Cyclic insertion on MZV's and the alternating block decomposition

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24 April 2018
Zahlentheorie Seminar, Köln

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Motivic MZV's and the cyclic insertion conjecture

Abstract: I will start by recalling two conjectural families of MZV identities proposed by Borwein-Bradley-Broadhurst-Lisonek, and by Hoffman. I will show how both of these conjectures can be unified into a larger conjectural family of identities by using the so-called block decomposition of iterated integrals introduced here.

Using the motivic MZV framework of Brown I will show that a symmetrised version of this conjecture holds up to $\mathbb{Q}$. This will give a proof of Hoffman's identity, up to $\mathbb{Q}$ and an improvement of the Bowman-Bradley theorem giving some progress towards the BBBL conjecture.

Outline

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1 Introduction to MZV's
2 Algebraic structure of MZV's
3 Cyclic insertion conjecture
4 Tools from motivic MZV's
5 The alternating block decomposition

Introduction to MZV's

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Mutiple zeta values

Definition (MZV)

Multiple zeta value $\zeta(s_1, s_2, \ldots, s_k)$ is defined by

$$\zeta(s_1, s_2, \ldots, s_k) := \sum_{0<n_1<n_2<\cdots<n_k} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}$$

"*Interesting* multi-variable version of $\zeta(s)$"

- Where $s_i \geq 1 \in \mathbb{Z}$
- For convergence $s_k \geq 2$

Also define

- Weight is sum $s_1 + \cdots + s_k$ of arguments
- Depth is number k of arguments

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Depth is number k of arguments

My convention on MZV's means that $\zeta(1,2) = \zeta(3)$. This fits better with the motivic MZV framework used later. But be aware the some people use the summation covention $n_1 > n_2 > \cdots > n_k$, so (essentially) have the arguments reversed.

Reasons for interest

- Arise naturally in physics calculations
- Connection to modular forms
- Periods of $\mathfrak{M}_{0,n}$ are in $\mathbb{Q}[\frac{1}{2\pi i}, \text{MZV's}]$

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1. Perhaps not such a good reason for pure mathematicians, but there has been a lot of interaction between physics and pure mathematics recently, with questions and results from one side leading to insights on the other side. Part of the recent HIM trimester on periods focused on the connections to amplitudes in pysics.
2. More inerestingly, is a connection to other areas of number theory. Somehow MZV's know about modular forms. In particualr there are certain 'exceptional' relations, which can be derived from cusp forms. I'll point one out in a moment.
3. Taking integrals of differential forms on $\mathfrak{M}_{0.n}$ leads to a special case of objects call periods. One can show that every period from $\mathfrak{M}_{0.n}$ is a $\mathbb{Q}[(2\pi i)^{-1}]$-linear combination of MZV's

Reasons for interest

More concretely:

- Have a surprising amount of structure
  - At weight 8, expect $2^{8-2} = 64$ MZV's
  - At most 4 $\mathbb{Q}$-linearly independent ones, e.g. $\{\zeta(8), \zeta(3,5), \zeta(5,3), \zeta(3,3,2)\}$
  - Implies lots of linear relations $\zeta(2,1,1,4) = -\frac{51}{40}\zeta(3,5) - \frac{3}{4}\zeta(5,3) + \frac{1}{4}\zeta(3,3,2) + \frac{539}{2880}\zeta(8)$
- Leads to difficult open questions
  - Euler $\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(2k) \in \pi^{2k}\mathbb{Q}$
  - Apéry (1970): $\zeta(3) \notin \mathbb{Q}$. What about $\zeta(5) \overset{?}{\in} \mathbb{Q}$?
  - Linear independence $\zeta(3)/\pi^3 \overset{?}{\in} \mathbb{Q}$?

Goal: Understand all $\mathbb{Q}$-linear relations

1. If one lists the number of ways of writing 8 as a composition, with last entry ≥ 2, one finds $2^{8-2} = 64$ possibilities.
2. But numerical evaluation finds lots of $\mathbb{Q}$-linear dependencies. Find checking `lindep` in Pari, we get small coefficients, which are stable after increasing precision. Can then check algebraically. One finds 4 MZV's which have no simple $\mathbb{Q}$-linear dependence; the coefficients continue to grow/change with increasing precision, suggesting no linear dependence. But this is not a proof!
3. Euler gives formula for $\zeta(2k) = (-1)^{k-1} B_{2k} (2\pi)^{2k} / (2 \cdot (2k)!)$
4. It is known that at least one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational (Zudilin, 2001). But nothing explicitly is known.
5. The evaluation of $\zeta(2k) \in \pi^{2k}\mathbb{Q}$, might suggest that $\zeta(3)$ should be in $\pi^3\mathbb{Q}$, but this does not appear to be the case. It is expected that they are linearly independent, and even algebraically independent. But nothing is known so far.

MZV Relations

- $\zeta(3) = \zeta(1,2)$
- $\zeta(\{1,3\}^n) = \frac{1}{2n+1}\frac{\pi^{4n}}{(4n+1)!} = \frac{1}{2n+1}\zeta(\overbrace{\{2\}^{2n}}^{\text{Repeat } 2,2,\ldots,2 \text{ total of } 2n \text{ times}})$
- $28\zeta(3,9) + 150\zeta(5,7) + 168\zeta(7,5) = \frac{5197}{691}\zeta(12)$
- $\zeta(\{3\}^n, 4) = \zeta(1,3,\{3\}^n) + \zeta(2,\{3\}^n,2)$

Conjecture (Weight grading)

Any $\mathbb{Q}$-linear relation between MZV's is weight graded.
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We already know that MZV's should satisfy a lot of relations. What is perhaps not as clear, is that the relations themselves can be highly structured and pretty.

1. This is perhaps one of the first relations between MZV's after Euler first defined them. Euler hoped that all *double* zeta value could be reducd to values of Riemann zeta. This is an example of duality.
2. The left hand side of this was conjectured by Zagier on the basis of numerical evidence. The proof was givey by Broadhurst using generating series methods, and hypergeometric functions.
3. This one perhaps does't look as pretty, but it is a very important relation. Firstly it has a connection to modular forms. At weight 12, we obtain the first non-trivial cusp form for $\mathrm{SL}_2(\mathbb{Z})$, namely the discriminant Δ. This MZV relation is a consequence. (Gangl, Kaneko, Zagier) Secondly, it gives an exceptional relation between $\zeta(odd, odd)$, which ruins a conjectural candidate for a basis. This is the 'depth defect phenomenon'.

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1. Comes from the 'cyclic derivations of Ohno, applied to $w = (yx^2)^n$.
2. Did you notice anything curious about these 3 relations? Maybe it is a coincidence because of the small sample size, maybe not? But every relation has the same weight on the left hand side, and the right hand side. Generally this is conjectured to hold.
   Currently, this conjecture is impossible to resolve: it we knew that all MZV relations were weight graded, then we would know automatically that $\zeta(5)$ is irrational. It has weight 5, but a rational number has weight 0. So currently there is no hope. This is the transcendentality problem which plagues MZV!

Algebraic structure of MZV's

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Integral representation; shuffle product

Definition (Iterated integral)

$$I(a_0; a_1, \ldots, a_N; a_{N+1}) \coloneqq \int\limits_{\substack{a_0 < t_1 < t_2 < \cdots \\ < t_N < a_{N+1}}} \frac{\mathrm{d}t_1}{t_1 - a_1} \wedge \cdots \wedge \frac{\mathrm{d}t_N}{t_N - a_N}$$

- Multiplication of iterated integrals gives shuffle product
  - Arrange $a_0 < t_i < a_{N+1}$ and $a_0 < s_j < a_{N+1}$ in all compatible ways $t_i < s_j$ or $t_i > s_j$.
  - $I(a; w_1; b) I(a; w_2; b) = I(a; w_1 ⧢ w_2; b)$ where

$$(xw_1) ⧢ (yw_2) \coloneqq x(w_1 ⧢ yw_2) + y(xw_1 ⧢ w_2)$$

Proposition (MZV as iterated integral, Kontsevich)

$$\zeta(s_1, \ldots, s_k) = (-1)^k I(0; 1, \{0\}^{s_1 - 1}, \ldots, 1, \{0\}^{s_k - 1}; 1)$$

We introduce a new object here, seemingly unmotivated. The way it interacts with MZV's makes it important. Later it is used extensively to create motivic MZV's.

