Clean single valued MPL's

ETH program on modular forms, periods and scattering amplitudes

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1. Motivation/goal

Want to define/introduce **class of MPL functions** which satisfy so-called **clean versions** of the function equations of normal MPL's.

By this we mean the functions somehow already incorporate the product terms which will appear in any functional equation, in some universal way.

Motivating example from polylog case

Example 1.1 (Single-valued dilog, Bloch-Wigner-Ramakrishnan/Zagier)**.** The dilog itself satisfies

$$\mathrm{Li}_2(z) + \mathrm{Li}_2\left(z^{-1}\right) = -\frac{\pi^2}{6} - \underbrace{\frac{1}{2}\log^2(-z)}_{\text{some product terms}}$$

But introduce

$$\mathcal{L}_2(z) = \mathrm{Im}(\mathrm{Li}_2(z) + \log(1-z)\log|z|)\,,$$

and this satisfies clean functional equation

$$\mathcal{L}_2(z) + \mathcal{L}_2(z^{-1}) = 0$$

for all $z \in \mathbb{C}$.

Similar for 5-term: Li_2 has many product terms, whereas $\mathcal{L}_2$ is constant 0.

Question was asked about whether something analogous works for higher depth multiply polylogs, by Gangl.

In particular, hope was maybe to obtain purely linear relations between iterated integrals, for the purpose of defining some analogues of the bloch groups but for multiple polylogarithms, via some symbols $\{x_1, x_2, \ldots\}_{n_1, n_2, \ldots}$, satisfying some relations.

Recall 1.2. Convention: recursively defined iterate integral

$$I(x_0; x_1, \ldots, x_n; x_{n+1}) = \int_{x_0}^{x_{n+1}} I(x_0; x_1, \ldots, x_{n-1}; t)\frac{dt}{t - x_n}\,.$$

And $I_{k_1,\ldots,k_r}(x_1, \cdots, x_r) = I(0; x_1, \{0\}^{k_1-1}, \ldots, x_r, \{0\}^{k_r-1}; 1)$.

Example 1.3. Following identity (Gangl) holds (on the symbol, some algebraic invariant) up to products of lower weight terms

$$I_{3,1}(1-x, 1-y) - I_{3,1}(x,y) + I_4\left(2\left[\frac{1-y}{1-x}\right] - 2\left[\frac{y}{x}\right] - \left[\frac{(1-x)y}{x(1-y)}\right] - \left[\frac{x}{x-1}\right] + \left[\frac{y}{y-1}\right]\right) = 0 \pmod{⧢}$$

Defining

$$I_4^{\text{clean}}(x) = I_4(x) + \frac{1}{4} I_3(x)\log(x)$$

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$$
\begin{aligned}
I_{3,1}^{\text{clean}}(x,y) = I_{3,1}(x,y) &+ \frac{1}{4}\log\left(\frac{x}{y}\right) I_{2,1}(x,y) + \frac{1}{4} I_2(x) I_2(y) \\
&+ \frac{1}{4} I_1(y) \left(-3 I_3\left(\frac{x}{y}\right) + I_2(x)\log(y) - I_2\left(\frac{x}{y}\right)\log\left(\frac{x}{y}\right) - I_3(x) \right) \\
&- \frac{1}{4} I_1(x) \left(-3 I_3\left(\frac{x}{y}\right) + I_2(y)\log(y) - I_2\left(\frac{x}{y}\right)\log\left(\frac{x}{y}\right) + 3 I_3(y) \right)
\end{aligned}
$$

Then one has the exact (up to constants and lower weight)

$$
\begin{aligned}
&I_{3,1}^{\text{clean}}(1-x,1-y) - I_{3,1}^{\text{clean}}(x,y) + 2 I_4^{\text{clean}}\left(\frac{1-y}{1-x}\right) - 2 I_4^{\text{clean}}\left(\frac{y}{x}\right) \\
&- I_4^{\text{clean}}\left(\frac{(1-x)y}{x(1-y)}\right) - I_4^{\text{clean}}\left(\frac{x}{x-1}\right) + I_4^{\text{clean}}\left(\frac{y}{y-1}\right) = 0 \pmod{\mathcal{S}}
\end{aligned}
$$

Moreover, for *any* other $I_{3,1}$ and I_4 identity, holding modulo products, these clean functions lift the identity to an exact one. The products are deal with in a universal way.

