Zagier's polylogarithm conjecture and an explicit 4-ratio

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Outline

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Dedekind zeta function and polylogarithms

Dedekind zeta

Throughout, let F be a number field.

Definition (Dedekind zeta function)

Dedekind zeta function is defined by

$$\zeta_F(s) \coloneqq \sum_{I \neq (0)} \frac{1}{N(I)^s}, \quad \mathrm{Re}(s) > 1$$

- $I \subset \mathcal{O}_F$ non-zero ideal
- $N(I)$ the norm of I

Meromorphic on $\mathbb{C}$, simple pole at $s = 1$.

When $F = \mathbb{Q}$ obtain Riemann zeta $\zeta(s)$.

Analytic class number formula

Theorem (Analytic class number formula)

Have

$$\operatorname{Res}_{s=1} \zeta_F(s) \sim_{\mathbb{Q}^\times} \sqrt{|\Delta_F|} \pi^{r_2} \operatorname{Reg}_F \,,$$

where

- Δ_F *is the discriminant*
- r_2 *is the number of pairs of complex embeddings*
- Reg_F *is a determinant of logs of units of* F *(mysterious!)*

Analytic non-class number formula

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$$\operatorname{Res}_{s=1} \zeta_{\mathbb{Q}(\sqrt{5})}(s) = \frac{2}{5}\sqrt{5}\log\left(\frac{1+\sqrt{5}}{2}\right)$$

$$\operatorname{Res}_{s=1} \zeta_{\mathbb{Q}(\zeta_5)}(s) = \frac{4}{125}\pi^2\sqrt{5}\log\left(\frac{1+\sqrt{5}}{2}\right)$$

Polylogarithms

Analytic class number formula gives "$\zeta_F(1)$".

Higher values of ζ_F require *higher* logs.

Definition (m-th polylogarithm)

The m-th polylogarithm is

$$\mathrm{Li}_m(z) \coloneqq \sum_{m=1}^{\infty} \frac{z^n}{n^m}, \quad |z| < 1$$

Note: $\mathrm{Li}_1(z) = -\log(1-z)$.

Analytic continuation to $\mathbb{C}$ via $\mathrm{Li}_{m+1}(z) = \int_0^z \mathrm{Li}_m(t) \frac{dt}{t}$.

Single-valued polylogs

Definition (Bloch-Wigner-Ramakrishnan-Zagier polylogarithm)

A single-valued polylogarithm is defined by

$$\mathscr{L}_m(z) = \mathrm{Re}_m\left(\sum_{k=0}^{m-1} \frac{2^k B_k}{k!} \mathrm{Li}_{m-k}(z) \log^k(z)\right), \quad m \geq 2$$

- $\mathrm{Re}_m = \begin{cases} \mathrm{Re} & m \text{ odd} \\ \mathrm{Im} & m \text{ even} \end{cases}$
- B_k the m-th Bernoulli number

- $\mathscr{L}_1(z) = -\log|1-z|$
- $\mathscr{L}_2(z) = \mathrm{Im}(\mathrm{Li}_2(z) + \log(1-z)\log|z|)$
- $\mathscr{L}_3(z) = \mathrm{Re}(\mathrm{Li}_3(z) - \mathrm{Li}_2(z)\log|z| - \frac{1}{3}\log(1-z)\log^2|z|)$

Some notation

Write

$$d_m = \begin{cases} r_1 + r_2 & m \text{ odd} \\ r_2 & m \text{ even} \end{cases}$$

Conceptually: order of vanishing of $\zeta_F(-m)$

Extend $\mathscr{L}_m, \mathrm{Li}_m$ and $\sigma \colon F \to \mathbb{C}$ by linearity to

$$\mathbb{Z}[F] = \Big\{ \sum_i \lambda_i [x_i] \mid \lambda_i \in \mathbb{Z}, x_i \in F \Big\} ,$$

so

$$f\Big(\sum_i \lambda_i [x_i] \Big) = \sum_i \lambda_i f(x_i)$$

Zagier's conjecture

Conjecture (Zagier)

Let $m \geq 2$*, order embeddings of* F *so* $\sigma_i = \overline{\sigma_{i+r_1+r_2}}$.
Exists $y_1, \ldots, y_{d_m} \in \mathbb{Z}[F^\times]$ *so that*

$$\zeta_F(m) \sim_{\mathbb{Q}^\times} \sqrt{|\Delta_F|}\pi^{md_{m+1}} \det\left(\mathscr{L}_m(\sigma_i(y_j))\right)_{i,j=1}^{d_m}.$$

Recipe to find y_i inductively, using *numerical* algorithm.