1. When multiplying two such integrals, the domains are independent, so the t_i and s_j do not interact. We are free to decompose into subdomains where $t_i < s_j$, $t_i > s_j$, etc. We don't need to include $t_i = s_j$ since this set has measure 0.
2. This was first observed by Kontsevich. The proof is not complicated, one simply expands out the the fractions $\frac{1}{t_1 - a_1}$ as a geometric series. The result can be recognised as the series definition of the MZV.

Properties of iterated integrals

Properties

- $I(a;b) = 1$ *for any* a, b *(Unit)*
- $I(0; a_1, \ldots, a_N; 0) = 0$ *for* $N \geq 1$ *(Equal boundaries)*
- $I(a_0; a_1, \ldots, a_N; a_{N+1}) = I(1 - a_0; 1 - a_1 \ldots, 1 - a_N; 1 - a_{N+1})$ *(Functoriality)*
- $I(a_0; a_1, \ldots, a_N; a_{N+1}) = (-1)^N I(a_{N+1}; a_N, \ldots, a_1; a_0)$ *(Reversal of paths)*

Corollary (MZV duality)

$$I(0; a_1, \ldots, a_N; 1) = (-1)^N I(0; 1 - a_N, \ldots, 1 - a_1; 1)$$

$$\rightsquigarrow \underbrace{\zeta(2,1,5)}_{-I(0;10110000;1)} = \underbrace{\zeta(1,1,1,3,2)}_{-I(0;11110010;1)}$$

Properties of iterated integrals

1. Well known formula due to Kontsevich shows how to write MZV's as iterated integrals over $0, 1$, of forms dx/x and $dx/(1-x)$
2. These iterated integrals satisfy a number of properties. For us some of the more useful ones are as follows. If the two bounds of integration are equal, then the integral (obviously) vanishes. The integral satisfies functoriality under the map $t \mapsto 1 - t$ (and more generally). If we reverse the path of integration, we pick up a sign.
3. Combining functionality and reversal of paths leads to the duality of MZV's: the result of reverse and $0 \leftrightarrow 1$ leads to an integral of MZV type, so we get equality between pairs of MZV's, which is otherwise not easy to see.
4. Finally we have the shuffle product multiplication of these integrals. $w \sqcup\!\sqcup v$ is the 'riffle shuffle' of w with v: all ways of interleaving the two words while preserving their original orders. By shuffling out leading $a_1 = 0$ variables, we obtain the shuffle regularisation of divergent integrals with $a_1 = 0$ or $a_N = 1$.

Series representation; stuffle product

- Multiply series gives stuffle product $*$
  - Arrange n_i, and m_j in all compatible ways $n_i < m_j$, or $n_i = m_j$ or $n_i > m_j$.
- Simplest case $\zeta(s) * \zeta(t) = \zeta(s,t) + \zeta(t,s) + \zeta(s+t)$.

Example (Comparing ⧢ and $*$)

$$2\zeta(2,2) + 4\zeta(1,3) \overset{⧢}{=} \zeta(2)\zeta(2) \overset{*}{=} 2\zeta(2,2) + \zeta(4)$$
$$\implies \zeta(1,3) = \frac{1}{4}\zeta(4) = \frac{1}{3}\frac{\pi^4}{5!}$$

Conjecture (Extended double shuffle)

All $\mathbb{Q}$-linear relations on MZV's arise by comparing ⧢ $- *$.
(Must allow divergent $\zeta(1)$; formally cancels using regularisation.)

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1. In terms of words in iterated integrals, a recursive definition is more complicated. But the idea is is simple: Shuffle the arguments $s_1, \ldots, s_k$ and $s'_1, \ldots, s'_\ell$. Also *stuff* two into one slot $s_i + s'_j$. This arises by arranging the independent summation indices n_i and m_j in all possible ways. Here we do need to include $n_i = m_j$ since this does not have measure 0.
2. We have two different ways to multiply MZV's now, so we should compare them.
3. This conjecture is even more hopeless than the previous one: it implies all relations are weight graded. The shuffle product of integrals write weight k times weight l as a sum of weight $k + \ell$ integrals. Similarly the series gives weight $k + \ell$, so the linear relation relation which results has weight $k + \ell$. This conjecture does pass extensive numerical testing: any numerically true relation on MZV's can (so far) be written as ⧢ $- *$.

Cyclic insertion conjecture

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Zagier-Broadhurst Identity

Theorem (Zagier-Broadhurst, BBBL 2001)

For $n \geq 0 \in \mathbb{Z}$, *have*

$$\zeta(\{1,3\}^n) = \frac{1}{2n+1} \frac{\pi^{4n}}{(4n+1)!}$$

Proof (Sketch).

- Generalise to single variable *multiple polylogarithms*.
- Generating series satisfies a differential equation.
- Explicit solution in terms of ${}_2F_1$. Compare coefficients.

Combinatorial proofs have also been given. □

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Explicit solution in terms of ${}_2F_1$. Compare coefficients.
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1. This identity was spotted by Zagier after some numerical computations of MZV (which in particular lead to the famous conjecture on the dimension d_k of the weight k MZV's, namely $d_k = d_{k-2} + d_{k-3}$, $d_0 = 1, d_1 = 0$.
2. The first proof of this identity was given by Broadhurst, by lifting the putative identity to certain single variable multiple polylogarithms, and assembling the results in the a generating series.
3. One can check that this generating series satisfies a certain differential equation. One can give an explicit solution to this differential equation in terms of a product of ${}_2F_1$ hypergeometric functions. This product can be simplified into Γ's and then $\sin(x)/x$ using Gauss's hypergeomtric summation theorem. This gives a formula for the coefficient.
4. More combinatorial proofs have been given by using the shuffle algebra of iterated integrals.

"Dressed with 2's"

Theorem (BBBL, 1998)

Let $n \geq 0 \in \mathbb{Z}$, *write*

$$I = \{ \text{ all } 2n+1 \text{ ways of inserting 2 into } \{1,3\}^n \ \} \ .$$

Then

$$\sum_{\mathbf{s}\in I} \zeta(\mathbf{s}) = \frac{\pi^{4n+2}}{(4n+3)!}$$

Example

For $n = 2$, have

$$\zeta(2,1,3,1,3) + \zeta(1,2,3,1,3) + \zeta(1,3,2,1,3) +$$
$$\zeta(1,3,1,2,3) + \zeta(1,3,1,3,2) = \frac{\pi^{10}}{11!}$$

1. Borwein, Bradley, Broadhurst and Lisonek were able to generalise some of the combinatorics of the proof of the Zagier identity to obtain a version 'dressed with 2's'. (The preprint proving Zagier's identity was published later than this.)
2. The identity is given by inserting into the gaps between the argument string $\{1, 3, 1, 3, \ldots\}$ a single 2, in all possible ways. One gets $2n + 1$ new MZV's, and the sum of these is proven to be $\pi^{4n+2}/(4n + 3)!$.

Cyclic insertion conjecture

Numerical experimentation lead to conjectural generalisation.

