading)

Any $\mathbb{Q}$-linear relation between MZV's is weight graded.
"There are no relations between MZV's of different weights."

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Algebraic structure of MZV's

Integral representation; shuffle product

Definition (Iterated integral)

$$I(a_0; a_1, \dots, a_N; a_{N+1}) \coloneqq \int_{\substack{a_0<t_1<t_2<\cdots \\ <t_N<a_{N+1}}} \frac{\mathrm{d}t_1}{t_1 - a_1} \wedge \cdots \wedge \frac{\mathrm{d}t_N}{t_N - a_N}$$

- Multiplication of iterated integrals gives shuffle product
  - Arrange $a_0 < t_i < a_{N+1}$ and $a_0 < s_j < a_{N+1}$ in all compatible ways $t_i < s_j$ or $t_i > s_j$.
  - $I(a; w_1; b) I(a; w_2; b) = I(a; w_1 \sqcup\!\sqcup\, w_2; b)$ where

$$(x w_1) \sqcup\!\sqcup\, (y w_2) \coloneqq x(w_1 \sqcup\!\sqcup\, y w_2) + y(x w_1 \sqcup\!\sqcup\, w_2)$$

Proposition (MZV as iterated integral, Kontsevich)

$$\zeta(s_1, \dots, s_k) = (-1)^k I(0; 1, \{0\}^{s_1-1}, \dots, 1, \{0\}^{s_k-1}; 1)$$

Properties of iterated integrals

Properties

- $I(a;b) = 1$ *for any* a, b *(Unit)*
- $I(0; a_1, \ldots, a_N; 0) = 0$ *for* $N \geq 1$ *(Equal boundaries)*
- $I(a_0; a_1, \ldots, a_N; a_{N+1}) = I(1-a_0; 1-a_1 \ldots, 1-a_N; 1-a_{N+1})$ *(Functoriality)*
- $I(a_0; a_1, \ldots, a_N; a_{N+1}) = (-1)^N I(a_{N+1}; a_N, \ldots, a_1; a_0)$ *(Reversal of paths)*

Corollary (MZV duality)

$$I(0; a_1, \ldots, a_N; 1) = (-1)^N I(0; 1-a_N, \ldots, 1-a_1; 1)$$
$$\rightsquigarrow \quad \underbrace{\zeta(2,1,5)}_{-I(0;10110000;1)} = \underbrace{\zeta(1,1,1,3,2)}_{-I(0;11110010;1)}$$

Series representation; stuffle product

- Multiply series gives stuffle product $*$
  - Arrange n_i, and m_j in all compatible ways $n_i < m_j$, or $n_i = m_j$ or $n_i > m_j$.
- Simplest case $\zeta(s) * \zeta(t) = \zeta(s,t) + \zeta(t,s) + \zeta(s+t)$.

Example (Comparing $ш$ and $*$)

$$2\zeta(2,2) + 4\zeta(1,3) \overset{ш}{=} \zeta(2)\zeta(2) \overset{*}{=} 2\zeta(2,2) + \zeta(4)$$
$$\implies \zeta(1,3) = \frac{1}{4}\zeta(4) = \frac{1}{3}\frac{\pi^4}{5!}$$

Conjecture (Extended double shuffle)

All $\mathbb{Q}$-linear relations on MZV's arise by comparing $ш - *$.
(Must allow divergent $\zeta(1)$; formally cancels using regularisation.)

Cyclic insertion conjecture

Zagier-Broadhurst Identity

Theorem (Zagier-Broadhurst, BBBL 2001)

For $n \geq 0 \in \mathbb{Z}$, *have*

$$\zeta(\{1,3\}^n) = \frac{1}{2n+1} \frac{\pi^{4n}}{(4n+1)!}$$

Proof (Sketch).

- Generalise to single variable *multiple polylogarithms*.
- Generating series satisfies a differential equation.
- Explicit solution in terms of ${}_2F_1$. Compare coefficients.