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Recipe to find y_i inductively, using *numerical* algorithm.

$$\zeta_{\mathbb{Q}(\sqrt{5})}(3) \overset{?}{=} \frac{24}{125}\sqrt{5}\det\begin{pmatrix} \mathscr{L}_3(1) & \mathscr{L}_3(\frac{1+\sqrt{5}}{2}) \\ \mathscr{L}_3(1) & \mathscr{L}_3(\frac{1-\sqrt{5}}{2}) \end{pmatrix} \approx 1.0275480117\ldots$$

$$\zeta_{\mathbb{Q}(\zeta_5)}(2) \overset{?}{=} -\frac{2^3\sqrt{5}}{3\cdot 5^4}\pi^4\det\begin{pmatrix} \mathscr{L}_2(\zeta_5) & \mathscr{L}(\zeta_5^2) \\ \mathscr{L}_2(\zeta_5^2) & \mathscr{L}(\zeta_5^4) \end{pmatrix} \approx 1.0923496617\ldots$$

Status

- $n = 2$: Zagier (weak version)
  Bloch-Suslin $\sim$1981
  Goncharov (subtle fix)
- $n = 3$: $\sim$1993 Goncharov via Li_3 breakthrough
- $n = 4$: 2018 Goncharov-Rudenko via $\mathbf{Q}_4$ new geometric identity

Also known for special classes of field F

- Cyclotomic fields
- Abelian fields(?)

Goncharov has a *strategy* which can prove specific m via Borel's theorem

Caveat: requires heavy input of *currently unknown* $\mathscr{L}_m$-functional equations and identities

Cohomology and canonical classes

Borel's Theorem

1977 Borel defined regulator from K-theory

$$R_m^{\mathrm{bo}} \colon K_{2m-1}(\mathbb{C}) \to \mathbb{R}(m-1) =: \mathbb{R}(2\pi i)^{m-1}$$

and proved

Theorem (Borel)

Consider

$$\phi \colon K_{2m-1}(F) \to \bigoplus_{\sigma_i} K_{2m-1}(\mathbb{C}) \to \mathbb{Z}^{\mathrm{Hom}(F,\mathbb{C})} \otimes \mathbb{R}(m-1)$$

1. *ϕ is injective (mod torsion)*
2. *image is a lattice Λ_m^F (in invariants under cx conjugation)*
3. $$\zeta_F(m) \sim_{\mathbb{Q}^\times} \sqrt{|\Delta_F|}\pi^{md_{m+1}} \operatorname{covol}(\Lambda_F^m)$$

Canonical classes

Strategy: find formula for R_m^{bo} via $\mathscr{L}_m$.

Can rephrase via canonical cohomology classes.

$$K_m(F) \cong \pi_n(\mathrm{BGL}^+ F) \overset{\text{Suslin}}{\cong} \pi_n(\mathrm{BGL}_n^+ F) \xrightarrow{\text{Hurewicz}} H_n(\mathrm{BGL}_n^+ F)$$

This is injective mod torsion, so

$$K_n(F) \subset H_n(\mathrm{GL}_n, \mathbb{Q})$$

Fact

Borel regulator R_m^{bo} arises from certain canonical element

$$c_{2m-1} \in H_{\mathrm{cts}}^{2m-1}(\mathrm{GL}_m(\mathbb{C}), \mathbb{R}(m-1))$$

How to construct c_{2m-1} via $\mathscr{L}_m$?

Construction of c_{2m-1}

Construction of c_1

Represent $c \in H^{m-1}_{\text{cts}}(G, \mathbb{R})$ via cochain $\phi \colon G^m \to \mathbb{R}$.

Fact

$$\phi_1 \;\; \mathrm{GL}_1(\mathbb{C})^2 \to \mathbb{R}$$
$$\phi_1(g_1, g_2) = \log(\det(g_1^{-1} g_2))$$

defines 1-cocycle, and represents c_1.