Notation

Let $a_1, \ldots, a_{2n+1} \in \mathbb{Z}_{\geq 0}$. Write

$$Z(a_1, \ldots, a_{2n+1}) = \zeta(\{2\}^{a_1}, 1, \{2\}^{a_2}, 3, \ldots, 1, \{2\}^{a_{2n}}, 3, \{2\}^{a_{2n+1}})$$

Conjecture (Cyclic insertion - BBBL, 1998)

$$\sum_{\sigma \in \mathbb{Z}/n\mathbb{Z}} Z(a_{\sigma(1)}, \ldots, a_{\sigma(2n+1)}) \stackrel{?}{=} \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$$

Shorthand: "wt" is weight of MZV's on the LHS

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1. Much numerical experimentation by these authors lead to a large conjectural generalisation of this 'dressed with 2's identity. One can see the dressed with 2's identity as a 2 in the first slot, then cyclically shifting it. This is the right direction to generalise.
2. One takes some blocks of 2's of lengths $a_1, \ldots, a_{2n+1}$. Insert $\{2\}^{a_i}$ into the i-th gap. The sum all cyclic permutations of these blocks. The result always seems to be $\pi^{\mathrm{wt}}/(\mathrm{wt}+1)$, in particular independent of the sizes of the blocks. It only depends on their sum and number.
3. It becomes rather messy to always write the precise formula for the weight of the MZV's, so I will use the shorthand 'wt' instead. Here the weight is $2(a_1 + \ldots + a_{2n+1}) + 4n$.

Bowman-Bradley

Best result so far is

Theorem (Bowman-Bradley, 2002)

Let $n, t \geq 0 \in \mathbb{Z}$*, then*

$$\sum_{\substack{a_1+\cdots+a_{2n+1}=t \\ a_i \geq 0}} Z(a_1, \ldots, a_{2n+1}) = \frac{1}{2n+1} \binom{t+2n}{t} \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$$

Remark

Compatible with cyclic insertion: Any permutation of a composition $a_1 + \cdots + a_{2n+1} = t$ is still a composition.

Will use the motivic MZV framework to improve on this, up to $\mathbb{Q}$.

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1. The best result so far generalised the combinatorics of the Zagier-with-2's identity. It views this as inserting all possible blocks of 2's which have total length 1, and so generalises to all compositions of lengths.
2. This result is compatible with cyclic insertion: there are $\binom{t+2n}{t}$ compositions of t into $2n+1$ parts. We obtain all permutations of a fixed composition, so can combine them with cyclic insertion to get $\frac{\pi^{\mathrm{wt}}}{(|wt+1)!}$. This means each term contributes on average $\frac{1}{2n+1}\frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$, giving the total above.
3. Later we see how to use the motivic MZV framework to improve this to a sum over a smaller index set, at the expense of getting an identity up to $\mathbb{Q}$.

Hoffman's conjecture

Separate conjecture, with a similar flavour

Conjecture (Hoffman, MZV Infopage, 2000)

For $m \geq 0 \in \mathbb{Z}$,

$$2\,\zeta(3,3,\{2\}^m) - \zeta(3,\{2\}^m,1,2) \overset{?}{=} -\zeta(\{2\}^{m+3}) = -\frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$$

Remark

Verified up to weight 22, $m = 8$ using MZV datamine, Vermaseren (2009).

Will show this up to $\mathbb{Q}$, using the motivic framework

Goal: connect these two conjectures, and work towards proofs.

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1. Now for something completely different. Hoffman lists the following conjectural identity on his homepage, as an example of identities which can be discovered with the EZ-Face engine. The identity has a rather similar flavour to the previous: zetas of 1's, 2's and 3's summing up to some multiple of $\pi^{\mathrm{wt}}/(\mathrm{wt}+1)!$. But it is different enough that there doesn't seem to be much of a connection.
2. Hoffman notes that this identity has been checked up to weight 22 using the datamine, but the general case is unproven.
3. Later we see a motivic proof of this (indeed even a generalisation), but only up to $\mathbb{Q}$.

Tools from motivic MZV's

Tools from motivic MZV's

1. To be able to motivate, state and prove these generalisations we need to introduce some algebraic tools, namely the motivic MZV's defined by Brown.

Brown's motivic MZV's

Solve transcendence problems with algebraic version of MZV's:

- Graded algebra $\mathcal{H}_\bullet$ of motivic MZV's

  $$\zeta^{\mathfrak{m}}(s_1, \ldots, s_r) := [\mathcal{O}(\pi_1^{\mathrm{un}}(\mathbb{P}^1 \backslash \{ 0, 1, \infty \}, \vec{1}_0, -\vec{1}_1)), \overbrace{\mathrm{dch}}^{\text{straight line}}, \underbrace{\Omega}_{\text{integrand}}]^{\mathfrak{m}} .$$

  Contains all motivic iterated integrals

  $$I^{\mathfrak{m}}(a_0; a_1, \ldots, a_N; a_{N+1}), a_i \in \{ 0, 1 \}$$

- Projection to algebra $\mathcal{A}_\bullet$ of de Rham motivic MZV's

  $$\zeta^{\mathfrak{a}}(s_1, \ldots, s_r) := [\mathcal{O}(\pi_1^{\mathrm{un}}(\mathbb{P}^1 \setminus \{ 0, 1, \infty \}, \vec{1}_0, -\vec{1}_1)), \underbrace{\varepsilon}_{\text{augmentation ideal}}, \Omega]^{\mathfrak{m}} ,$$

  kernel generated by $\zeta^{\mathfrak{m}}(2)$.

- Coaction

  $$\Delta \colon \mathcal{H} \to \mathcal{A} \otimes_{\mathbb{Q}} \mathcal{H}$$

  lifts Goncharov's 'semicircular' coproduct on $\mathcal{A}$. $\mathcal{H}$ Hopf algebra comodule over $\mathcal{A}$.

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Coaction
lifts Goncharov's 'semicircular' coproduct on $\mathcal{A}$. $\mathcal{H}$ Hopf algebra comodule over $\mathcal{A}$.

1. Motivic MZV's are defined as certain triples, which give objects in the ring of motivic periods. Integrating the previous integrand on the straight line path in the motivic fundamental group of $\mathbb{P}^1 - \{\infty, 0, 1\}$ gives us the motivic zeta. Period map to $\mathbb{C}$.
2. Get motivic analogues of the iterated integrals here, as divergent iterated integrals can be regularised to MZV's. Motivic integrals satisfy the same nice properties as before.
3. Moreover, we have the de Rham version of this construction, which gives de Rham motivic MZV's. There is a projection map from motivic MZV's to de Rham motivic MZV's whose kernel is exactly $\zeta^{\mathfrak{m}}(2)$, so one can think of just killing $\zeta^{\mathfrak{m}}(2)$.
4. The resulting motivic MZV's naturally form a Hopf algebra comodule, so there is a coaction. This coaction is given by some lifting of Goncharov's semicircular coproduct formula. Introduce infinitesimal version of it next, so perhaps can indicate how Δ looks like.