Combinatorial proofs have also been given. □

"Dressed with 2's"

Theorem (BBBL, 1998)

Let $n \geq 0 \in \mathbb{Z}$*, write*

$$I = \{ \text{ all } 2n+1 \text{ ways of inserting 2 into } \{1,3\}^n \ \} .$$

Then

$$\sum_{\mathbf{s} \in I} \zeta(\mathbf{s}) = \frac{\pi^{4n+2}}{(4n+3)!}$$

Example

For $n = 2$, have

$$\zeta(2,1,3,1,3) + \zeta(1,2,3,1,3) + \zeta(1,3,2,1,3) + \zeta(1,3,1,2,3) + \zeta(1,3,1,3,2) = \frac{\pi^{10}}{11!}$$

Cyclic insertion conjecture

Numerical experimentation lead to conjectural generalisation.

Notation

Let $a_1, \ldots, a_{2n+1} \in \mathbb{Z}_{\geq 0}$. Write

$$Z(a_1, \ldots, a_{2n+1}) = \zeta(\{2\}^{a_1}, 1, \{2\}^{a_2}, 3, \ldots, 1, \{2\}^{a_{2n}}, 3, \{2\}^{a_{2n+1}})$$

Conjecture (Cyclic insertion - BBBL, 1998)

$$\sum_{\sigma \in \mathbb{Z}/n\mathbb{Z}} Z(a_{\sigma(1)}, \ldots, a_{\sigma(2n+1)}) \stackrel{?}{=} \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$$

Shorthand: "wt" is weight of MZV's on the LHS

Bowman-Bradley

Best result so far is

Theorem (Bowman-Bradley, 2002)

Let $n, t \geq 0 \in \mathbb{Z}$, *then*

$$\sum_{\substack{a_1+\cdots+a_{2n+1}=t \\ a_i \geq 0}} Z(a_1, \ldots, a_{2n+1}) = \frac{1}{2n+1} \binom{t+2n}{t} \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$$

Remark

Compatible with cyclic insertion: Any permutation of a composition $a_1 + \cdots + a_{2n+1} = t$ is still a composition.

Will use the motivic MZV framework to improve on this, up to $\mathbb{Q}$.

Hoffman's conjecture

Separate conjecture, with a similar flavour

Conjecture (Hoffman, MZV Infopage, 2000)

For $m \geq 0 \in \mathbb{Z}$*,*

$$2\,\zeta(3,3,\{2\}^m) - \zeta(3,\{2\}^m,1,2) \stackrel{?}{=} -\zeta(\{2\}^{m+3}) = -\frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$$

Remark

Verified up to weight 22, $m = 8$ using MZV datamine, Vermaseren (2009).

Will show this up to $\mathbb{Q}$, using the motivic framework

Goal: connect these two conjectures, and work towards proofs.

Tools from motivic MZV's

Brown's motivic MZV's

Solve transcendence problems with algebraic version of MZV's:

- Graded algebra $\mathcal{H}_\bullet$ of motivic MZV's

$$\zeta^{\mathfrak{m}}(s_1,\ldots,s_r) := [\mathcal{O}(\pi_1^{\mathrm{un}}(\mathbb{P}^1\setminus\{0,1,\infty\},\vec{1}_0,-\vec{1}_1)),\overbrace{\mathrm{dch}}^{\text{straight line}},\underbrace{\Omega}_{\text{integrand}}]^{\mathfrak{m}}.$$

  Contains all motivic iterated integrals

$$I^{\mathfrak{m}}(a_0;a_1,\ldots,a_N;a_{N+1}), a_i\in\{0,1\}$$

- Projection to algebra $\mathcal{A}_\bullet$ of de Rham motivic MZV's

$$\zeta^{\mathfrak{a}}(s_1,\ldots,s_r) := [\mathcal{O}(\pi_1^{\mathrm{un}}(\mathbb{P}^1\setminus\{0,1,\infty\},\vec{1}_0,-\vec{1}_1)),\underbrace{\varepsilon}_{\text{augmentation ideal}},\Omega]^{\mathfrak{m}},$$

  kernel generated by $\zeta^{\mathfrak{m}}(2)$.