Goal is somehow to explain the origin of these functions, and give some applications. By using the single valued MPL's, obtained via the single valued map one can lift these symbol identities to numerically verifiable functional equations.

2. Algebraic setup

Our setting: let $H = \bigoplus_{n=0}^{\infty} H_n$ be a graded, connected Hopf algebra over $\mathbb{Q}$.

Meaning: $H_0 = \mathbb{Q}$ (connected), and we have coproduct Δ and multiplication μ, with usual compatibility conditions. Write $\Delta' = \Delta - 1 \otimes \mathrm{id} - \mathrm{id} \otimes$ for the reduced coproduct.

Main object:

Definition 2.1 (R operator, ,,Reinigungsmap")**.** Define linear map $R_n \colon H_n \to H_n$ in weight n by

$$R_n = n\,\mathrm{id} - \mu(\mathrm{id} \otimes R_\bullet)\Delta'$$

and $R_0 = \mathrm{id}$. Note: $R_1 = \mathrm{id}$ since $\Delta' H_1 = 0$.

[To emphasis the algebra, write $R_\bullet^H$, as necessary.]

Proposition 2.2. *$\ker R_\bullet = H_{>0} \cdot H_{>0}$, i.e. R kills all linear combinations of non-trivial products.*

Proof. Clear $\ker R_\bullet \subset H_{>0} \cdot H_{>0}$. If $a \in H_m$, then $0 = R_m a = na - \mu(\mathrm{id} \otimes R_\bullet)\Delta' a$ i.e.

$$a = \frac{1}{n}\mu(\mathrm{id} \otimes R_\bullet)\Delta' a \in H_{>0} \cdot H_{>0}\,.$$

By direct calculation, we see $x = ab \in H_2$, where $a, b \in H_1$ necessarily, has

$$
\begin{aligned}
R_2 x &= 2ab - \mu(\mathrm{id} \otimes \overbrace{R_1}^{\mathrm{id}}) \underbrace{\Delta'(ab)}_{a\otimes b + b \otimes a + \cancel{ab \otimes 1} + \cancel{1 \otimes ab}} \\
&= 0
\end{aligned}
$$

For higher weights, use induction. Let $x = a \cdot b \in H_{n_a} \cdot H_{n_b}$. Recall Sweedler's notation for coproduct

$$\Delta(a) = 1 \otimes a + a \otimes 1 + \sum_a a_{(1)} \otimes a_{(2)}\,.$$

Writing out

$$\Delta'(ab) = (\mu \otimes \mu)(\mathrm{id} \otimes \overbrace{\tau}^{\text{swap tensors}} \otimes \mathrm{id})(\Delta(a) \otimes \Delta(b)) - 1 \otimes ab - ab \otimes 1$$

gives

$$
\begin{aligned}
=& a \otimes b + \sum_b (b_{(1)} \otimes ab_{(2)} + ab_{(1)} \otimes b_{(2)}) \\
&+ b \otimes a + \sum_a (a_{(1)} \otimes a_{(2)} b + a_{(1)} b \otimes a_{(2)}) + \sum_{a,b} a_{(1)} b_{(1)} \otimes a_{(2)} b_{(2)}
\end{aligned}
$$

Applying $R_\bullet$ in second slot, by assumption kills lower weight products

$$(\mathrm{id} \otimes R_\bullet)\Delta'(ab) = a \otimes R(b) + \sum_b (\cancel{b_{(1)} \otimes R(ab_{(2)})} + ab_{(1)} \otimes R(b_{(2)})) + b \otimes R(a) + \sum_a (\cancel{a_{(1)} \otimes R(a_{(2)}b)} + a_{(1)}b \otimes R(a_{(2)})) + \sum_{a,b} \cancel{a_{(1)}b_{(1)} \otimes R(a_{(2)}b_{(2)})}$$