Results from motivic MZV's

- $\zeta^{\mathfrak{a}}(2k+1)$ are linearly independent
  - $\zeta^{\mathfrak{a}}(2k+1) \neq 0 \in \mathcal{A}_{2k+1}(\mathbb{Q})$
  - So have different gradings
- $\zeta^{\mathfrak{a}}(2k+1)$ are *algebraically* independent
  - Suppose some $\zeta^{\mathfrak{a}}(2k+1)$ satisfy a polynomial
  - Use coproduct Δ to show all coefficients are 0
- $\zeta^{\mathfrak{a}}(3,5)$ is irreducible (i.e. not in $\mathbb{Q}[\zeta(n)]$)
  - $(\Delta - \Delta^{\mathrm{op}})\zeta^{\mathfrak{a}}(3,5) = -5\zeta^{\mathfrak{a}}(3) \wedge \zeta^{\mathfrak{a}}(5)$
  - $(\Delta - \Delta^{\mathrm{op}})\zeta^{\mathfrak{a}}(n_1) \cdots \zeta^{\mathfrak{a}}(n_k) = 0$

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1. This follows trivially from the fact that $\mathcal{H}_\bullet$ is an object grade by the weight weight. So each $\zeta^{\mathfrak{m}}(n)$ is in a component of different weight n, therefore must be linearly independent.
2. This also follows reasonably easily. One can take set of such Riemann zeta values, and support they satisfy a only polynomail of some minimal degree. Applying the coproduct allows us to extract a polynomial of lower degree that they satisfy. Hence conclude the lower degree polynomial vanishes, and this tells us about the coefficients of our polynomial. This is enough to establish our starting polynomial also vanishes.
3. The proof of this is rather cute. The reduced coproduct of $\zeta^{\mathfrak{m}}(3,5)$ is (a mulitple of) $\zeta^{\mathfrak{m}}(3) \otimes \zeta^{\mathfrak{m}}(5)$, which gives $\zeta^{\mathfrak{m}}(3) \wedge \zeta^{\mathfrak{m}}(5)$ under antisymmetrisation $\Delta - \Delta^{op}$.
   On the other hand, the reduced coproduct of $\zeta(n)$ is 0. In such a case $\Delta(xy) = x \otimes y + y \otimes x$, so a $\zeta(n)$'s vanishes under antisymmetrisation.

Infinitesimal coproduct

Definition (Derivations D_k)

Let $\mathcal{L} \coloneqq \mathcal{A}/(\mathcal{A}_{>0} \cdot \mathcal{A}_{>0})$, which kills products and $\zeta^{\mathfrak{m}}(2)$. For k odd define

$$\begin{aligned} D_k \colon \quad & \mathcal{H} \to \mathcal{L}_k \otimes_{\mathbb{Q}} \mathcal{H} \\ & I^{\mathfrak{m}}(w) \mapsto (\pi \otimes \mathrm{id}) \circ (\Delta - 1 \otimes \mathrm{id}) \, I^{\mathfrak{m}}(w) \end{aligned}$$

$$D_k I^{\mathfrak{m}}(a_0; a_1, \ldots, a_N; a_{N+1}) =$$

$$\sum_{p=0}^{N-k} I^{\mathfrak{L}}(a_p; a_{p+1}, \ldots, a_{p+k}; a_{p+k+1}) \otimes \qquad \leftsquigarrow \text{Subsequence}$$

$$I^{\mathfrak{m}}(a_0; a_1, \ldots, a_p, a_{p+k+1}, \ldots, a_N; a_N + 1) \qquad \leftsquigarrow \text{Quoient sequence}$$

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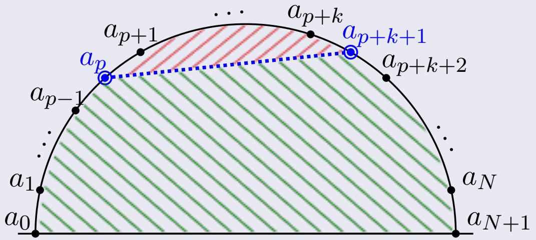
1. The full coaction is given by a long formula, and is difficult to work with in general. It has order n^2 terms. Instead Brown introduces an infinitesimal (linearised) version of it which only has a linear number of terms, as follows.
2. First go to the Lie coalgebra of indecomposables $\mathcal{L}$ by killing products. Then we can project the weight k part of the coaction to this, to define the operator D_k.
3. Using the (not given) formula for Δ, one can compute how D_k acts on any integral $I^{\mathfrak{m}}(w)$, giving this explicit formula.
4. It is useful to introduce some terminology here for the two terms appearing in D_k. The term on the left is the subsequence (as the arguments form one). The term on the right is the quotient sequence, since we kill the subsequence to get it.

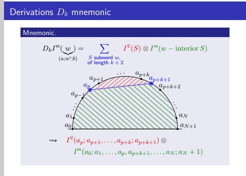
Derivations D_k mnemonic

Mnemonic.

$$D_k I^{\mathfrak{m}}(\underbrace{w}_{(a;w';b)}) = \sum_{\substack{S \text{ subword } w, \\ \text{of length } k+2}} I^{\mathfrak{L}}(S) \otimes I^{\mathfrak{m}}(w - \text{interior } S)$$



$$\rightsquigarrow \quad I^{\mathfrak{L}}(a_p; a_{p+1}, \ldots, a_{p+k}; a_{p+k+1}) \otimes I^{\mathfrak{m}}(a_0; a_1, \ldots, a_p, a_{p+k+1}, \ldots, a_N; a_N + 1)$$



1. A much better way to interpret/use this formula for D_k is using a pictorial mnemonic.
2. Schematically the formula for D_k is given by a sum over all subwords of w, and forming the sub/quotient sequence. If we arrange the points of w around a 'semicircular' polygon, then the terms are formed by cutting off segments containing k points. This splits the polygon into two parts: a main polygon in green, and a cut off polygon in red. From the main polygon we form the right hand term, and from the quotient polygon we form the left hand term.
3. This is the same mnemonic that can be used for Goncharov's coproduct/the full coaction: but we take all contiguous n-tuples of segments starting from a_0 and ending at a_{N+1}. Form the product of every cut-off polygon to get the left hand factor.

Transcendental Galois Theory

Theorem (Brown, 2012)

Let $D_{<N} = \bigoplus_{1<2r+1<N} D_{2r+1}$. *In weight* N,

$$\ker D_{<N} = \zeta^{\mathfrak{m}}(N)\mathbb{Q}\,.$$

$\rightsquigarrow$ *'exact-numerical' algorithm for decomposing motivic MZV's*

Example

Can show $\zeta^{\mathfrak{m}}(\{2\}^n) = \pm I^{\mathfrak{m}}(0; \underbrace{1,0,1,0,\ldots,1,0}_{n \text{ times}}; 1) \in \zeta^{\mathfrak{m}}(2n)\mathbb{Q}$

- Integral word alternates 0 and 1
- Odd length subsequence has same boundaries, vanishes
- Therefore all D_{2r+1} vanish

Conclude $\zeta^{\mathfrak{m}}(\{2\}^n) \in \ker D_{<2n} = \zeta^{\mathfrak{m}}(2n)\mathbb{Q}$.

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Therefore all D_{2r+1} vanish
Conclude $\zeta^{\mathfrak{m}}(\{2\}^n) \in \ker D_{<2n} = \zeta^{\mathfrak{m}}(2n)\mathbb{Q}$.

1. The real upshot of this construction comes with the following theorem, a kind of transcendental Galois theory.
2. We can check that $\zeta(N)$ vanishes under all D_{odd} simultaneously. Brown shows that this actual characterises $\zeta^{\mathfrak{m}}(N)$. Any motivic MZV of weight N which vanishs under all D_{odd} must be a rational multiple of $\zeta(N)$. This result forms the basis of an exact-numerical algorithm for decomposing MZV's into a chosen basis.
3. For us, this result will be used to check/prove identities up to $\mathbb{Q}$. Here is a quick example, which shows that $\zeta(\{2\}^n)$ is a rational multiple of $\zeta(2n)$. (Confirming one of the known special cases of cyclic insertion.)
4. $\zeta(\{2\}^n)$ is described by the integral $I(0,(1,0)^n,1)$. Any subsequence of odd length must start and end with the same symbols since the word w has period 2. This means every subsequence vanishes trivially: the bounds of integration are the same. The result follows immediately.