- Coaction

$$\Delta\colon\mathcal{H}\to\mathcal{A}\otimes_{\mathbb{Q}}\mathcal{H}$$

  lifts Goncharov's 'semicircular' coproduct on $\mathcal{A}$. $\mathcal{H}$ Hopf algebra comodule over $\mathcal{A}$.

Results from motivic MZV's

- $\zeta^{\mathfrak{a}}(2k+1)$ are linearly independent
  - $\zeta^{\mathfrak{a}}(2k+1) \neq 0 \in \mathcal{A}_{2k+1}(\mathbb{Q})$
  - So have different gradings
- $\zeta^{\mathfrak{a}}(2k+1)$ are *algebraically* independent
  - Suppose some $\zeta^{\mathfrak{a}}(2k+1)$ satisfy a polynomial
  - Use coproduct Δ to show all coefficients are 0
- $\zeta^{\mathfrak{a}}(3,5)$ is irreducible (i.e. not in $\mathbb{Q}[\zeta(n)]$)
  - $(\Delta - \Delta^{\mathrm{op}})\zeta^{\mathfrak{a}}(3,5) = -5\zeta^{\mathfrak{a}}(3) \wedge \zeta^{\mathfrak{a}}(5)$
  - $(\Delta - \Delta^{\mathrm{op}})\zeta^{\mathfrak{a}}(n_1)\cdots\zeta^{\mathfrak{a}}(n_k) = 0$

Infinitesimal coproduct

Definition (Derivations D_k)

Let $\mathcal{L} := \mathcal{A}/(\mathcal{A}_{>0} \cdot \mathcal{A}_{>0})$, which kills products and $\zeta^{\mathfrak{m}}(2)$. For k odd define

$$\begin{aligned} D_k : \quad & \mathcal{H} \to \mathcal{L}_k \otimes_{\mathbb{Q}} \mathcal{H} \\ & I^{\mathfrak{m}}(w) \mapsto (\pi \otimes \mathrm{id}) \circ (\Delta - 1 \otimes \mathrm{id})\, I^{\mathfrak{m}}(w) \end{aligned}$$

$$D_k I^{\mathfrak{m}}(a_0; a_1, \dots, a_N; a_{N+1}) =$$

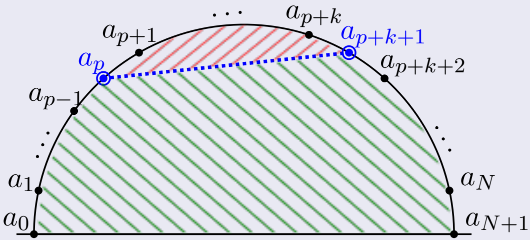
$$\sum_{p=0}^{N-k} I^{\mathfrak{L}}(a_p; a_{p+1}, \dots, a_{p+k}; a_{p+k+1}) \otimes \qquad \rightsquigarrow \text{ Subsequence}$$

$$I^{\mathfrak{m}}(a_0; a_1, \dots, a_p, a_{p+k+1}, \dots, a_N; a_N + 1) \qquad \rightsquigarrow \text{ Quoient sequence}$$

Derivations D_k mnemonic

Mnemonic.

$$D_k I^{\mathfrak{m}}(\underbrace{w}_{(a;w';b)}) = \sum_{\substack{S \text{ subword } w, \\ \text{of length } k+2}} I^{\mathfrak{L}}(S) \otimes I^{\mathfrak{m}}(w - \text{interior } S)$$



$$\rightsquigarrow \quad I^{\mathfrak{L}}(a_p; a_{p+1}, \dots, a_{p+k}; a_{p+k+1}) \otimes$$
$$I^{\mathfrak{m}}(a_0; a_1, \dots, a_p, a_{p+k+1}, \dots, a_N; a_N + 1)$$

Transcendental Galois Theory

Theorem (Brown, 2012)

Let $D_{<N} = \bigoplus_{1<2r+1<N} D_{2r+1}$. *In weight* N,

$$\ker D_{<N} = \zeta^{\mathfrak{m}}(N)\mathbb{Q}\,.$$

$\rightsquigarrow$ *'exact-numerical' algorithm for decomposing motivic MZV's*

Example

Can show $\zeta^{\mathfrak{m}}(\{2\}^n) = \pm I^{\mathfrak{m}}(0; \underbrace{1,0,1,0,\ldots,1,0}_{n \text{ times}}; 1) \in \zeta^{\mathfrak{m}}(2n)\mathbb{Q}$

- Integral word alternates 0 and 1
- Odd length subsequence has same boundaries, vanishes
- Therefore all D_{2r+1} vanish

Conclude $\zeta^{\mathfrak{m}}(\{2\}^n) \in \ker D_{<2n} = \zeta^{\mathfrak{m}}(2n)\mathbb{Q}$.

