

Cluster functions, depth 1, 2

$$\mathcal{C}_n^{(1)}(x_1, \dots, x_4) := \text{Cyc}_4^{(-1)^{n-1}} I_n \left(\begin{array}{c} 2 \quad 1 \\ \diagdown \quad \diagup \\ 1 \\ \diagup \quad \diagdown \\ 3 \quad 4 \end{array} \right)$$

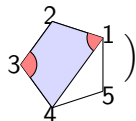
$$\mathcal{C}_n^{(2)}(x_1, \dots, x_6) := \text{Cyc}_6^{(-1)^{n-1}} \left(I_{n-1,1}^N \left(\begin{array}{c} 2 \\ \diagdown \quad \diagup \\ 3 \quad 1 \\ \diagdown \quad \diagup \\ 4 \quad 2 \\ \diagdown \quad \diagup \\ 5 \quad 6 \end{array} \right) - \frac{2(n-1)}{3} I_n \left(\begin{array}{c} 2 \\ \diagdown \quad \diagup \\ 3 \quad 1 \\ \diagdown \quad \diagup \\ 4 \quad 2 \\ \diagdown \quad \diagup \\ 5 \quad 6 \end{array} \right) \right)$$

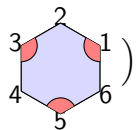
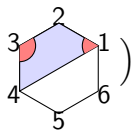
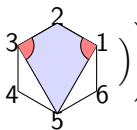
Cluster function, depth 3

$$\mathcal{C}_n^{(3)}(x_1, \dots, x_8) :=$$

$$\begin{aligned} & \text{Cyc}_8^{(-1)^{n-1}} \left[I_{n-2,1,1}^N \left(\begin{array}{c} 3 \quad 2 \\ 4 \quad 1 \quad 1 \\ 2 \\ 5 \quad 3 \quad 8 \\ 6 \quad 7 \end{array} \right) - I_{n-2,1,1}^N \left(\begin{array}{c} 3 \quad 2 \\ 4 \quad 1 \quad 1 \\ 2 \\ 5 \quad 2 \quad 3 \quad 8 \\ 6 \quad 7 \end{array} \right) \right. \\ & - I_{n-2,1,1}^N \left(\begin{array}{c} 3 \quad 2 \\ 4 \quad 1 \quad 1 \\ 2 \\ 5 \quad 3 \quad 8 \\ 6 \quad 7 \end{array} \right) - 2I_{n-2,2}^N \left(\begin{array}{c} 3 \quad 2 \\ 4 \quad 1 \quad 1 \\ 2 \\ 5 \quad 2 \quad 3 \quad 8 \\ 6 \quad 7 \end{array} \right) \\ & \left. - I_{n-1,1}^N \left(\begin{array}{c} 3 \quad 2 \\ 4 \quad 2 \quad 1 \\ 1 \\ 5 \quad 1 \quad 8 \\ 6 \quad 7 \end{array} \right) - \frac{5(n-1)(n-2)}{8} I_n^N \left(\begin{array}{c} 3 \quad 2 \\ 4 \quad 1 \quad 1 \\ 1 \\ 5 \quad 1 \quad 8 \\ 6 \quad 7 \end{array} \right) \right] \end{aligned}$$

Q_2 (five-term), weight 2, depth 2 'off-diagonal' (CGR)

$$\text{Cyc}_5^+ C_2^{(1)} \left(\begin{array}{c} 2 \\ \text{3} \quad \text{1} \\ \text{4} \quad \text{5} \end{array} \right) = 0 \pmod{\text{products}}$$


$$\text{Cyc}_6^- \left(C_2^{(2)} \left(\begin{array}{c} 2 \\ \text{3} \quad \text{1} \\ \text{4} \quad \text{5} \quad \text{6} \end{array} \right) - \frac{2}{3} C_2^{(1)} \left(\begin{array}{c} 2 \\ \text{3} \quad \text{1} \\ \text{4} \quad \text{5} \quad \text{6} \end{array} \right) - \frac{2}{3} C_2^{(1)} \left(\begin{array}{c} 2 \\ \text{3} \quad \text{1} \\ \text{4} \quad \text{5} \quad \text{6} \end{array} \right) \right) = 0 \pmod{\text{products}}$$