$\zeta^{\mathfrak{m}}(\{1,3\}^n)$

More interesting: $\zeta^{\mathfrak{m}}(\{1,3\}^n) = I^{\mathfrak{m}}(0;(1100)^n;1) \in \zeta^{\mathfrak{m}}(4n)\mathbb{Q}$

- Word has period 4, so length $1 \pmod 4$ subsequence vanish
- For length $3 \pmod 4$, look at starting position

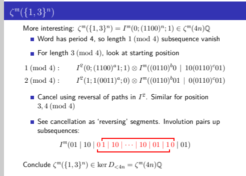
$$1 \pmod 4: \quad I^{\mathfrak{L}}(0;(1100)^a 1;1) \otimes I^{\mathfrak{m}}((0110)^b 0 \mid 10(0110)^c 01)$$

$$2 \pmod 4: \quad I^{\mathfrak{L}}(1;1(0011)^a;0) \otimes I^{\mathfrak{m}}((0110)^b 01 \mid 0(0110)^c 01)$$

- Cancel using reversal of paths in $I^{\mathfrak{L}}$. Similar for position $3,4 \pmod 4$
- See cancellation as 'reversing' segments. Involution pairs up subsequences:

$$I^{\mathfrak{m}}(01 \mid 10 \mid 0\,1 \mid 10 \mid \cdots \mid 10 \mid 01 \mid 1\,0 \mid 01)$$

Conclude $\zeta^{\mathfrak{m}}(\{1,3\}^n) \in \ker D_{<4n} = \zeta^{\mathfrak{m}}(4n)\mathbb{Q}$



1. A more interesting case is $\zeta(\{1,3\}^n)$, the Zagier-Broadhurst identity. This time the word w is $0(1100)^n 1$, periodic with period 4.
2. If the subsequence has length $1 \pmod 4$, then the we see that the subsequence again trivially vanishes. However if the subsequence has length $3 \pmod 4$, things don't just vanish, and we have to look at where the subsequence starts.
3. If the subsequence starts at position $1 \pmod 4$, we get a contribute to D_k which looks as indicated. And starting at $2 \pmod 4$ gives the other term.
4. Using reversal of paths, we can revere the second $I^{\mathfrak{L}}$, shows the two terms are equal up to a -1. So they cancel in D_k. Something similar happens for terms starting $3,4 \pmod 4$.
5. In some sense we can see this cancellation as a pairing up of terms indicted by a reversal/reflection of segments. Either way, this shows all terms in D_k cancel, and by Brown we get the result. We are going to generalise this cancellation observation.

The alternating block decomposition

The alternating block decomposition

Alternating blocks

Observation

In $\zeta^{\mathfrak{m}}(\{1,3\}^n)$ proof, points 00 and 11 in w are 'somehow' significant.

- Splitting here decomposes a word into *alternating blocks* $0101\cdots$ or $1010\cdots$.

Definition (Block decomposition)

Let w be a word starting with $\varepsilon_1 \in \{0,1\}$. Write w as alternating blocks, with lengths $\ell_1, \ldots, \ell_k$. The block decomposition of w is

$$\mathrm{bl}(w) = (\varepsilon_1; \ell_1, \ldots, \ell_k) \,.$$

Example

$$\mathrm{bl}(\underbrace{0}_{1} \mid \underbrace{01}_{2} \mid \underbrace{10}_{2} \mid \underbrace{01010}_{5} \mid \underbrace{0}_{1} \mid \underbrace{01}_{2}) = (0; 1, 2, 2, 5, 1, 2)$$

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Example
$\mathrm{bl}(0 \mid 01 \mid 10 \mid 01010 \mid 0 \mid 01) = (0;1,2,2,5,1,2)$

1. Somehow the 00 and 11 points determined the cancellation pairing i the $\zeta(\{1,3\}^n)$ proof, so they play a significant role. When cut at these points, we get segments of the form 01 or 10, which consist of alternating 0's and 1's. Let's do this in general.
2. Cut any word w at points 00 and 11, and we get an expression for w as a concatenation of alternating blocks 010101 and 1010101. Make a note of the lengths of these blocks, and this defines the block decomposition of w, as in the example. In order to recover w from this construction, we definitely need to know the starting point, so this is part of the block decomposition too.

Alternating blocks

Can recover w from $(\varepsilon_1; \ell_1, \ldots, \ell_k)$: blocks arise from $00 \to 0 \mid 0$ or $11 \to 1 \mid 1$.

Notation

Write $I_{\mathrm{bl}}(\varepsilon_1; \ell_1, \ldots, \ell_k) = I(\mathrm{bl}^{-1}(\varepsilon_1; \ell_1, \ldots, \ell_k))$. If $\varepsilon_1 = 0$, just write $(\ell_1, \ldots, \ell_k)$.

- Weight of $I_{\mathrm{bl}}(\varepsilon_1; \ell_1, \ldots, \ell_k)$ is $-2 + \sum_i \ell_i$. (Bounds of integration are counted!)
- If $\mathrm{wt} \equiv k \pmod 2$ then $I_{\mathrm{bl}} = 0$. (End points are equal!)
- I_{bl} is divergent iff $\ell_1 = 1$ or $\ell_k = 1$.

Example

$$I_{\mathrm{bl}}(1, 2, 2, 5, 1, 2) = I(0; 01100101000; 1)$$

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Example
$I_{\mathrm{bl}}(1, 2, 2, 5, 1, 2) = I(0; 01100101000; 1)$

1. But once we known the starting point and the block lengths, it is easy to see how to recover w from them. The blocks arise from cutting $00 \to 0 \mid 0$ and $11 \to 1 \mid 1$. So consecutive blocks end/start with the same symbol. If we know where to start we can just write down blocks of the appropriate lengths to get w.
2. This means bl is invertible, and from this we can define the block integral: it is just the integral given by the word with corresponding block decomposition.
3. Some properties of this: the weight of the integral is $\sum \ell_i$ minus 2, because we also count the end points of integration in the block decomposition.
4. If the weight and number of blocks are the same mod 2, then the integral is trivially 0 because the end points are equal. We integrate from 0 to 0, or from 1 to 1.
5. Divergent integrals start with 00 or end with 11, so integrals are divergent iff the first block has $\ell_1 = 1$, or the last has $\ell_k = 1$.

Block structure of BBBL conjecture

- Write the BBBL identity as iterated integrals

$$\sum_{\text{cycle } a_i} \zeta(\{2\}^{a_1}, 1, \{2\}^{a_2}, 3, \ldots, 1, \{2\}^{a_{2n}}, 3, \{2\}^{a_{2n+1}})$$

$$\rightsquigarrow \pm \sum_{\text{cycle } a_i} I(0(10)^{a_1}1(10)^{a_2}100\cdots01(10)^{a_{2n}}100(10)^{a_{2n+1}}1)$$

- Split into 'alternating blocks' at $00 \to 0 \mid 0$ or $11 \to 1 \mid 1$

$$= \pm \sum_{\text{cycle } a_i} I(0(10)^{a_1}1 \mid (10)^{a_2}10 \mid 0\cdots01 \mid (10)^{a_{2n}}10 \mid 0(10)^{a_{2n+1}}1)$$

- Record lengths of the blocks

$$= \pm \sum_{\text{cycle } a_i} I_{\text{bl}}(2a_1+2, 2a_2+2, \ldots, 2a_{2n+1}+2)$$

- Right hand side is $\zeta(\{2\}^{\text{wt}/2}) = \pm I_{\text{bl}}(\text{wt}+2)$.

Block structure of BBBL conjecture
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1. Now let us apply this construction to gain a new understand the structure of the BBBL identity
2. When converting to integrals we have a sign corresponding to the depth. For the left hand side the sign is constantly $(-1)^{(}2n + \sum a_i) = (-1)^{(}\text{wt}/2)$.
3. Each term gives an integral with even sized blocks $2a_i + 2$.
4. The right hand side of $\zeta(\{2\}^{\text{wt}/2})$, so again we pick up sign $(-1)^{(}\text{wt}/2)$.
5. So in the resulting integral identity, we can cancel the sign to make every term positive. We get the structure: sum over cyclic shifts of block lengths is block of weight$+2$.