Then applying μ gives

$$\mu(\mathrm{id} \otimes R_\bullet)\Delta'(ab) = a\underbrace{\Big(R(b) + \sum_b b_{(1)} \otimes R(b_{(2)}) \Big)}_{R(b)+\mu(\mathrm{id} \otimes R)\Delta'(b)=n_b b} + b\underbrace{\Big(R(a) + \sum_a a_{(1)} \otimes R(a_{(2)}) \Big)}_{n_a a}$$
$$= (n_a + n_b)(ab)$$

So $R(ab) = (n_a + n_b)\,\mathrm{id}(ab) - \mu(\mathrm{id} \otimes R_\bullet)\Delta'(ab) = 0$ □

3. Symbols and integrals

Apply this to Goncharov's Hopf $\mathcal{A}$ algebra of (motivic) iterated integrals, and to symbols thereof.

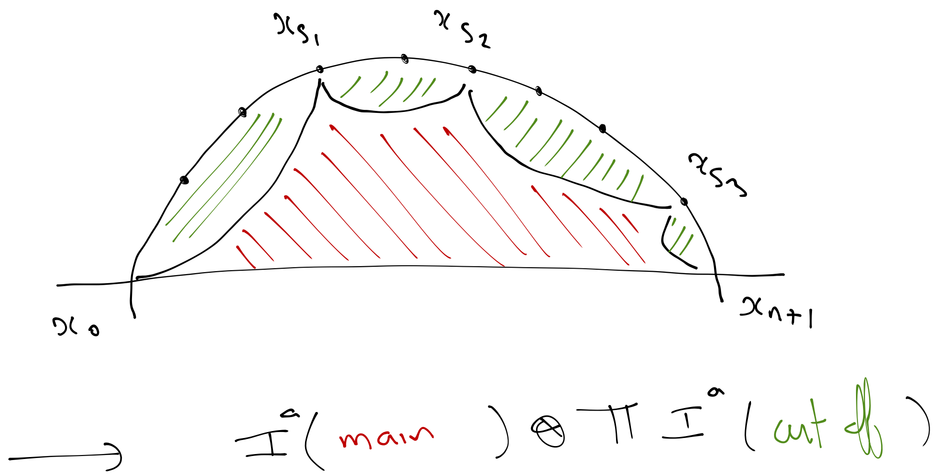
Recall 3.1. Hopf algebra spanned by $I^{\mathfrak{a}}(x_0; x_1, \ldots, x_n; x_{n+1})$ some formal/combinatorial version of $I(x_0; x_1, \ldots, x_n; x_{n+1})$, satisfying the same relations.

Coproduct formula:

$$\Delta I^{\mathfrak{a}}(x_0; x_1, \ldots, x_n; x_{n+1}) = \sum_{0=i_1<i_2<\cdots<i_{k+1}=n} I^{\mathfrak{a}}(x_0; x_{i_1}, \cdots, x_{i_k}; a_{n+1}) \otimes \prod_{p=0}^{k} I^{\mathfrak{a}}(x_{i_p}; x_{i_p+1}; \cdots; x_{i_{p+1}-1}; x_{i_{p+1}})$$

Semicircular coproduct

$$\Delta I^{\mathfrak{a}}(a; \vec{x}; b) = \sum_{S \subset \vec{x}} I^{\mathfrak{a}}(a; S; b) \otimes \prod I^{\mathfrak{a}}(\text{cuff-off segments})$$



Symbol is obtained as maximal iteration of this coproduct. Giving algebra morphism $\mathcal{S} \colon \mathcal{A} \to T^c(V)$, to the tensor coalgebra equipped with shuffle product $\sqcup\!\sqcup$ and Δ^{dec} deconcatenation coproduct.

