Multiple polylogarithms in weight 5

ε towards Zagier's conjecture on $\zeta_F(5)$?

Steven Charlton

21 April 2021

Arithmetische Geometrie
und Zahlentheorie Seminar

28 April 2021: The revenge of the polylogs.

Outline

Ongoing work with H. Gangl & D. Radchenko

17 pg → 4 lines

1. Definitions and motivation <— HEP, Hyperbolic geometry, zeta functions
2. Zagier's conjecture on $\zeta_F(n)$ = det(Li_n's).
3. Functional equations for Li_n
4. Depth reduction and the Lie coalgebra
5. Strategy to prove Zagier's conjecture
6. Construction of c_{2m-1}
7. Higher ratios
8. Explicit reduction of Gr_4 and a 4-ratio

Can talk about our work on $\zeta_F(4)$.

Classical polylogarithms

Definition (Polylogarithm)

The weight n polylogarithm is

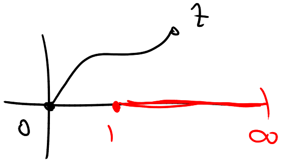
$$\mathrm{Li}_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n}, \quad |z| < 1$$

$$\mathrm{Li}_1(z) = -\log(1-z)$$

$$\frac{\partial}{\partial z} \mathrm{Li}_n(z) = \frac{1}{z} \mathrm{Li}_{n-1}(z), \quad n \geq 2.$$

Analytically continue to $\mathbb{C} \setminus [1, \infty)$

$$\mathrm{Li}_n(z) = \int_0^z \mathrm{Li}_{n-1}(t) \frac{dt}{t}$$



Multiple polylogarithms

Natural multi-variable generalisation

Definition (Multiple polylogarithms)

A multiple polylogarithm (MPL) is

unit polydisc

$$\mathrm{Li}_{\underbrace{n_1,\ldots,n_r}}(\underbrace{z_1,\ldots,z_r}) := \sum_{k_1<k_2<\cdots<k_r} \frac{z_1^{k_1}\cdots z_r^{k_r}}{k_1^{n_1}\cdots k_r^{n_r}}, \quad |z_1\cdots z_r|<1$$

$|z_2\cdots z_r|<1$

$\vdots$

$|z_r|<1$

use $<$ not $\le$, to get interesting result.

- The *depth* is r, and *weight* is $n_1+\cdots+n_r$

Multiple polylogarithms

Natural multi-variable generalisation

Definition (Multiple polylogarithms)

A multiple polylogarithm (MPL) is

$$\mathrm{Li}_{n_1,\ldots,n_r}(z_1,\ldots,z_r) \coloneqq \sum_{k_1<k_2<\cdots<k_r} \frac{z_1^{k_1}\cdots z_r^{k_r}}{k_1^{n_1}\cdots k_r^{n_r}}$$

- The *depth* is r, and *weight* is $n_1+\cdots+n_r$

Expression by integrals

$$\mathrm{Li}_{n_1,\ldots,n_r}(z_1,\ldots,z_r) = (-1)^r I(0; \tfrac{1}{z_1\cdots z_r}, \overbrace{0,\ldots,0}^{n_1-1}, \tfrac{1}{z_2\cdots z_r}, \overbrace{0,\ldots,0}^{n_2-1}, \ldots, \tfrac{1}{z_r}, \overbrace{0,\ldots,0}^{n_r-1}; 1),$$

where

$$I(x_0; x_1,\ldots,x_n; x_{n+1}) = \int_{x_0<t_1<\cdots<t_n<x_{n+1}} \frac{\mathrm{d}t_1}{t_1-x_1}\cdots\frac{\mathrm{d}t_n}{t_n-x_n}$$

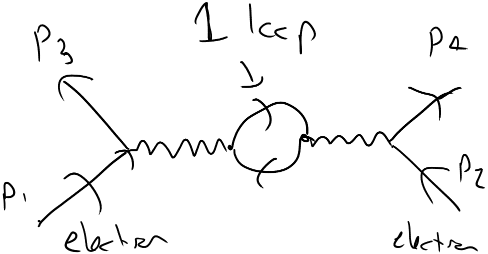
- stuffle via Σ — shuffle via $\int$

Appearances and applications of polylogarithms

- High-energy physics
  - Computation of Feynman diagrams and scattering amplitudes
  - Cross-fertilisation NT↔HEP, to understand structure of results