Block structure of Hoffman's conjecture

- Write Hoffman's identity as iterated integrals

$$\begin{aligned} & 2\zeta(3,3,\{2\}^n) && - \zeta(3,\{2\}^n,1,2) \\ &= \zeta(3,3,\{2\}^n) && - \zeta(3,\{2\}^n,1,2) \quad + \zeta(\{2\}^n,1,2,1,2) \\ \rightsquigarrow & \pm \big(I(0100100(10)^n1) + I(0100(10)^n1101) + I(0(10)^n1101101)\big) \end{aligned}$$

- Split into 'alternating blocks' at $00 \to 0 \mid 0$ or $11 \to 1 \mid 1$

$$= \pm \big(I(010 \mid 010 \mid 0(10)^n1) + I(010 \mid 0(10)^n1 \mid 101) \\ + I(0(10)^n1 \mid 101 \mid 101)\big)$$

- Record lengths of the blocks

$$= \pm \big(I_{\text{bl}}(3,3,2n+2) + I_{\text{bl}}(3,2n+2,3) + I_{\text{bl}}(2n+2,3,3)\big)$$

- Right hand side is $-\zeta(\{2\}^{n+3}) = \pm I_{\text{bl}}(\text{wt}+2)$

Block structure of Hoffman's conjecture
- Write Hoffman's identity as iterated integrals
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1. Now let us apply this construction to gain a new understand the sturcture of the Hoffman identity
2. We a first step, I want to split up the coefficient 2 term using duality. Maybe this is unmotivated currently, but it is useful in a moment.
3. When converting to integrals we have a sign corresponding to the depth. For the left hand side the sign is overall $(-1)^(n)$, as some terms have one extra argument, and one extra minus sign.
4. Each term gives block of size $3, 3, 2n+2$ in some order. Surprising?
5. The right hand side of $-\zeta(\{2\}^{n+3})$, so again we pick up sign $-(-1)^{n+3} = (-1)^(n)$.
6. So in the resulting integral identity, we can cancel the sign to make every term positive. We get the structure: sum over cyclic shifts of block lengths is block of weight+2.

Common structure and generalisation

Both conjectures have same structure: cyclic permutations of block lengths ℓ_i.

Conjecture (Cyclic insertion, C., 2017, arXiv 1703.03784)

For any $(\ell_1, \ldots, \ell_k)$ *with all* $\ell_i > 1$,

$$\sum_{\text{cycle } \ell_i} I_{\text{bl}}(\ell_1, \ldots, \ell_k) \overset{?}{=} I_{\text{bl}}(\text{wt}+2) = \begin{cases} \frac{\pi^{\text{wt}}}{(\text{wt}+1)!} & \text{wt } \textit{even} \\ 0 & \text{wt } \textit{odd} \end{cases}$$

- Numerically tested all cases weight ≤ 18, to 500 decimal places
- Can prove a symmetrised version, up to $\mathbb{Q}$
- Can prove *some* special cases, up to $\mathbb{Q}$

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Numerically tested all cases weight ≤ 18, to 500 decimal places
Can prove a symmetrised version, up to $\mathbb{Q}$
Can prove some special cases, up to $\mathbb{Q}$

1. We see that both identities have exactly the same structure, so hopefully it is not too much of a leap to think a generalisation like this may hold.
2. Of course, we need more than just two conjectural cases to justify this pattern. So I justify this conjecture by noting that I have checked it for all cases up to weight 18, to 500 decimal places. (Could also use the datamine.) Moreover, I can prove a symmetrisation of this holds up to $\mathbb{Q}$ using motivic MZV's, and can even prove special cases on the nose, up to $\mathbb{Q}$.
3. We can also produce new candidate identities that we can individually check to very high weight numerically.

Examples

Example

Let $(\ell_1, \ldots, \ell_k) = (2m+2, 2, 3, 2, 3)$, then we obtain

$$\zeta(\{2\}^m, 1, 3, 3, 1, 2) + \zeta(3, 1, 2, 1, \{2\}^m, 3) - \zeta(1, 2, 1, \{2\}^m, 3, 1, 2) + \\ + \zeta(1, 2, 1, 3, 3, \{2\}^m) - \zeta(3, \{2\}^m, 1, 3, 3) \overset{?}{=} \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$$

Proposition (C., 2017, arXiv 1703.03784)

The above identity holds up to $\mathbb{Q}$

Proof (Sketch).

Lift the identity to $\zeta^{\mathfrak{m}}$, and compute $D_{<2m+10}$. A (tedious) calculation shows $D_{<2m+10}$ vanishes. □

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Proof (Sketch).
Lift the identity to $\zeta^{\mathfrak{m}}$, and compute $D_{<2m+10}$. A (tedious) calculation shows $D_{<2m+10}$ vanishes. □

1. If we start with the indicated block lengths, we produce this candidate identity. One can verify it numerically up to weight 100 or so, using gp/pari
2. Moreover, I claim that one can give a motivic proof to show the result holds up to $\mathbb{Q}$. However this is by a very tedious calculation writing down every single term in $D_{<N}$, and showing explicitly that they cancel. There don't seem to be any nice properties to exploit to simplify this.

Progress and results

Theorem (Symmetric insertion, C., 2017, arXiv 1703.03784)

For any $(\ell_1, \ldots, \ell_k)$, *with even weight,*

$$\sum_{\text{permute } \ell_i} I_{\text{bl}}(\ell_1, \ldots, \ell_k) \in I_{\text{bl}}(\text{wt} + 2)\mathbb{Q}$$

(Odd weight holds trivially, by duality)

Proof (Strategy).

- Lift to motivic version $I^{\mathfrak{m}}$.
- Define a reflection $\mathcal{R}$ on non-trivial subsequences
- Use $\mathcal{R}$ to cancel terms in $D_{<N}$
- Conclude $\in \zeta^{\mathfrak{m}}(\text{wt})\mathbb{Q} = I^{\mathfrak{m}}_{\text{bl}}(\text{wt} + 2)\mathbb{Q}$ using Brown.

2018-04-30

Cyclic insertion on MZV's and the alternating block decomposition
└The alternating block decomposition
└Progress and results

Progress and results
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(Odd weight holds trivially, by duality)
Proof (Strategy).
Lift to motivic version $I^{\mathfrak{m}}$.
Define a reflection $\mathcal{R}$ on non-trivial subsequences
Use $\mathcal{R}$ to cancel terms in $D_{<N}$
Conclude $\in \zeta^{\mathfrak{m}}(\text{wt})\mathbb{Q} = I^{\mathfrak{m}}_{\text{bl}}(\text{wt} + 2)\mathbb{Q}$ using Brown.

1. The general result is the following. If one sums over all permutations of the block lengths, then one is guaranteed to get are rational multiple of $I_{\text{bl}}(\text{wt} + 2)$. This is still far away from the cyclic insertion conjecture generally, but as we'll see in a moment already produces non-trivial new results.
2. One remark though: in the odd weight case this symmetrisation will trivially vanish, since duality reverses the block lengths. This means the permutation which reverses the block lengths gives the dual term, which we pick up an extra minus sign from the when converting back to MZV's.
3. The strategy of the proof is to use the motivic MZV framework, and generalise the $\zeta(\{1,3\}^n)$ proof. We lift this to a motivic version, and show that $D_{<N}$ vanishes by setting up a cancellation of terms.