In $T^c(V)$, the map $R_\bullet^T$ reduces to the shuffle product projection operator ρ_n defined by Duhr, Gangl, Rhodes.

$$\rho_n(a_1 \otimes \cdots \otimes a_n) = \rho_{n-1}(a_1 \otimes \cdots \otimes a_{n-1}) \otimes a_n - \rho_{n-1}(a_2 \otimes \cdots \otimes a_n) \otimes a_1 \,.$$

Why? Relation

$$\sum_{i=1}^{n-1}(a_1 \otimes \cdots \otimes a_i) \sqcup\!\sqcup \rho_{n-i}(a_{i+1} \otimes \cdots \otimes a_n) = n a_1 \otimes \cdots \otimes a_n$$

gives operator identity

$$\sqcup\!\sqcup(\mathrm{id} \otimes \rho_\bullet)(\Delta' + 1 \otimes \mathrm{id}) = n\,\mathrm{id} \implies \rho_n = n\,\mathrm{id} - \sqcup\!\sqcup\,(\mathrm{id} \otimes \rho_\bullet)\Delta'$$

I.e. same recursion and same initial conditions.

Remark 3.2. This operator identity form of ρ was the motivation for Duhr to make the construction in a general Hopf algebra

Applying R_n^H, we only need to keep the terms in Δ which are not (non-trivial) products. These occur only when we have a bunch of consecutive points (giving empty products), then a single long cut-off segment, and more consecutive points, i.e. $i_1, \ldots, i_l$ and $i_{l+n}, \ldots, i_k$ are consecutive (giving empty products on the right hand side, and a single cut-off segment $x_{i_l}, \cdots, x_{i_{l+n}}$).

Obtain recursion

$$\begin{aligned}
&R_\bullet I^{\mathfrak{a}}(x_0; x_1, \ldots, x_n; x_{n+1}) = n I^{\mathfrak{a}}(x_0; x_1, \ldots, x_n; x_{n+1}) \\
&- \sum_{\substack{\text{proper substring} \\ S \text{ of } x_0, \cdots, x_{n+1}}} I^{\mathfrak{a}}(x_0 \cdots x_{n+1}/S) R I^{\mathfrak{a}}(S) \\
&= n I^{\mathfrak{a}}(x_0; x_1, \ldots, x_n; x_{n+1}) \\
&- \sum_{k=1}^{n-1} \sum_{l=1}^{k+1} I^{\mathfrak{a}}(x_0; x_1, \ldots, x_{l-1}, x_{l+n-k}, \ldots, x_n; x_{n+1}) R I^{\mathfrak{a}}(x_{l-1}; x_l, \ldots, x_{l_n-k-1}; x_{l+n-k})
\end{aligned}$$

i.e. all proper substrings of length ≥ 1 in right hand term, with quotient of the original in the left.

Definition 3.3 (Clean, symbol level)**.** Recursive definition

$$\begin{aligned}
&I^{\text{clean}}(x_0; x_1, \ldots, x_n; x_{n+1}) = \frac{1}{n} R_\bullet I^{\mathfrak{a}}(x_0; x_1, \ldots, x_n; x_{n+1}) \\
&= I^{\mathfrak{a}}(x_0; x_1, \ldots, x_n; x_{n+1}) \\
&- \sum_{k=1}^{n-1} \sum_{l=1}^{k+1} \frac{n-k}{n} I^{\mathfrak{a}}(x_0; x_1, \ldots, x_{l-1}, x_{l+n-k}, \ldots, x_n; x_{n+1}) I^{\text{clean}}(x_{l-1}; x_l, \ldots, x_{l_n-k-1}; x_{l+n-k})
\end{aligned}$$

Claim 3.4. These functions satisfy clean symbol level functional equations.

Lemma 3.5. *The symbol map intertwines R^A of iterated integrals and $R^T = \rho$ of tensor symbols.*
True for $n = 1$ where both sides are id. *Next:*

$$\begin{aligned}
\rho_n \mathcal{S} &= n\mathcal{S} - \sqcup\!\sqcup(\mathrm{id} \otimes \rho_\bullet)(\Delta^{dec})'\mathcal{S} \\
&= n\mathcal{S} - \sqcup\!\sqcup(\mathcal{S} \otimes \rho_\bullet \mathcal{S})\Delta' \\
&= n\mathcal{S} - \sqcup\!\sqcup(\mathcal{S} \otimes \mathcal{S} R_\bullet)\Delta' \\
&= n\mathcal{S} - \mathcal{S}\mu(\mathrm{id} \otimes \mathcal{R}_\bullet)\Delta' \\
&= \mathcal{S} R_n
\end{aligned}$$

