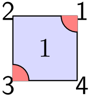
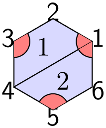
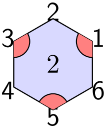
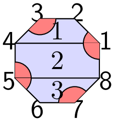
Cluster functions, depth 1, 2

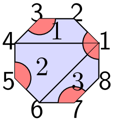
$\mathcal{C}_n^{(1)}(x_1,\ldots,x_4) \coloneqq \mathrm{Cyc}_4^{(-1)^{n-1}}\, I_n\Big($  $\Big)$

$\mathcal{C}_n^{(2)}(x_1,\ldots,x_6) \coloneqq \mathrm{Cyc}_6^{(-1)^{n-1}}\, \Big(I^N_{n-1,1}\Big($  $\big) - \frac{2(n-1)}{3} I_n\big($  $\big)\Big)$

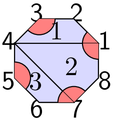
Cluster function, depth 3

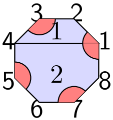
$$\mathcal{C}_n^{(3)}(x_1,\ldots,x_8) \coloneqq$$

$$\mathrm{Cyc}_8^{(-1)^{n-1}}\Big[I^N_{n-2,1,1}\Big($$


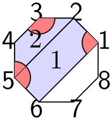
$$\Big) - I^N_{n-2,1,1}\Big($$


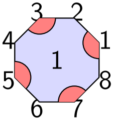
$$\Big)$$

$$- I^N_{n-2,1,1}\Big($$


$$\Big) - 2I^N_{n-2,2}\Big($$


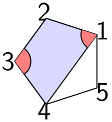
$$\Big)$$

$$- I^N_{n-1,1}\Big($$


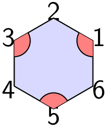
$$\Big) - \frac{5(n-1)(n-2)}{8} I^N_n\Big($$


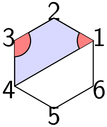
$$\Big)\Big]$$

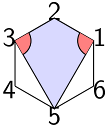
Q_2 (five-term), weight 2, depth 2 'off-diagonal' (CGR)

$\mathrm{Cyc}_5^+ \, \mathcal{C}_2^{(1)} \Big($ 

$\Big) = 0$ (mod products)

$\mathrm{Cyc}_6^- \Bigg(\mathcal{C}_2^{(2)} \Big($ 

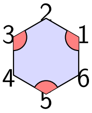
$\Big) - \frac{2}{3} \mathcal{C}_2^{(1)} \Big($ 

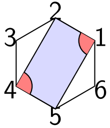
$\Big) - \frac{2}{3} \mathcal{C}_2^{(1)} \Big($ 

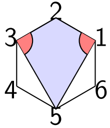
$\Big) \Bigg)$

$= 0$ (mod products)

Q_3, Q_4 identities (Goncharov-Rudenko)

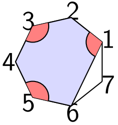
$\mathrm{Cyc}_6^+ \Bigg(\mathcal{C}_3^{(2)}\Big($ 

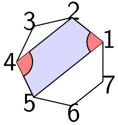
$\Big) + \mathcal{C}_3^{(1)}\Big($ 

$\Big) - 2\mathcal{C}_3^{(1)}\Big($ 

$\Big)\Bigg)$

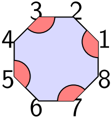
$= 0$ (mod products) .

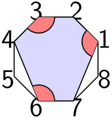
$\mathrm{Cyc}_7^+ \Bigg(\mathcal{C}_4^{(2)}\Big($ 

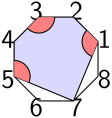
$\Big) - \frac{1}{3}\mathcal{C}_4^{(1)}\Big($ 

$\Big)\Bigg) = 0$ (mod products) .