$\zeta^{\mathfrak{m}}(\{1,3\}^n)$

More interesting: $\zeta^{\mathfrak{m}}(\{1,3\}^n) = I^{\mathfrak{m}}(0;(1100)^n;1) \in \zeta^{\mathfrak{m}}(4n)\mathbb{Q}$

- Word has period 4, so length $1 \pmod 4$ subsequence vanish
- For length $3 \pmod 4$, look at starting position

$$1 \pmod 4: \qquad I^{\mathfrak{L}}(0;(1100)^a 1;1) \otimes I^{\mathfrak{m}}((0110)^b 0 \mid 10(0110)^c 01)$$
$$2 \pmod 4: \qquad I^{\mathfrak{L}}(1;1(0011)^a;0) \otimes I^{\mathfrak{m}}((0110)^b 01 \mid 0(0110)^c 01)$$

- Cancel using reversal of paths in $I^{\mathfrak{L}}$. Similar for position $3, 4 \pmod 4$
- See cancellation as 'reversing' segments. Involution pairs up subsequences:

$$I^{\mathfrak{m}}(01 \mid 10 \mid 0\,1 \mid 10 \mid \cdots \mid 10 \mid 01 \mid 1\,0 \mid 01)$$

Conclude $\zeta^{\mathfrak{m}}(\{1,3\}^n) \in \ker D_{<4n} = \zeta^{\mathfrak{m}}(4n)\mathbb{Q}$

The alternating block decomposition

Alternating blocks

Observation

In $\zeta^{\mathfrak{m}}(\{1,3\}^n)$ proof, points 00 and 11 in w are 'somehow' significant.

- Splitting here decomposes a word into *alternating blocks* $0101\cdots$ or $1010\cdots$.

Definition (Block decomposition)

Let w be a word starting with $\varepsilon_1 \in \{0,1\}$. Write w as alternating blocks, with lengths $\ell_1, \ldots, \ell_k$. The block decomposition of w is

$$\mathrm{bl}(w) = (\varepsilon_1; \ell_1, \ldots, \ell_k)\,.$$

Example

$$\mathrm{bl}(\underbrace{0}_{1} \mid \underbrace{01}_{2} \mid \underbrace{10}_{2} \mid \underbrace{01010}_{5} \mid \underbrace{0}_{1} \mid \underbrace{01}_{2}) = (0; 1, 2, 2, 5, 1, 2)$$

Alternating blocks

Can recover w from $(\varepsilon_1; \ell_1, \ldots, \ell_k)$: blocks arise from $00 \to 0 \mid 0$ or $11 \to 1 \mid 1$.

Notation

Write $I_{\mathrm{bl}}(\varepsilon_1; \ell_1, \ldots, \ell_k) = I(\mathrm{bl}^{-1}(\varepsilon_1; \ell_1, \ldots, \ell_k))$. If $\varepsilon_1 = 0$, just write $(\ell_1, \ldots, \ell_k)$.

- Weight of $I_{\mathrm{bl}}(\varepsilon_1; \ell_1, \ldots, \ell_k)$ is $-2 + \sum_i \ell_i$. (Bounds of integration are counted!)
- If $\mathrm{wt} \equiv k \pmod 2$ then $I_{\mathrm{bl}} = 0$. (End points are equal!)
- I_{bl} is divergent iff $\ell_1 = 1$ or $\ell_k = 1$.