Corollary 3.6. *$\rho_\bullet \mathcal{S} I$ (the symbol of something reduce under ρ) is itself always an integrable symbol: it is the symbol of $R_n I$*

Corollary 3.7. *An identity which for symbols modulo products, can be lifted to an exact identity on symbols, even in a universal way. In particular, the kernel of ρ for integrable symbols consists of shuffle products of* integrable *symbols (a priori, only known shuffle products of some symbols).*

Proof. Suppose in weight n,

$$\rho_n \mathcal{S}\Big(\sum_i \lambda_i f_i\Big) = 0\,,$$

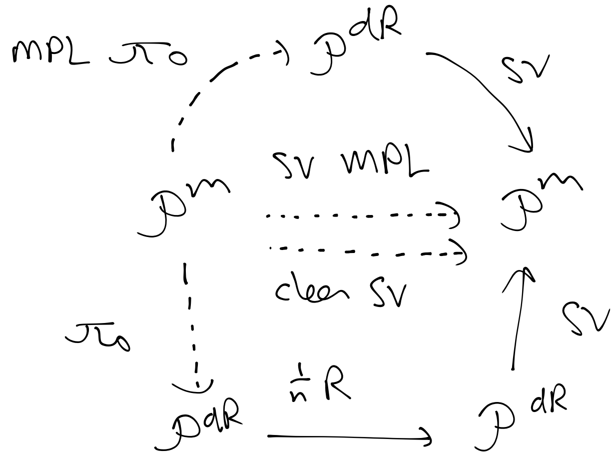
and define $f^{\text{clean}} = \frac{1}{n} R_n f$. Then

$$\begin{aligned}\mathcal{S}(\textstyle\sum_i \lambda_i (f_i)^{\text{clean}}) &= \mathcal{S}(\textstyle\sum_i \lambda_i \frac{1}{n} R_n f_i) \\ &= \frac{1}{n} \mathcal{S} R_n (\textstyle\sum_i \lambda_i f_i) \\ &= \frac{1}{n} \rho_n \mathcal{S}(\textstyle\sum_i \lambda_i f_i) = 0\end{aligned}$$

Since $f^{\text{clean}} - f = \frac{1}{n}\mu(\text{id} \otimes R_\bullet)\Delta' f$ is only formed from products, the identity for $\sum \lambda_i f_i^{\text{clean}}$ adds product terms make identity $\sum \lambda_i f_i = 0 \pmod{ш}$ hold exactly. □

4. Clean single-valued MPL's

More interesting/useful case is when we apply this to the de Rham images of motivic MPL's, and use the single valued map.



Recall 4.1. Motivic periods form a Hopf algebra comodule over de Rham motivic periods. The motivic MPL's $I^{\mathfrak{m}}(x_0; x_1, \ldots, x_n; x_{m+1})$ are mixed Tate, so form a graded comodule $\mathcal{H}$ over de Rham MPL's, whose weight 0 component is just $\overline{\mathbb{Q}}$. So we can define a projection to de Rham MPL's $\pi^{dr} \colon \mathcal{H} \to \mathcal{A}$ via the projection to weight 0 components.

$$\pi^{dr} = (\pi_0 \otimes \text{id})\Delta$$

where

$$\Delta \colon \mathcal{P}^{\mathfrak{m}} \to \mathcal{P}^{\mathfrak{dr}} \otimes \mathcal{P}^{\mathfrak{m}}$$

The de Rham MPL's form a connected graded Hopf algebra $\mathcal{A}$, with coproduct given by (suitable version of) Goncharov's semicircular coproduct formula. In particular, can define R_n on de Rham MPL's.

The single valued map generally exists in this setting. Defined to be the unique element of $G_{dR}(\mathcal{P}^{\mathfrak{m}})$ such that $sv \circ \sigma = \text{id}$, where σ is some twist of the real Frobenius (complex conjugation).

Get a map $\mathbf{s} \colon \mathcal{A} \to \mathcal{H}$, sending a de Rham MPL to a single-valued version.