Q_3, Q_4 identities (Goncharov-Rudenko)

$$\text{Cyc}_6^+ \left(c_3^{(2)} \left(\begin{array}{c} 2 \\ 3 \quad 1 \\ 4 \quad 6 \\ 5 \end{array} \right) + c_3^{(1)} \left(\begin{array}{c} 2 \\ 3 \quad 1 \\ 4 \quad 6 \\ 5 \end{array} \right) - 2c_3^{(1)} \left(\begin{array}{c} 2 \\ 3 \quad 1 \\ 4 \quad 6 \\ 5 \end{array} \right) \right) \\ = 0 \text{ (mod products).}$$

The diagrammatic equation for Cyc_6^+ involves three configurations of a 6-gon with vertices labeled 1 through 6. The first configuration, $c_3^{(2)}$, shows a shaded quadrilateral with vertices 1, 2, 3, and 4, and red angles at vertices 3, 4, and 5. The second configuration, $c_3^{(1)}$, shows a shaded quadrilateral with vertices 1, 2, 3, and 4, and red angles at vertices 1, 4, and 5. The third configuration, $2c_3^{(1)}$, shows a shaded quadrilateral with vertices 1, 2, 3, and 4, and red angles at vertices 1, 3, and 5.

$$\text{Cyc}_7^+ \left(c_4^{(2)} \left(\begin{array}{c} 3 \quad 2 \\ 4 \quad 1 \\ 5 \quad 6 \\ 7 \end{array} \right) - \frac{1}{3} c_4^{(1)} \left(\begin{array}{c} 3 \quad 2 \\ 4 \quad 1 \\ 5 \quad 6 \\ 7 \end{array} \right) \right) = 0 \text{ (mod products).}$$

The diagrammatic equation for Cyc_7^+ involves two configurations of a 7-gon with vertices labeled 1 through 7. The first configuration, $c_4^{(2)}$, shows a shaded pentagon with vertices 1, 2, 3, 4, and 5, and red angles at vertices 3, 4, 5, and 6. The second configuration, $\frac{1}{3}c_4^{(1)}$, shows a shaded pentagon with vertices 1, 2, 3, 4, and 5, and red angles at vertices 1, 4, and 5.

Weight 4: depth 3 'off-diagonal' identity (CGR)

$$\begin{aligned}
 & \text{Cyc}_8^- \left(\mathcal{C}_4^{(3)} \left(\begin{array}{c} 3 \quad 2 \\ 4 \quad 1 \\ 5 \quad 8 \\ 6 \quad 7 \end{array} \right) - \frac{3}{2} \mathcal{C}_4^{(2)} \left(\begin{array}{c} 3 \quad 2 \\ 4 \quad 1 \\ 5 \quad 8 \\ 6 \quad 7 \end{array} \right) + 3 \mathcal{C}_4^{(2)} \left(\begin{array}{c} 3 \quad 2 \\ 4 \quad 1 \\ 5 \quad 8 \\ 6 \quad 7 \end{array} \right) \right. \\
 & + \frac{1}{2} \mathcal{C}_4^{(2)} \left(\begin{array}{c} 3 \quad 2 \\ 4 \quad 1 \\ 5 \quad 8 \\ 6 \quad 7 \end{array} \right) - \frac{1}{2} \mathcal{C}_4^{(1)} \left(\begin{array}{c} 3 \quad 2 \\ 4 \quad 1 \\ 5 \quad 8 \\ 6 \quad 7 \end{array} \right) - \frac{1}{2} \mathcal{C}_4^{(1)} \left(\begin{array}{c} 3 \quad 2 \\ 4 \quad 1 \\ 5 \quad 8 \\ 6 \quad 7 \end{array} \right) \\
 & \left. - \frac{1}{2} \mathcal{C}_4^{(1)} \left(\begin{array}{c} 3 \quad 2 \\ 4 \quad 1 \\ 5 \quad 8 \\ 6 \quad 7 \end{array} \right) \right) = 0 \pmod{\text{products}}
 \end{aligned}$$