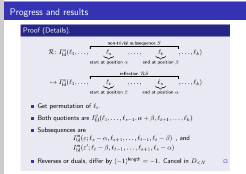
Progress and results

Proof (Details).

$$\mathcal{R}\colon I^{\mathfrak{m}}_{\mathrm{bl}}(\ell_1, \ldots, \overbrace{\underbrace{\ell_s}_{\text{start at position } \alpha}, \ldots, \underbrace{\ell_t}_{\text{end at position } \beta}}^{\text{non-trivial subsequence } S}, \ldots, \ell_k)$$

$$\mapsto I^{\mathfrak{m}}_{\mathrm{bl}}(\ell_1, \ldots, \overbrace{\underbrace{\ell_t}_{\text{start at position } \beta}, \ldots, \underbrace{\ell_s}_{\text{end at position } \alpha}}^{\text{reflection } \mathcal{R}S}, \ldots, \ell_k)$$

- Get permutation of ℓ_i.
- Both quotients are $I^{\mathfrak{L}}_{\mathrm{bl}}(\ell_1, \ldots, \ell_{s-1}, \alpha + \beta, \ell_{t+1}, \ldots, \ell_k)$
- Subsequences are
$$I^{\mathfrak{m}}_{\mathrm{bl}}(\varepsilon; \ell_s - \alpha, \ell_{s+1}, \ldots, \ell_{t-1}, \ell_t - \beta) \text{ , and}$$
$$I^{\mathfrak{m}}_{\mathrm{bl}}(\varepsilon'; \ell_t - \beta, \ell_{t-1}, \ldots, \ell_{s+1}, \ell_s - \alpha)$$
- Reverses or duals, differ by $(-1)^{\text{length}} = -1$. Cancel in $D_{<N}$ □



1. More precisely, $\mathcal{R}$ takes a non-trivial subsequence which starts at position α in block ℓ_s, and ends at position β from the end block ℓ_t. It maps this to a subsequence on another integral by reversing the blocks, and carrying the subsequence with it. (It starts and ends in the same blocks, but now the lengths ℓ_i are some permutation. However, the start and end positions are changed.)
2. We now see how the quotient and subsequences compare.
3. The quotient sequences are exactly equal. Outside of S the blocks are identical. Once we cut S out from the sequence, these blocks are joined by an alternating sequence 01010 of length $\alpha + \beta$. (Why alternating? Well the two end points of S are different, so when we just from the start to the end we go $0 \to 1$ or $1 \to 0$.)
4. Finally the subsequences? Well it is clear that the block lengths are reversed, so it depends on only the starting letter of the subsequence. If end of ℓ_t is start ℓ_s, we get reverse. Otherwise we have $0 \mapsto 1$, and get dual.

Corollaries of symmetric insertion

Corollary (Generalisation of Hoffman, up to $\mathbb{Q}$)

For $(\ell_1, \ell_2, \ell_3) = (2a+3, 2b+3, 2c+2)$, *we obtain*

$$\mathrm{Sym}_{a,b}\left(\zeta(\{2\}^a, 3, \{2\}^b, 3, \{2\}^c) - \zeta(\{2\}^b, 3, \{2\}^c, 1, 2, \{2\}^a) + \zeta(\{2\}^c, 1, 2, \{2\}^a, 1, 2, \{2\}^b)\right) \in \pi^{\mathrm{wt}}\mathbb{Q}$$

Duality shows cyclic insertion already holds up to $\mathbb{Q}$

$$\zeta(\{2\}^a, 3, \{2\}^b, 3, \{2\}^c) - \zeta(\{2\}^b, 3, \{2\}^c, 1, 2, \{2\}^a) + \zeta(\{2\}^c, 1, 2, \{2\}^a, 1, 2, \{2\}^b)) \in \pi^{\mathrm{wt}}\mathbb{Q}$$

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In particular, $a = b = 0$ is Hoffman's identity up to $\mathbb{Q}$.

1. We can take the following block lengths generalising Hoffman's case, and immediately conclude the sum of these 6 MZV's is a rational multiple of π^{wt}.
2. In fact duality allows us to combine pairs of terms $\zeta(\{2\}^a, 3, \{2\}^b, 3\{2\}^c) = \zeta(\{2\}^c, 1, 2, \{2\}^b, 1, 2, \{2\}^a)$, etc to get a 3-term identity which is already cyclic insertion in this case.
3. Further specialising to $a = b = 0$ gives Hoffman's identity up to $\mathbb{Q}$.

Corollaries of symmetric insertion

Corollary (Improvement of Bowman-Bradley, up to $\mathbb{Q}$)

For $\ell_i = 2a_i + 2$, obtain

$$\sum_{\text{permute } a_i} \zeta(\{2\}^{a_1}, 1, \{2\}^{a_2}, 3, \ldots, 1, \{2\}^{a_{2n}}, 3, \{2\}^{a_{2n+1}}) \in \pi^{\mathrm{wt}}\mathbb{Q}$$

"Only need permutations of a single composition."

In particular, for $a_1 = \cdots = a_n = m$

Corollary (Evaluable MZV)

The following MZV is evaluable

$$\zeta(\{\{2\}^m, 1, \{2\}^m, 3\}^n, \{2\}^m) \in \pi^{\mathrm{wt}}\mathbb{Q}$$

Up to $\mathbb{Q}$, proves conjecture of Borwein-Bradley-Broadhurst, 1997

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1. If we apply this to the BBBL case, we need only to sum over all permutations of the block lengths. This gives an improvement over the Bowman-Bradley theorem where all compositions were needed. The unfortunate thing is this is a non-explicit version: I can't find the rational coefficient exactly.
2. Nevertheless, putting all the $a_i = m$ shows that the previously conjectural MZV evauation does hold, again up to $\mathbb{Q}$. This MZV is definitely some rational multiple of π^{wt}.

Further progress?

Complete motivic proof of cyclic insertion is not (yet?) possible

- Cyclic insertion has a stability under D_k
- Odd weight implies $D_{<N}(\text{even weight}) = 0$
- Problem: Must fix rational multiple of $\zeta^{\mathfrak{m}}(\mathrm{wt})$ somehow $\rightsquigarrow$ analytically or numerically...
- $D_{<N}(\text{odd weight})$ involves $I^{\mathfrak{L}}$ explicitly

$$D_7 \sum\nolimits_{\text{cycle}} I^{\mathfrak{m}}_{\mathrm{bl}}(2,10,3,2) =$$
$$\underbrace{(I^{\mathfrak{L}}_{\mathrm{bl}}(6,3) + I^{\mathfrak{L}}_{\mathrm{bl}}(3,3,2,1) + I^{\mathfrak{L}}_{\mathrm{bl}}(2,3,2,2) + I^{\mathfrak{L}}_{\mathrm{bl}}(1,2,2,4))}_{-\zeta^{\mathfrak{L}}(2)\zeta^{\mathfrak{L}}(2,3) - 2\zeta^{\mathfrak{L}}(2)\zeta^{\mathfrak{L}}(3,2) + 2\zeta^{\mathfrak{L}}(3)\zeta^{\mathfrak{L}}(2,2) = 0} \otimes I^{\mathfrak{m}}_{\mathrm{bl}}(10)$$

- In general only have

$$\text{odd weight} = \sum\nolimits_k \alpha_k \zeta(2k+1)\zeta(\{2\}^{\mathrm{wt}/2-k}) \,, \quad \alpha_k \in \mathbb{Q}$$

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1. Unlikely that one can prove this hold result purely motivically. However, cyclic insertion has stability under D_k. Computation of D_k reduces to $I^{\mathfrak{L}}(\cdot) \times$ cyclic insertion, so hope of a recursive/inductive partial proof.
2. In particular, odd weight holds implies D_k even vanishes, so that we know the even weight case holds up to a rational if all lower weight odd caes hold. Unfortunately the rational is not visible motivically: need a numerical evaluation, or an exact formula to continue.
3. Moreover, the odd weight case has more problems. Computing D_k odd weight leads to explicit computations of $I^{\mathfrak{L}}$. For example the following. We need to recognise the factor is a sum of products and $\zeta(2)$'s in order to see $D_7 = 0$.
4. However, one can say from the special form of D_k odd weight (namely only $\zeta(2)^l$ appears on the right), that the odd weight identity can be expressed as a sum $\zeta(odd)\zeta(\{2\}^k)$, with rational coefficients.