Definition 4.2. Define the clean single valued (motivic) MPL.

$$C(x_0; x_1, \ldots, x_n; x_{n+1}) = \frac{1}{n} \mathbf{s}\, R_n \pi^{dr} I^{\mathfrak{m}}(x_0; x_1, \ldots, x_n; x_{n+1})$$

Write also $\mathcal{I}(x_0; x_1, \ldots, x_n; x_{n+1}) = \mathbf{s}\, \pi^{dr} I^{\mathfrak{m}}(x_0; x_1, \ldots, x_n; x_{n+1})$, for the single-valued projection of motivic integral $I^{\mathfrak{m}}$.

Theorem 4.3. *Every functional relation among MPL's gives a linear relation between the C-functions*

Proof. General functional relation

$$\sum_i \lambda_i I^{\mathfrak{m}}(\vec{x}_i) + f_n + \sum_{k=0}^{n-1} (2\pi i)^k f_k = 0$$

where $f_n \in \mathcal{H}_{>0} \cdot \mathcal{H}_{>0}$, $f_k \in \mathcal{H}_k$.

Project to de Rham, and apply the single valued map:

$$\sum_i \lambda_i \mathcal{I}(\vec{x}_i) + \mathbf{s}(\pi^{dr} f_n) = 0$$

where $\mathbf{s}(\pi^{dr} f_n) \in \mathcal{H}^{sv}_{>0} \cdot \mathcal{H}^{sv}_{>0}$.

Now apply R^{sv}, which kills product terms

$$\sum_i \lambda_i C(\vec{x}_i) = 0$$

□

The single valued projections $J = \mathbf{s}(\pi^{dr} I)$, some I, form a Hopf algebra $\mathcal{H}^{sv}$ via $\Delta^{sv}\,\mathbf{s}(\pi^{dr} I) = (\mathbf{s} \otimes \mathbf{s})\Delta(\pi^{dr} I)$. In particular, get same semicircular coproduct formula, with $\mathcal{I}$.

Lemma 4.4. *R^{sv} in $\mathcal{H}^{sv}$ and R^H are intertwined by $\mathbf{s}$. [Same recursive proof as for R^T and R^A under $\mathcal{S}$.]*

Corollary 4.5. *Recursive formula:*

$$\begin{aligned} C(x_0; x_1, \ldots, x_n; x_{n+1}) &= \frac{1}{n} R^{sv}_n \mathcal{I}(x_0; x_1, \ldots, x_n; x_{n+1}) \\ &= \mathcal{I}(x_0; x_1, \ldots, x_n; x_{n+1}) \\ &\quad - \sum_{k=1}^{n-1} \sum_{l=1}^{k+1} \frac{n-k}{n} \mathcal{I}(x_0; x_1, \ldots, x_{l-1}, x_{l+n-k}, \ldots, x_n; x_{n+1}) C(x_{l-1}; x_l, \ldots, x_{l_n-k-1}; x_{l+n-k}) \end{aligned}$$

Proposition 4.6. *The total (holomorphic) differential of C satisfies (for $n >= 2$):*

$$\begin{aligned} \partial C(x_0; x_1, \ldots, x_n; x_{n+1}) = \frac{n-1}{n} \Bigg[&\sum_{k=1}^{n} C(x_0; x_1, \ldots, \widehat{x_k}, \ldots, x_n; x_{n+1}) dI(x_{k-1}; x_k; x_{k+1}) \\ &- C(x_0; x_1, \ldots, x_{n-1}; x_n, \widehat{x_{n+1}}) dI(x_0; x_n; x_{n+1}) \\ &- C(\widehat{x_0}, x_1; x_2 \ldots, x_n; x_{n+1}) dI(x_0; x_1; x_{n+1}) \Bigg] \end{aligned}$$

Compare with:

$$dI(x_0; x_1, \ldots, x_n; x_{n+1}) = \sum_{k=1}^{n} I(x_0; x_1, \ldots, \widehat{x_k}, \ldots, x_n; x_{n+1}) dI(x_{k-1}; x_k; x_{k+1})$$

Remark 4.7. Somehow, we expect that every such relation for C comes from linearising an MPL relation in such a way. Should be able to prove this, if we can show that there are no relations among products of MPL's that are not already products of lower weight identities.