Q_5 identity (CGR)

$$\begin{aligned}
 & \text{Cyc}_8^+ \left(\mathcal{C}_5^{(3)} \left(\begin{array}{c} 3 \quad 2 \\ 4 \quad 1 \\ 5 \quad 8 \\ 6 \quad 7 \end{array} \right) + 3\mathcal{C}_5^{(2)} \left(\begin{array}{c} 3 \quad 2 \\ 4 \quad 1 \\ 5 \quad 8 \\ 6 \quad 7 \end{array} \right) - 6\mathcal{C}_5^{(2)} \left(\begin{array}{c} 3 \quad 2 \\ 4 \quad 1 \\ 5 \quad 8 \\ 6 \quad 7 \end{array} \right) \right. \\
 & \left. + 6\mathcal{C}_5^{(2)} \left(\begin{array}{c} 3 \quad 2 \\ 4 \quad 1 \\ 5 \quad 8 \\ 6 \quad 7 \end{array} \right) - \frac{1}{2}\mathcal{C}_5^{(1)} \left(\begin{array}{c} 3 \quad 2 \\ 4 \quad 1 \\ 5 \quad 8 \\ 6 \quad 7 \end{array} \right) \right) = 0 \text{ (mod products)}
 \end{aligned}$$

Also for $\mathcal{C}_3^{(\bullet)}$, 'off-diagonal' identity.

Reduction of $I_{3,2}$

$$\begin{aligned} I_{3,2}(x, y) = & -I_{4,1}\left(1-x, -\frac{x-y}{y}\right) + I_{4,1}\left(\frac{1}{x}, \frac{1}{y}\right) - I_{4,1}\left(-\frac{1-x}{x}, \frac{y}{x}\right) - I_{4,1}\left(x, -\frac{x-y}{y}\right) \\ & - I_{4,1}\left(\frac{1}{1-y}, \frac{x-y}{x(1-y)}\right) - I_{4,1}\left(\frac{1-x}{1-y}, \frac{1}{1-y}\right) - I_{4,1}\left(\frac{1-x}{1-y}, -\frac{y}{1-y}\right) + I_{4,1}\left(\frac{x(1-y)}{x-y}, -\frac{1-y}{y}\right) \\ & - I_{4,1}\left(\frac{x}{y}, \frac{1}{y}\right) - I_{4,1}\left(1, \frac{1}{1-x}\right) + I_{4,1}\left(\frac{1}{1-y}, \frac{1}{1-y}\right) - I_{4,1}\left(1-y, -\frac{1-y}{y}\right) \\ & + I_5\left(-\frac{1-x}{x-y}\right) + 3I_5\left(\frac{x}{x-y}\right) - I_5\left(\frac{x(1-y)}{x-y}\right) + 8I_5\left(-\frac{1-x}{y}\right) - 5I_5\left(\frac{x}{y}\right) \\ & + 3I_5\left(-\frac{x-y}{(1-x)y}\right) + 4I_5\left(-\frac{x-y}{xy}\right) + 6I_5\left(\frac{1}{1-x}\right) + I_5\left(\frac{1}{x}\right) + I_5\left(\frac{1}{1-y}\right) \\ & - 4I_5\left(-\frac{1}{y}\right) - 4I_5\left(\frac{1}{y}\right) \quad (\text{mod products}) \end{aligned}$$