Example

$$I_{\mathrm{bl}}(1, 2, 2, 5, 1, 2) = I(0; 01100101000; 1)$$

Block structure of BBBL conjecture

- Write the BBBL identity as iterated integrals

$$\sum_{\text{cycle } a_i} \zeta(\{2\}^{a_1}, 1, \{2\}^{a_2}, 3, \ldots, 1, \{2\}^{a_{2n}}, 3, \{2\}^{a_{2n+1}})$$

$$\rightsquigarrow \pm \sum_{\text{cycle } a_i} I(0(10)^{a_1} 1(10)^{a_2} 100 \cdots 01(10)^{a_{2n}} 100(10)^{a_{2n+1}} 1)$$

- Split into 'alternating blocks' at $00 \to 0 \mid 0$ or $11 \to 1 \mid 1$

$$= \pm \sum_{\text{cycle } a_i} I(0(10)^{a_1} 1 \mid (10)^{a_2} 10 \mid 0 \cdots 01 \mid (10)^{a_{2n}} 10 \mid 0(10)^{a_{2n+1}} 1)$$

- Record lengths of the blocks

$$= \pm \sum_{\text{cycle } a_i} I_{\text{bl}}(2a_1 + 2, 2a_2 + 2, \ldots, 2a_{2n+1} + 2)$$

- Right hand side is $\zeta(\{2\}^{\text{wt}/2}) = \pm I_{\text{bl}}(\text{wt} + 2)$.

Block structure of Hoffman's conjecture

- Write Hoffman's identity as iterated integrals

$$\begin{aligned} & 2\zeta(3,3,\{2\}^n) \quad - \zeta(3,\{2\}^n,1,2) \\ = {} & \zeta(3,3,\{2\}^n) \quad - \zeta(3,\{2\}^n,1,2) \quad + \zeta(\{2\}^n,1,2,1,2) \\ \rightsquigarrow {} & \pm \big(I(0100100(10)^n 1) + I(0100(10)^n 1101) + I(0(10)^n 1101101) \big) \end{aligned}$$

- Split into 'alternating blocks' at $00 \to 0 \mid 0$ or $11 \to 1 \mid 1$

$$\begin{aligned} = \pm \big(& I(010 \mid 010 \mid 0(10)^n 1) + I(010 \mid 0(10)^n 1 \mid 101) \\ & + I(0(10)^n 1 \mid 101 \mid 101) \big) \end{aligned}$$

- Record lengths of the blocks

$$= \pm \big(I_{\mathrm{bl}}(3,3,2n+2) + I_{\mathrm{bl}}(3,2n+2,3) + I_{\mathrm{bl}}(2n+2,3,3) \big)$$

- Right hand side is $-\zeta(\{2\}^{n+3}) = \pm I_{\mathrm{bl}}(\mathrm{wt}+2)$

Common structure and generalisation

Both conjectures have same structure: cyclic permutations of block lengths ℓ_i.

Conjecture (Cyclic insertion, C., 2017, arXiv 1703.03784)

For any $(\ell_1, \ldots, \ell_k)$ *with all* $\ell_i > 1$,

$$\sum_{\text{cycle } \ell_i} I_{\mathrm{bl}}(\ell_1, \ldots, \ell_k) \overset{?}{=} I_{\mathrm{bl}}(\mathrm{wt} + 2) = \begin{cases} \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!} & \mathrm{wt} \textit{ even} \\ 0 & \mathrm{wt} \textit{ odd} \end{cases}$$

- Numerically tested all cases weight ≤ 18, to 500 decimal places
- Can prove a symmetrised version, up to $\mathbb{Q}$
- Can prove *some* special cases, up to $\mathbb{Q}$

Examples

Example

Let $(\ell_1, \ldots, \ell_k) = (2m+2, 2, 3, 2, 3)$, then we obtain

$$\zeta(\{2\}^m, 1, 3, 3, 1, 2) + \zeta(3, 1, 2, 1, \{2\}^m, 3) - \zeta(1, 2, 1, \{2\}^m, 3, 1, 2) + \\ + \zeta(1, 2, 1, 3, 3, \{2\}^m) - \zeta(3, \{2\}^m, 1, 3, 3) \overset{?}{=} \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$$

Proposition (C., 2017, arXiv 1703.03784)

The above identity holds up to $\mathbb{Q}$

Proof (Sketch).