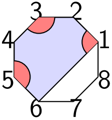
Weight 4: depth 3 ‘off-diagonal’ identity (CGR)

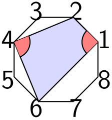
$\mathrm{Cyc}_8^- \Big(\mathcal{C}_4^{(3)} \Big($ 

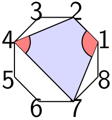
$\Big) - \frac{3}{2}\mathcal{C}_4^{(2)} \Big($ 

$\Big) + 3\mathcal{C}_4^{(2)} \Big($ 

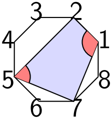
$\Big)$

$+ \frac{1}{2}\mathcal{C}_4^{(2)} \Big($ 

$\Big) - \frac{1}{2}\mathcal{C}_4^{(1)} \Big($ 

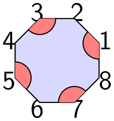
$\Big) - \frac{1}{2}\mathcal{C}_4^{(1)} \Big($ 

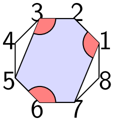
$\Big)$

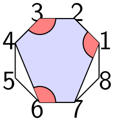
$- \frac{1}{2}\mathcal{C}_4^{(1)} \Big($ 

$\Big)\Big) = 0$ (mod products)

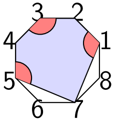
Q_5 identity (CGR)

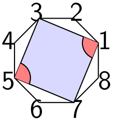
$\mathrm{Cyc}_8^+ \Bigg(\mathcal{C}_5^{(3)} \Big($ 

$\Big) + 3\mathcal{C}_5^{(2)} \Big($ 

$\Big) - 6\mathcal{C}_5^{(2)} \Big($ 

$\Big)$

$+6\mathcal{C}_5^{(2)} \Big($ 

$\Big) - \frac{1}{2}\mathcal{C}_5^{(1)} \Big($ 

$\Big) \Bigg) = 0$ (mod products)

Also for $\mathcal{C}_3^{(\bullet)}$, ‘off-diagonal’ identity.

Reduction of $I_{3,2}$

$$
\begin{aligned}
&I_{3,2}(x,y) = \\
&-I_{4,1}\left(1-x,-\tfrac{x-y}{y}\right)+I_{4,1}\left(\tfrac{1}{x},\tfrac{1}{y}\right)-I_{4,1}\left(-\tfrac{1-x}{x},\tfrac{y}{x}\right)-I_{4,1}\left(x,-\tfrac{x-y}{y}\right)\\
&-I_{4,1}\left(\tfrac{1}{1-y},\tfrac{x-y}{x(1-y)}\right)-I_{4,1}\left(\tfrac{1-x}{1-y},\tfrac{1}{1-y}\right)-I_{4,1}\left(\tfrac{1-x}{1-y},-\tfrac{y}{1-y}\right)+I_{4,1}\left(\tfrac{x(1-y)}{x-y},-\tfrac{1-y}{y}\right)\\
&-I_{4,1}\left(\tfrac{x}{y},\tfrac{1}{y}\right)-I_{4,1}\left(1,\tfrac{1}{1-x}\right)+I_{4,1}\left(\tfrac{1}{1-y},\tfrac{1}{1-y}\right)-I_{4,1}\left(1-y,-\tfrac{1-y}{y}\right)\\
&+I_5\left(-\tfrac{1-x}{x-y}\right)+3I_5\left(\tfrac{x}{x-y}\right)-I_5\left(\tfrac{x(1-y)}{x-y}\right)+8I_5\left(-\tfrac{1-x}{y}\right)-5I_5\left(\tfrac{x}{y}\right)\\
&+3I_5\left(-\tfrac{x-y}{(1-x)y}\right)+4I_5\left(-\tfrac{x-y}{xy}\right)+6I_5\left(\tfrac{1}{1-x}\right)+I_5\left(\tfrac{1}{x}\right)+I_5\left(\tfrac{1}{1-y}\right)\\
&-4I_5\left(-\tfrac{1}{y}\right)-4I_5\left(\tfrac{1}{y}\right)\qquad\text{(mod products)}
\end{aligned}
$$

Reduction of $I_{3,1,1}$

$$I_{3,1,1}(x,y,z) \quad = \quad I_{3,2}(x,y)$$

$$- I_{4,1}\left(1-x, -\tfrac{y-z}{z}\right) - I_{4,1}\left(\tfrac{1}{x}, \tfrac{1}{y}\right) - I_{4,1}\left(\tfrac{1}{x}, \tfrac{1}{z}\right) - I_{4,1}\left(x, \tfrac{y-z}{1-z}\right) + I_{4,1}\left(x, -\tfrac{z}{1-z}\right)$$

$$- I_{4,1}\left(\tfrac{1}{1-y}, -\tfrac{y-z}{(1-y)z}\right) + I_{4,1}\left(\tfrac{x}{z}, \tfrac{y}{z}\right) + I_{4,1}\left(\tfrac{1-x}{1-y}, \tfrac{y(1-z)}{(1-y)z}\right) + I_{4,1}\left(\tfrac{1-x}{1-y}, -\tfrac{y-z}{(1-y)z}\right)$$