Proposition 4.8. *In particular, very general set of linear relation for C:*

- *Shuffle product:* $C(a; x \sqcup\!\sqcup y; b) = 0$
- *Path composition* $C(a, x, b) + C(b, x, c) = C(a, x, c)$
- *Path reversal* $C(a; x_1, \ldots, x_n; b) = (-1)^n C(b; x_n, \ldots, x_1; a)$
- *Shuffle antipode* $C(a_0; a_n, \ldots, a_1; a_{n+1}) = (-1)^{n+1} C(a_0; a_1, \ldots, a_n; a_{n+1})$
- *Shuffle regularisation of* $0's$

$$\begin{aligned} &C(0; 0^k, a_1, 0^{n_1-1}, \ldots, a_r, 0^{n_r-1}; a_{r+1}) \\ &= (-1)^k \sum_{i_1+\cdots+i_r=k} \binom{n_1+i_1-1}{i_1} \cdots \binom{n_r+i_r-1}{i_r} C(0; a_1, 0^{n_1+i_1}, \ldots, a_r, 0^{n_r+i_r}; a_{r+1}) \end{aligned}$$

- *Rescaling (even for $a_1 = 0$)*

$$C(za_0; za_1, \ldots, za_n; za_{n+1}) = C(a_0; a_1, \ldots, a_n; a_{n+1})$$

Remark 4.9. Lifting of symbol mod products level identities, to numerically testable identities among clean single valued MPL's, by replacing I with C.

Idea: take symbol mod products identity, lift to symbol identity. Integrate to a functional equation. Top level slice will be original combination of integrals (up to products). Cleaning the result gives the linear combination of C functions.

Remark 4.10. Duhr has lifted various identities found by Gangl and me with the symbol to numerically testable identities among the C functions. Passes numerical tests to high accuracy, results are always constant.

[MZV constants can appear in pure weight n piece of above relation.]

Theorem 4.11. *In the case of depth 1, can relate $C(0;1,0,\dots,0;z)$ to Zagier's single valued polylog:*

$$-C(0;1,\{0\}^n;z) = \mathcal{P}_n(z) + \sum_{k=2}^{n-1} \frac{1}{n-k+1} \frac{k-1}{n} \frac{\log^{n-k}(z\bar{z})}{(n-k)!} \mathcal{P}_k(z)\,,$$

where

$$\mathcal{P}_n(z) = \sum_{r=0}^{n-1} \frac{B_r}{r!} \log^r(z\bar{z})(\mathrm{Li}_{n-r}(z) - (-1)^n \mathrm{Li}_{n-r}(\bar{z}))\,.$$

So that $\mathcal{P}_n(z) = \begin{cases} 2P_n(z) & n \text{ odd} \\ 2iP_n(z) & n \text{ even} \end{cases}$ *in terms of Zagier's single valued MPL.*

5. Follow up work

- Can equip the vector space spanned by $C(a_0;a_1,\dots,a_n;a_{n+1})$ with a co lie bracket. Maybe somehow reminiscent of Goncharov's colie bracket on (generating series of) iterated integrals, but somehow only worked modulo lower depth integrals.

$$\delta C(a_0;a_1,\dots,a_n;a_{n+1}) = \sum_{0\le p<q\le n+1} C(a_0;a_1,\dots,a_p;a_{q+1},\dots,a_n;a_{n+1}) \wedge C(a_p;a_{p+1},\dots,a_q,a_{q+1})$$

- By taking the values of these functions are 1, can obtain some kind of clean single valued MZV. Do these satisfy any good properties? Some kind of generators for the space of indecomposible motivic MZV's. Unfortunately, need to go to high weight to get non-trivial structure (weight 11, to get 2 basis elements?), and current code is not optimised for this.
- Applications elsewhere? Elliptic case? Procedure requires two ingredients: mixed Tate to get the splitting/projection to de Rham. Then also require graded, connected Hopf algebra structure, to define the cleaning map.

  For the case of mixed elliptic motives, the first bit is okay. For elliptic MZV's, should also have the second requirement.