Lift the identity to $\zeta^{\mathfrak{m}}$, and compute $D_{<2m+10}$. A (tedious) calculation shows $D_{<2m+10}$ vanishes. □

Progress and results

Theorem (Symmetric insertion, C., 2017, arXiv 1703.03784)

For any $(\ell_1, \ldots, \ell_k)$*, with even weight,*

$$\sum_{\text{permute } \ell_i} I_{\mathrm{bl}}(\ell_1, \ldots, \ell_k) \in I_{\mathrm{bl}}(\mathrm{wt} + 2)\mathbb{Q}$$

(Odd weight holds trivially, by duality)

Proof (Strategy).

- Lift to motivic version $I^{\mathfrak{m}}$.
- Define a reflection $\mathcal{R}$ on non-trivial subsequences
- Use $\mathcal{R}$ to cancel terms in $D_{<N}$
- Conclude $\in \zeta^{\mathfrak{m}}(\mathrm{wt})\mathbb{Q} = I^{\mathfrak{m}}_{\mathrm{bl}}(\mathrm{wt}+2)\mathbb{Q}$ using Brown.

Progress and results

Proof (Details).

$$\mathcal{R}\colon I^{\mathfrak{m}}_{\mathrm{bl}}(\ell_1, \ldots, \overbrace{\underbrace{\ell_s}_{\text{start at position } \alpha}, \ldots, \underbrace{\ell_t}_{\text{end at position } \beta}}^{\text{non-trivial subsequence } S}, \ldots, \ell_k)$$

$$\mapsto I^{\mathfrak{m}}_{\mathrm{bl}}(\ell_1, \ldots, \overbrace{\underbrace{\ell_t}_{\text{start at position } \beta}, \ldots, \underbrace{\ell_s}_{\text{end at position } \alpha}}^{\text{reflection } \mathcal{R}S}, \ldots, \ell_k)$$

- Get permutation of ℓ_i.
- Both quotients are $I^{\mathfrak{L}}_{\mathrm{bl}}(\ell_1, \ldots, \ell_{s-1}, \alpha + \beta, \ell_{t+1}, \ldots, \ell_k)$
- Subsequences are
  $$I^{\mathfrak{m}}_{\mathrm{bl}}(\varepsilon; \ell_s - \alpha, \ell_{s+1}, \ldots, \ell_{t-1}, \ell_t - \beta) \text{ , and}$$
  $$I^{\mathfrak{m}}_{\mathrm{bl}}(\varepsilon'; \ell_t - \beta, \ell_{t-1}, \ldots, \ell_{s+1}, \ell_s - \alpha)$$
- Reverses or duals, differ by $(-1)^{\text{length}} = -1$. Cancel in $D_{<N}$ □

Corollaries of symmetric insertion

Corollary (Generalisation of Hoffman, up to $\mathbb{Q}$)

For $(\ell_1, \ell_2, \ell_3) = (2a+3, 2b+3, 2c+2)$, *we obtain*

$$\mathrm{Sym}_{a,b}\left(\zeta(\{2\}^a, 3, \{2\}^b, 3, \{2\}^c) - \zeta(\{2\}^b, 3, \{2\}^c, 1, 2, \{2\}^a) + \zeta(\{2\}^c, 1, 2, \{2\}^a, 1, 2, \{2\}^b)\right) \in \pi^{\mathrm{wt}}\mathbb{Q}$$

Duality shows cyclic insertion already holds up to $\mathbb{Q}$

$$\zeta(\{2\}^a, 3, \{2\}^b, 3, \{2\}^c) - \zeta(\{2\}^b, 3, \{2\}^c, 1, 2, \{2\}^a) + \zeta(\{2\}^c, 1, 2, \{2\}^a, 1, 2, \{2\}^b)) \in \pi^{\mathrm{wt}}\mathbb{Q}$$

In particular, $a = b = 0$ is Hoffman's identity up to $\mathbb{Q}$.