Cocycle condition corresponds to log functional equation.

$$\log(x) - \log(y) = \log\left(\frac{x}{y}\right)$$

Construction of c_3

Introduce coordinates

$$\mathrm{Conf}_m(n) = \{(v_1 \ldots, v_n) | v_i \in \mathbb{C}^m\} \Big/ \mathrm{GL}_m$$

Write

$$\langle i_1, \ldots, i_m \rangle = \det(v_{i_1} \cdots v_{i_m})$$

Classical cross-ratio is a function on $\mathrm{Conf}_4(2)$

$$\begin{aligned} \mathrm{cr}(v_1, \ldots, v_4) &= \frac{\langle 13 \rangle \langle 24 \rangle}{\langle 14 \rangle \langle 23 \rangle} \\ &= \frac{z_1 - z_3}{z_1 - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_3} \end{aligned}$$

where $z_i \in \mathbb{P}^1(\mathbb{C}) \leftrightarrow v_i \in \mathbb{C}^2$

Construction of c_3

Theorem (Bloch)

$$\phi_2 : \mathrm{GL}_2(\mathbb{C})^4 \to \mathbb{R}$$
$$\phi_2(g_1, \ldots, g_4) = \mathscr{L}_2(\mathrm{cr}(g_1 v, \ldots, g_4 v)))$$

defines 3-cocycle, and represents c_3.

Get Zagier's Conjecture for $n = 2$ via Borel.

Cocycle condition corresponds to non-trivial $\mathscr{L}_2$ functional equation

$$\mathscr{L}_2 \left([x] + [y] + \left[\frac{1-x}{1-xy}\right] + [1-xy] + \left[\frac{1-y}{1-xy}\right] \right) = 0$$

Famous five-term relation.

Construction of c_5

Goncharov defines a (pre-)triple-ratio

$$\mathrm{cr}_3(v_1, \ldots, v_6) = \frac{\langle 124 \rangle \langle 235 \rangle \langle 316 \rangle}{\langle 125 \rangle \langle 236 \rangle \langle 314 \rangle}$$

Theorem (Goncharov)

$$\phi_3 : \mathrm{GL}_3(\mathbb{C})^6 \to \mathbb{R}$$
$$\phi_3(g_1, \ldots, g_6) = \mathrm{Alt}_6 \, \mathscr{L}_3(\mathrm{cr}_3(g_1 v, \ldots, g_6 v))$$

defines 5-cocycle, and represents c_5.

Get Zagier's Conjecture for $n = 3$ via Borel.

Cocycle condition corresponds to 840-term Li_3 functional equation. Related 22-term functional equation

22-term

Theorem (22-term relation, Goncharov)

$$\mathscr{L}_3\left(\operatorname{Cyc}\left([z]+\left[-\frac{x(yz-z+1)}{xz-x+1}\right]+\left[\frac{yz-z+1}{y(xz-x+1)}\right]-\left[\frac{yz-z+1}{yz(xz-x+1)}\right]\right.\right.$$
$$\left.\left.+[xz-x+1]-\left[\frac{xz-x+1}{z}\right]+\left[\frac{xz-x+1}{xz}\right]\right)+[-xyz]\right)=3\,\mathscr{L}_3(1)$$

How to generalise the cross-ratio and triple-ratio? Naive guesses like

$$\operatorname{cr}_4(v_1,\ldots,v_8)=\frac{\langle 1235\rangle\langle 2346\rangle\langle 3457\rangle\langle 4518\rangle}{\langle 1238\rangle\langle 2345\rangle\langle 3456\rangle\langle 4517\rangle}$$

fail. Not functional equations for $\mathscr{L}_4$!

Higher ratios and Grassmannian polylogs

m-ratio

Despite failure of naive generalisation, Goncharov conjectures some generalisation exists

Conjecture

For $m \geq 2$, there exists

$$\sum_i \lambda_i [r_i] \, , \quad r_i \in \mathbb{Q}(\mathrm{Conf}_{2m}(m)) \tag{1}$$

such that

$$\phi_m(g_1, \ldots, g_{2m}) = \mathrm{Alt}_{2m} \sum_i \lambda_i \, \mathscr{L}_m(r_i(g_1 v, \ldots, g_{2m} v))$$

is a $(2m-1)$-cocycle and represents c_{2m-1}.

Formal linear combination (1) is called an m-ratio

Goncharov-Rudenko show 4-ratio exists, but do not construct it.