$$+ I_{4,1}\left(1-y, \tfrac{(1-y)z}{y(1-z)}\right) - I_{4,1}\left(\tfrac{x(1-y)}{x-y}, \tfrac{(1-y)z}{y(1-z)}\right) - I_{4,1}\left(\tfrac{x(1-y)}{x-y}, -\tfrac{(1-y)z}{y-z}\right) + I_{4,1}\left(\tfrac{x}{y}, \tfrac{y-z}{y(1-z)}\right)$$

$$- I_{4,1}\left(\tfrac{x}{y}, \tfrac{x(1-z)}{(1-x)z}\right) - I_{4,1}\left(\tfrac{1-x}{1-z}, \tfrac{y}{z}\right) - I_{4,1}\left(1-z, \tfrac{z}{y}\right) + I_{4,1}\left(\tfrac{x(1-z)}{x-z}, \tfrac{y-z}{x-z}\right)$$

$$- I_{4,1}\left(\tfrac{x(1-z)}{x-z}, -\tfrac{z}{x-z}\right) + I_{4,1}\left(\tfrac{x(1-z)}{x-z}, \tfrac{(x-y)z}{y(x-z)}\right) - I_{4,1}\left(\tfrac{x-z}{y-z}, -\tfrac{x-z}{(1-x)z}\right) + I_{4,1}\left(\tfrac{y}{z}, \tfrac{1}{z}\right)$$

$$+ I_{4,1}\left(-\tfrac{x-z}{z}, -\tfrac{x-z}{(1-x)z}\right) - I_{4,1}\left(\tfrac{y(x-z)}{(x-y)z}, -\tfrac{x-z}{(1-x)z}\right)$$

$$- I_5\left(\tfrac{1}{1-x}\right) - I_5\left(\tfrac{1}{1-y}\right) + 2I_5\left(\tfrac{x(1-y)}{x-y}\right) + 2I_5\left(\tfrac{x}{y}\right) + I_5\left(-\tfrac{x-y}{(1-x)y}\right)$$

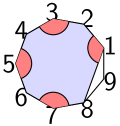
$$+ I_5\left(\tfrac{1-x}{1-z}\right) + I_5\left(\tfrac{1-z}{y-z}\right) - 4I_5\left(\tfrac{x(1-z)}{y-z}\right) - I_5\left(-\tfrac{(x-y)(1-z)}{(1-x)(y-z)}\right) + I_5\left(\tfrac{x-z}{y-z}\right)$$

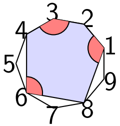
$$+ 3I_5\left(\tfrac{x}{z}\right) - I_5\left(\tfrac{y}{z}\right) - I_5\left(-\tfrac{1-z}{z}\right) + 4I_5\left(-\tfrac{x(1-z)}{z}\right) + I_5\left(\tfrac{x(1-z)}{(1-x)z}\right) + 4I_5\left(\tfrac{y(1-z)}{z}\right)$$

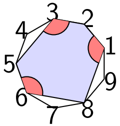
$$- 4I_5\left(\tfrac{xy(1-z)}{(x-y)z}\right) - I_5\left(-\tfrac{x-z}{z}\right) + I_5\left(-\tfrac{x-z}{(1-x)z}\right) + I_5\left(\tfrac{y(x-z)}{(x-y)z}\right) + 3I_5\left(-\tfrac{y-z}{z}\right)$$

$$+ I_5\left(-\tfrac{y-z}{(1-y)z}\right) + I_5\left(-\tfrac{x(y-z)}{(x-y)z}\right) \quad \text{(mod products)}$$

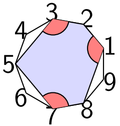
Q_6 identity (CGR)

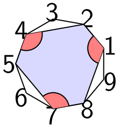
$\mathrm{Cyc}_9^+ \Big(\mathcal{C}_6^{(3)} \Big($ 

$\Big) + \frac{3}{4}\mathcal{C}_6^{(2)} \Big($ 

$\Big) - \frac{3}{4}\mathcal{C}_6^{(2)} \Big($ 

$\Big)$

$+ \frac{3}{4}\mathcal{C}_6^{(2)} \Big($ 

$\Big) - \frac{1}{4}\mathcal{C}_6^{(2)} \Big($ 

$\Big) \Big) = 0$ (mod products)