Corollaries of symmetric insertion

Corollary (Improvement of Bowman-Bradley, up to $\mathbb{Q}$)

For $\ell_i = 2a_i + 2$, obtain

$$\sum_{\text{permute } a_i} \zeta(\{2\}^{a_1}, 1, \{2\}^{a_2}, 3, \ldots, 1, \{2\}^{a_{2n}}, 3, \{2\}^{a_{2n+1}}) \in \pi^{\mathrm{wt}}\mathbb{Q}$$

"Only need permutations of a single composition."

In particular, for $a_1 = \cdots = a_n = m$

Corollary (Evaluable MZV)

The following MZV is evaluable

$$\zeta(\{\{2\}^m, 1, \{2\}^m, 3\}^n, \{2\}^m) \in \pi^{\mathrm{wt}}\mathbb{Q}$$

Up to $\mathbb{Q}$, proves conjecture of Borwein-Bradley-Broadhurst, 1997

Further progress?

Complete motivic proof of cyclic insertion is not (yet?) possible

- Cyclic insertion has a stability under D_k
- Odd weight implies $D_{<N}$(even weight) $= 0$
- Problem: Must fix rational multiple of $\zeta^{\mathfrak{m}}(\mathrm{wt})$ somehow $\rightsquigarrow$ analytically or numerically...
- $D_{<N}$(odd weight) involves $I^{\mathfrak{L}}$ explicitly

$$D_7 \sum\nolimits_{\text{cycle}} I^{\mathfrak{m}}_{\mathrm{bl}}(2,10,3,2) =$$

$$\underbrace{(I^{\mathfrak{L}}_{\mathrm{bl}}(6,3) + I^{\mathfrak{L}}_{\mathrm{bl}}(3,3,2,1) + I^{\mathfrak{L}}_{\mathrm{bl}}(2,3,2,2) + I^{\mathfrak{L}}_{\mathrm{bl}}(1,2,2,4))}_{-\zeta^{\mathfrak{L}}(2)\zeta^{\mathfrak{L}}(2,3) - 2\zeta^{\mathfrak{L}}(2)\zeta^{\mathfrak{L}}(3,2) + 2\zeta^{\mathfrak{L}}(3)\zeta^{\mathfrak{L}}(2,2) = 0} \otimes I^{\mathfrak{m}}_{\mathrm{bl}}(10)$$

- In general only have

$$\text{odd weight} = \sum\nolimits_k \alpha_k \zeta(2k+1)\zeta(\{2\}^{\mathrm{wt}/2-k}), \quad \alpha_k \in \mathbb{Q}$$

Recent work

Using iterated integrals over $\mathbb{P}^1 \setminus \{ \infty, 0, 1, z \}$ gives

Theorem (Hirose-Sato, 2017, arXiv 1704.06478)

The generalisation of Hoffman's identity holds exactly

$$\zeta(\{2\}^a, 3, \{2\}^b, 3, \{2\}^c) - \zeta(\{2\}^b, 3, \{2\}^c, 1, 2, \{2\}^a) + \zeta(\{2\}^c, 1, 2, \{2\}^a, 1, 2, \{2\}^b) = -\zeta(\{2\}^{a+b+c+3})$$

Theorem (Hirose-Sato, 2017/18)

A 'block-shuffle' identity holds, which implies the conjecture.

See HIM talk, in "Periods and Regulators Workshop", at 15:00 on 19 January 2018. Video `https://youtu.be/b83fkeUAWu0`

Extra material

(if necessary)

Full version of cyclic insertion

If some $\ell_i = 1$, the identity involves product term corrections.

$$\mathcal{L}_d = \{ (m_{d+1}, \ldots, m_k) \mid (\overbrace{1, \ldots, 1}^{d \text{ times}}, m_{d+1}, \ldots, m_k) \text{ is a cyclic permutation of } (\ell_1, \ldots, \ell_k) \}$$

"Take all cyclic permutations of $(\ell_1, \ldots, \ell_k)$ which start with d consecutive 1's. Then drop the initial 1's"

Conjecture (Cyclic insertion, C., 2017, arXiv 1703.03784)