Reduction of $I_{3,1,1}$

$$\begin{aligned}
 I_{3,1,1}(x, y, z) &= I_{3,2}(x, y) \\
 &- I_{4,1}\left(1-x, -\frac{y-z}{z}\right) - I_{4,1}\left(\frac{1}{x}, \frac{1}{y}\right) - I_{4,1}\left(\frac{1}{x}, \frac{1}{z}\right) - I_{4,1}\left(x, \frac{y-z}{1-z}\right) + I_{4,1}\left(x, -\frac{z}{1-z}\right) \\
 &- I_{4,1}\left(\frac{1}{1-y}, -\frac{y-z}{(1-y)z}\right) + I_{4,1}\left(\frac{x}{z}, \frac{y}{z}\right) + I_{4,1}\left(\frac{1-x}{1-y}, \frac{y(1-z)}{(1-y)z}\right) + I_{4,1}\left(\frac{1-x}{1-y}, -\frac{y-z}{(1-y)z}\right) \\
 &+ I_{4,1}\left(1-y, \frac{(1-y)z}{y(1-z)}\right) - I_{4,1}\left(\frac{x(1-y)}{x-y}, \frac{(1-y)z}{y(1-z)}\right) - I_{4,1}\left(\frac{x(1-y)}{x-y}, -\frac{(1-y)z}{y-z}\right) + I_{4,1}\left(\frac{x}{y}, \frac{y-z}{y(1-z)}\right) \\
 &- I_{4,1}\left(\frac{x}{y}, \frac{x(1-z)}{(1-x)z}\right) - I_{4,1}\left(\frac{1-x}{1-z}, \frac{y}{z}\right) - I_{4,1}\left(1-z, \frac{z}{y}\right) + I_{4,1}\left(\frac{x(1-z)}{x-z}, \frac{y-z}{x-z}\right) \\
 &- I_{4,1}\left(\frac{x(1-z)}{x-z}, -\frac{z}{x-z}\right) + I_{4,1}\left(\frac{x(1-z)}{x-z}, \frac{(x-y)z}{y(x-z)}\right) - I_{4,1}\left(\frac{x-z}{y-z}, -\frac{x-z}{(1-x)z}\right) + I_{4,1}\left(\frac{y}{z}, \frac{1}{z}\right) \\
 &+ I_{4,1}\left(-\frac{x-z}{z}, -\frac{x-z}{(1-x)z}\right) - I_{4,1}\left(\frac{y(x-z)}{(x-y)z}, -\frac{x-z}{(1-x)z}\right) \\
 &- I_5\left(\frac{1}{1-x}\right) - I_5\left(\frac{1}{1-y}\right) + 2I_5\left(\frac{x(1-y)}{x-y}\right) + 2I_5\left(\frac{x}{y}\right) + I_5\left(-\frac{x-y}{(1-x)y}\right) \\
 &+ I_5\left(\frac{1-x}{1-z}\right) + I_5\left(\frac{1-z}{y-z}\right) - 4I_5\left(\frac{x(1-z)}{y-z}\right) - I_5\left(-\frac{(x-y)(1-z)}{(1-x)(y-z)}\right) + I_5\left(\frac{x-z}{y-z}\right) \\
 &+ 3I_5\left(\frac{x}{z}\right) - I_5\left(\frac{y}{z}\right) - I_5\left(-\frac{1-z}{z}\right) + 4I_5\left(-\frac{x(1-z)}{z}\right) + I_5\left(\frac{x(1-z)}{(1-x)z}\right) + 4I_5\left(\frac{y(1-z)}{z}\right) \\
 &- 4I_5\left(\frac{xy(1-z)}{(x-y)z}\right) - I_5\left(-\frac{x-z}{z}\right) + I_5\left(-\frac{x-z}{(1-x)z}\right) + I_5\left(\frac{y(x-z)}{(x-y)z}\right) + 3I_5\left(-\frac{y-z}{z}\right) \\
 &+ I_5\left(-\frac{y-z}{(1-y)z}\right) + I_5\left(-\frac{x(y-z)}{(x-y)z}\right) \pmod{\text{products}}
 \end{aligned}$$

Q_6 identity (CGR)

$$\begin{aligned}
 & \text{Cyc}_9^+ \left(\mathcal{C}_6^{(3)} \left(\begin{array}{c} 3 \quad 2 \\ 4 \quad 1 \\ 5 \quad 9 \\ 6 \quad 8 \end{array} \right) \right) + \frac{3}{4} \mathcal{C}_6^{(2)} \left(\begin{array}{c} 3 \quad 2 \\ 4 \quad 1 \\ 5 \quad 9 \\ 6 \quad 8 \end{array} \right) - \frac{3}{4} \mathcal{C}_6^{(2)} \left(\begin{array}{c} 3 \quad 2 \\ 4 \quad 1 \\ 5 \quad 9 \\ 6 \quad 8 \end{array} \right) \\
 & + \frac{3}{4} \mathcal{C}_6^{(2)} \left(\begin{array}{c} 3 \quad 2 \\ 4 \quad 1 \\ 5 \quad 9 \\ 6 \quad 8 \end{array} \right) - \frac{1}{4} \mathcal{C}_6^{(2)} \left(\begin{array}{c} 3 \quad 2 \\ 4 \quad 1 \\ 5 \quad 9 \\ 6 \quad 8 \end{array} \right) \Bigg) = 0 \text{ (mod products)}
 \end{aligned}$$