Recent work

Using iterated integrals over $\mathbb{P}^1 \setminus \{ \infty, 0, 1, z \}$ gives

Theorem (Hirose-Sato, 2017, arXiv 1704.06478)

The generalisation of Hoffman's identity holds exactly

$$\zeta(\{2\}^a, 3, \{2\}^b, 3, \{2\}^c) - \zeta(\{2\}^b, 3, \{2\}^c, 1, 2, \{2\}^a) + \zeta(\{2\}^c, 1, 2, \{2\}^a, 1, 2, \{2\}^b) = -\zeta(\{2\}^{a+b+c+3})$$

Theorem (Hirose-Sato, 2017/18)

A 'block-shuffle' identity holds, which implies the conjecture.

See HIM talk, in "Periods and Regulators Workshop", at 15:00 on 19 January 2018. Video `https://youtu.be/b83fkeUAWu0`

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1. A more optimistic approach comes from recent work by Hirose and Sato, where they use iterated integrals over $\mathbb{P}^1 \setminus \{ \infty, 0, 1, z \}$ to give an exact proof of the generalised version of Hoffman's identity. Computer assistance was needed to find the right combination of integrals to work with, but then the proof is a simple exercise in computing the derivative.
2. Moreover, they are able to generalise the cyclic insertion conjecture to a 'block-shuffle' identity, and give a proof of this more general result.

Extra material

(if necessary)

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Full version of cyclic insertion

If some $\ell_i = 1$, the identity involves product term corrections.

$$\mathcal{L}_d = \{\, (m_{d+1}, \ldots, m_k) \mid (\overbrace{1, \ldots, 1}^{d \text{ times}}, m_{d+1}, \ldots, m_k) \text{ is a cyclic permutation of } (\ell_1, \ldots, \ell_k) \,\}$$

"Take all cyclic permutations of $(\ell_1, \ldots, \ell_k)$ which start with d consecutive 1's. Then drop the initial 1's"

Conjecture (Cyclic insertion, C., 2017, arXiv 1703.03784)

For any $(\ell_1, \ldots, \ell_k)$ of weight N,

$$\sum_{\text{cycle } \ell_i} I_{\mathrm{bl}}(\ell_1, \ldots, \ell_k) \overset{?}{=} I_{\mathrm{bl}}(N+2) - \sum_{d=1}^{\lfloor k/2 \rfloor} \frac{2(2\pi\mathrm{i})^{2d}}{(2d+2)!} \sum_{\mathbf{m} \in \mathcal{L}_{2d}} I_{\mathrm{bl}}(\mathbf{m}) \,.$$

1. The version of cyclic insertion I stated above required that all the $\ell_i > 1$, so that there was no shuffle-regularisation occurring. If any of the $\ell_i = 1$, then eventually it is cycled into the first (and/or last) position, and so the resulting integral is divergent and needs to be regularised.
2. One can regularise the resulting integrals, and give a generalisation of the identity which includes product term corrections.
3. These product terms are obtained by dropping the increasingly long divergent sequences $1, 1, \ldots, 1$ and complementing this with powers of π. One could perhaps say that cyclic insertion holds modulo π^2?

Full version of cyclic insertion

Example

With $(\ell_i) = (1,1,2,3)$, need only $\mathcal{L}_2 = \{\, (2,3) \,\}$. Get

$$I_{\text{bl}}(1,1,2,3) + I_{\text{bl}}(1,2,3,1) + I_{\text{bl}}(2,3,1,1) + I_{\text{bl}}(3,1,1,2) \stackrel{?}{=} I_{\text{bl}}(7) - \frac{2(2\pi \mathrm{i})^2}{4!} I_{\text{bl}}(2,3)$$

Shuffle regularisation gives

$$(3\zeta(1,1,3) + 2\zeta(1,2,2) + \zeta(2,1,2)) + (\zeta(2,3) - 6\zeta(1,1,3) - 4\zeta(1,2,2) - 2\zeta(2,1,2)) + (6\zeta(1,1,1,2)) + (-\zeta(5)) \stackrel{?}{=} 0 + \zeta(2)\zeta(1,2)$$

✓

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1. We should, in principle compute $\mathcal{L}_4$ also, but since there is no $1,1,1,1,$ subsequence it is empty.
2. The resulting block integral identity is given here. Shuffle regularising each divergent term gives the corresponding identity on MZV's which can be checked using the datamine. The order of the terms is preserved between the two lines, so one can match up the divergent integral with its shuffle regularisation.

Another block decomposition conjecture

Conjecture (BBBL 1998, rewritten)

Let $a_1, a_2, a_3, b_1, b_2 \in \mathbb{Z}_{\geq 0}$. *Then*

$$\sum_{\sigma \in S_3} \operatorname{sgn}(\sigma) \zeta(\{2\}^{a_{\sigma(1)}}, 1, \{2\}^{b_1}, 3, \{2\}^{a_{\sigma(2)}}, 1, \{2\}^{b_2}, 3, \{2\}^{a_{\sigma(3)}}) \stackrel{?}{=} 0$$

Generalising the block decomposition structure leads to

Conjecture (Alt-odd, C., 2017, arXiv 1703.03784)

For any $(\ell_1, \ldots, \ell_{2k+1})$ *of even weight, with all* $\ell_i > 1$,

$$\operatorname{Alt}_{\{\ \ell_i \mid i \textit{ odd }\}} I_{\mathrm{bl}}(\ell_1, \ldots, \ell_{2k+1}) \stackrel{?}{=} 0$$

"Alternating sum over odd-position blocks."

Remark

This conjecture is included in Hirose-Sato's generalisation too.

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1. Another block decomposition conjecture comes from generalising another conjecture from BBBL, which has not received much attention. They write their identity as a sum over the dihedral group D_3, which obscures the real structure and generalisation. One one notes that it can be written as an alternating sum over S_3 instead, a generalisation readily follows.
2. Have numerically checked this for weight ≤ 18, though I don't have any motivic proofs of this. Computing D_k seems to lead to messier and messier identities at lower weight, so this result is not stable under D_k. In particular, I don't have a good odd weight version of the conjecture.

Another block decomposition conjecture

Example

For block lengths $\ell_i = 2a_i + 2$, $1 \leq i \leq 7$, get

$$\mathrm{Alt}_{a_1,a_3,a_5,a_7}\,\zeta(\{2\}^{a_1}, 1, \{2\}^{a_2}, 3, \{2\}^{a_3}, 1, \{2\}^{a_4}, 3, \{2\}^{a_5}, 1, \{2\}^{a_6}, 3, \{2\}^{a_7}) \overset{?}{=} 0$$

Example

For block lengths $(2a_1 + 3, 2a_2 + 3, 2a_3 + 3, 2a_4 + 2, 2a_5 + 3)$, get

$$\mathrm{Alt}_{a_1,a_3,a_5}\,\zeta(\{2\}^{a_1}, 3, \{2\}^{a_2}, 3, \{2\}^{a_3}, 3, \{2\}^{a_4}, 1, 2, \{2\}^{a_5}) \overset{?}{=} 0$$

Another block decomposition conjecture

1. Immediately one gets the 'level' 3 generalisation to $\zeta(\{1,3\}^3)$ of the BBBL conjecture. Or one can apply it to a Hoffman identity type MZV, to get similar results. Can check each numerically to very high precision in various cases, so the conjecture looks promising.

Summary

- Defined block decomposition of an iterated integral
- Used block decomposition to unify/generalise BBBL and Hoffman's conjectures
- Used motivic MZV's to prove a symmetrised version holds
  - Improved Bowman-Bradley to only permutations
  - Proved Hoffman up to $\mathbb{Q}$,
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