Grassmannian polylogs

Key tool: Grassmannian polylogs Gr_m

Definition (Grassmannian polylog)

Gr_m is the multivalued analytic funtion defined by

$$d\,\mathrm{Gr}_m(v_1,\ldots,v_{2m}) = \mathrm{Alt}_{2m}\,\mathcal{A}(v_1,\ldots,v_m \mid v_{m+1},\ldots,v_{2m}) \cdot d\log\langle m+1,\ldots,2m\rangle$$

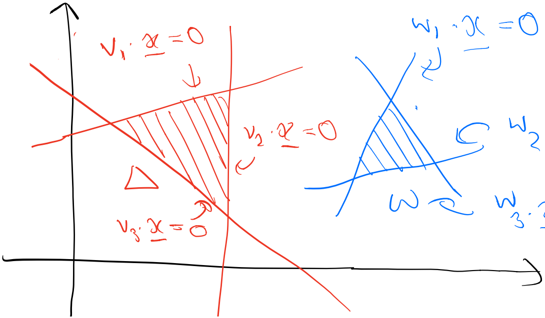
where $\mathcal{A}(v_1,\ldots,v_m \mid w_1,\ldots,w_m)$ is geometrically defined Aomoto polylogarithm.

Theorem (Goncharov)

A single-valued version of Gr_m *represents* c_{2m-1}

$\mathrm{Alt}_{2m+1}\,\mathrm{Gr}_m = 0$ because each symbol term depends on $2m-1$ points.

Aomoto polylogs



$$\mathcal{A}(v_1, \ldots, v_n \mid w_1, \ldots, w_n) = \int_\Delta d\log\left(\frac{w_2 \cdot \mathbf{x}}{w_1 \cdot \mathbf{x}}\right) \wedge d\log\left(\frac{w_n \cdot \mathbf{x}}{w_1 \cdot \mathbf{x}}\right)$$

Reduction of Gr_m

Goal: Rewrite Gr_m in terms of $\mathscr{L}_m$

Problem: An obstruction exists, meaning this is impossible

Fix: Can modify Gr_m by trivial coboundary terms depending on $\leq 2m-1$ points. Find trivial coboundary correction which kills obstruction.

Goncharov-Rudenko already do this in weight 4.

Explicit reduction of Gr_4 and a 4-ratio

$I_{3,1}$ and ρ-coordinates

Definition ($I_{3,1}$)

$I_{3,1}$ multiple polylog is defined by

$$I_{3,1}(x,y) = \mathrm{Li}_{3,1}\left(\frac{y}{x}, \frac{1}{y}\right) = \sum_{0<n<m} \frac{y^{n-m}x^{-m}}{n^3 m}$$

Definition (ρ-coordinates)

Coordinates on $\mathrm{Conf}_8(4)$

$$\rho_i = \underbrace{\rho_{i,i+1,i_2}}_{\text{mod } 6} = \frac{\langle i, i+1, i+2, 7\rangle}{\langle i, i+1, i+2, 8\rangle}$$

Shorthand $\rho_{i,j} = \rho_i - \rho_j$

Reduction of Gr_4

Theorem (CGR, 2019)

Modulo products

$$\frac{7}{144}\,\mathrm{Gr}_4 \;=\; \mathrm{Alt}_8\Big[I_{3,1}\Big(\frac{\rho_{1,2}\rho_{3,4}}{\rho_{3,2}\rho_{1,4}}, \frac{\rho_1}{\rho_{1,4}}\Big) + 2I_{3,1}\Big(\frac{\rho_{1,2}}{\rho_1}, \frac{\rho_{3,2}}{\rho_{3,4}}\Big) + 6\,\mathrm{Li}_4\Big(\frac{\rho_1\rho_{3,2}}{\rho_{1,2}\rho_{3,4}}\Big)\Big].$$

Proof.

Found with computer assistance. Explicit calculation of the symbol by hand. □

Note: some structure in this reduction.

Makes explicit first step of Goncharov-Rudenko.