For any $(\ell_1, \ldots, \ell_k)$ of weight N,

$$\sum_{\text{cycle } \ell_i} I_{\mathrm{bl}}(\ell_1, \ldots, \ell_k) \overset{?}{=} I_{\mathrm{bl}}(N+2) - \sum_{d=1}^{\lfloor k/2 \rfloor} \frac{2(2\pi \mathrm{i})^{2d}}{(2d+2)!} \sum_{\mathbf{m} \in \mathcal{L}_{2d}} I_{\mathrm{bl}}(\mathbf{m}) \,.$$

Full version of cyclic insertion

Example

With $(\ell_i) = (1,1,2,3)$, need only $\mathcal{L}_2 = \{\,(2,3)\,\}$. Get

$$I_{\mathrm{bl}}(1,1,2,3) + I_{\mathrm{bl}}(1,2,3,1) + I_{\mathrm{bl}}(2,3,1,1) + I_{\mathrm{bl}}(3,1,1,2)$$
$$\overset{?}{=} I_{\mathrm{bl}}(7) - \frac{2(2\pi \mathrm{i})^2}{4!} I_{\mathrm{bl}}(2,3)$$

Shuffle regularisation gives

$$\big(3\zeta(1,1,3) + 2\zeta(1,2,2) + \zeta(2,1,2)\big) +$$
$$\big(\zeta(2,3) - 6\zeta(1,1,3) - 4\zeta(1,2,2) - 2\zeta(2,1,2)\big) +$$
$$\big(6\zeta(1,1,1,2)\big) + \big(-\zeta(5)\big) \overset{?}{=} 0 + \zeta(2)\zeta(1,2) \qquad \checkmark$$

Another block decomposition conjecture

Conjecture (BBBL 1998, rewritten)

Let $a_1, a_2, a_3, b_1, b_2 \in \mathbb{Z}_{\geq 0}$. *Then*

$$\sum_{\sigma \in S_3} \operatorname{sgn}(\sigma) \zeta(\{2\}^{a_{\sigma(1)}}, 1, \{2\}^{b_1}, 3, \{2\}^{a_{\sigma(2)}}, 1, \{2\}^{b_2}, 3, \{2\}^{a_{\sigma(3)}}) \stackrel{?}{=} 0$$

Generalising the block decomposition structure leads to

Conjecture (Alt-odd, C., 2017, arXiv 1703.03784)

For any $(\ell_1, \ldots, \ell_{2k+1})$ *of even weight, with all* $\ell_i > 1$,

$$\operatorname{Alt}_{\{ \ell_i \mid i \text{ odd } \}} I_{\mathrm{bl}}(\ell_1, \ldots, \ell_{2k+1}) \stackrel{?}{=} 0$$

"Alternating sum over odd-position blocks."

Remark

This conjecture is included in Hirose-Sato's generalisation too.

Another block decomposition conjecture

Example

For block lengths $\ell_i = 2a_i + 2$, $1 \leq i \leq 7$, get

$$\mathrm{Alt}_{a_1,a_3,a_5,a_7}\, \zeta(\{2\}^{a_1}, 1, \{2\}^{a_2}, 3, \{2\}^{a_3}, 1, \{2\}^{a_4}, 3, \{2\}^{a_5}, 1, \{2\}^{a_6}, 3, \{2\}^{a_7}) \overset{?}{=} 0$$

Example

For block lengths $(2a_1 + 3, 2a_2 + 3, 2a_3 + 3, 2a_4 + 2, 2a_5 + 3)$, get

$$\mathrm{Alt}_{a_1,a_3,a_5}\, \zeta(\{2\}^{a_1}, 3, \{2\}^{a_2}, 3, \{2\}^{a_3}, 3, \{2\}^{a_4}, 1, 2, \{2\}^{a_5}) \overset{?}{=} 0$$

Summary

- Defined block decomposition of an iterated integral
- Used block decomposition to unify/generalise BBBL and Hoffman's conjectures
- Used motivic MZV's to prove a symmetrised version holds
  - Improved Bowman-Bradley to only permutations
  - Proved Hoffman up to $\mathbb{Q}$,
  - Proved other identities up to $\mathbb{Q}$
- Recent work by Hirose-Sato proves the generalised conjecture