Behaviour of $I_{3,1}$

Heuristic: modulo explicit Li_4 terms

$$I_{3,1}(x, y) \sim \mathrm{Li}_2(x) \wedge \mathrm{Li}_2(y)\,.$$

Related to the obstruction to $\mathrm{Gr}_4 = \mathrm{Li}_4$'s

Theorem (Gangl, 2012)

Exist $f_i(x, y, z)$ rational functions, so that modulo products

$$I_{3,1}\Big(z, [x] + [y] + \Big[\frac{1-x}{1-xy}\Big] + [1-xy] + \Big[\frac{1-y}{1-xy}\Big]\Big) \\ = \sum_{i=1}^{122} c_i \,\mathrm{Li}_4(f_i(x, y, z)) =: V(z, [x, y])$$

Found with computer assistance. Goncharov-Rudenko have a geometric derivation.

Gr_4 coboundary

Goncharov gives Gr_4 coboundary as

$$\mathrm{Alt}_8\, I_{3,1}(\mathrm{cr}(34 \mid 2567), \mathrm{cr}(67 \mid 1345))\,,$$

with projected cross-ratio

$$\mathrm{cr}(ab \mid cdef) = \frac{\langle abce \rangle}{\langle abcf \rangle} \frac{\langle abdf \rangle}{\langle abde \rangle}\,.$$

Symmetrise $I_{3,1}$ for convenience using

$$I_{3,1}(x,y) + I_{3,1}(x^{-1}, y) = \mathrm{Li}_4\text{'s}\,,$$
$$I_{3,1}(x,y) + I_{3,1}(1-x, y) = \mathrm{Li}_4\text{'s}\,.$$

Write $\mathrm{Sym}_{36}(x,y)$ for these extra Li_4 terms.

Gr_4 to Li_4's

Theorem (Explicit 4-ratio, CGR, 2019)

$$\frac{7}{144}\,\mathrm{Gr}_4 + 2\,\mathrm{Alt}_8\, I_{3,1}^{\mathrm{sym}}(\mathrm{cr}(34|2567), \mathrm{cr}(67|1345)) =$$

$$\mathrm{Alt}_8 \Big\{ - V\big(\tfrac{\rho_4}{\rho_1};\ [\tfrac{\rho_{4,2}}{\rho_{4,1}}; \tfrac{\rho_{4,1}}{\rho_{4,3}}] - [\mathrm{cr}(43|2685); \mathrm{cr}(48|7653)]$$

$$+ \tfrac{1}{4}[\mathrm{cr}(43|1256); \mathrm{cr}(43|1268)] - \tfrac{1}{12}[\mathrm{cr}(43|1256); \mathrm{cr}(42|1365)]\big)$$

$$+ V(\tfrac{\rho_2}{\rho_1};\ -[\mathrm{cr}(43|2685; \mathrm{cr}(48|7653)] + [\mathrm{cr}(48|7235; \mathrm{cr}(48|7263)]$$

$$+ \tfrac{1}{2}[\mathrm{cr}(46|5238; 43|2568)])$$

$$+ \mathrm{Sym}_{36}(\tfrac{\rho_{1,2}\rho_{3,4}}{\rho_{1,4}\rho_{3,2}}, \tfrac{\rho_1}{\rho_{1,4}}) + 2\,\mathrm{Sym}_{36}(\tfrac{\rho_{1,2}}{\rho_1}, \tfrac{\rho_{3,2}}{\rho_{3,4}}) + 6\,\mathrm{Li}_4(\tfrac{\rho_1\rho_{3,2}}{\rho_{1,2}\rho_{3,4}}) \Big\}.$$

Corollary

Symmetrising over 9 points gives a new Li_4 functional equation with 1775 S_8-orbits. Compute assistance gives 368 S_8 orbits.

Higher weight

Theorem (CGR, 2019)

For any $m \geq 2$, have

$$-\frac{2m-1}{m!(m-1)!}\,\mathrm{Gr}_m = \mathrm{Alt}_{2m}\, I(0; 0, \rho_1, \rho_2, \ldots, \rho_{m-1}; \rho_m)\,,$$

with generalised ρ-coordinates.

Theorem (CGR, 2019)

Expression for Gr_5 in terms of four $I_{4,1}$ terms and 2 Li_5, under Alt_{10}. Coboundary correction term expressed via two $I_{4,1}$ terms.

This is a starting point for reduction in weight ≥ 5. (In progress.)

Summary

- Statement of Zagier's polylogarithm conjecture on $\zeta_F(m)$
- Goncharov's expected strategy for proof involving m-ratios
- Expressions for Grassmannian polylogs
- Explicit reduction of Gr_4 and 4-ratio
- New functional equations for Li_4
- Progress in weight 5