17 pages of MPLs in wt 4

↳ 3 lines of Li_4



$$\longrightarrow \int_{\mathbb{C}^4} \frac{P_1(p_1,\dots,p_4)}{P_2(p_1,\dots,p_4)}\, d^4x$$

$$\longrightarrow Li_{n_1 \dots n_2}(x, x)$$

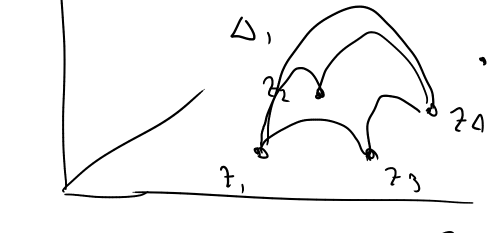
Appearances and applications of polylogarithms

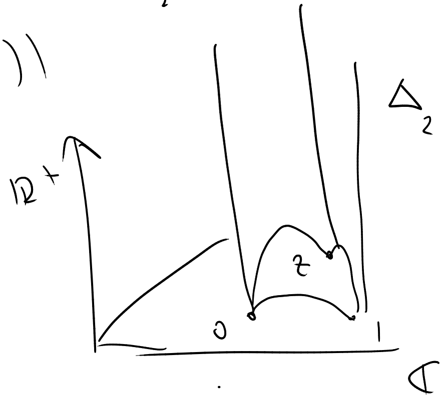
- Hyperbolic geometry
  - Ideal tetrahedron via Li_2
  - Volumes of hyperbolic polytopes

$\mathrm{Li}_2 = \text{dilog}$

$\mathrm{vol}\,\Delta_2 = \mathcal{L}i_2(z)$

$\mathrm{Vol}\,\Delta_1 = \mathcal{L}_2(cr(z_1, z_2, z_3, z_4))$





$z_1 \,..\, z_4 \in \mathbb{C}.$

$z_1 \dots z_5 \longrightarrow \sum (-1)^i \mathcal{L}_2(cr(z_1 .. \hat{z}_i .. z_5)) = 0$

Appearances and applications of polylogarithms

- Zeta values and zeta functions
  - Multiple zeta values $\zeta(n_1, \ldots, n_r) = \mathrm{Li}_{n_1,\ldots,n_r}(1, \ldots, 1)$
  - Values of Dedekind zeta function $\zeta_F(n)$

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{n^s}$$

$$\zeta(2) = \frac{\pi^2}{6} \qquad \zeta(4) = \frac{\pi^4}{90}$$

$$\zeta(2k) \in \mathbb{Q}\pi^{2k} \qquad \zeta(3) \notin \mathbb{Q}$$

$\zeta(s) \in / \notin \mathbb{Q}$? Unknown.

$$\zeta(\overbrace{1,3,1,3,\ldots}^{n}) = \frac{2\pi^{4n}}{(4n+2)!}$$

$$\sum \mathrm{Li}_{13\ldots13}(1,\ldots,1,z)\, t^{4n}$$

Satisfies PDE. Solution via ${}_2F_1$

Appearances and applications of polylogarithms

- High-energy physics
  - Computation of Feynman diagrams and scattering amplitudes
  - Cross-fertilisation NT$\leftrightarrow$HEP, to understand structure of results
- Hyperbolic geometry
  - Ideal tetrahedron via Li_2
  - Volumes of hyperbolic polytopes
- Zeta values and zeta functions
  - Multiple zeta values $\zeta(n_1, \ldots, n_r) = \mathrm{Li}_{n_1,\ldots,n_r}(1, \ldots, 1)$
  - Values of Dedekind zeta function $\zeta_F(n)$

Unifying viewpoint/property: identities, functional equations and functional relations

between MPL's.

Zagier's conjecture on $\zeta_F(n)$

Dedekind zeta function

Let F be a number field, $\mathcal{O}_F$ its ring of integers

Definition (Dedekind zeta function)

The Dedekind zeta function of F is

$$\zeta_F(s) = \sum_{\substack{I \neq (0) \subset \mathcal{O}_F \\ \text{ideal}}} \frac{1}{N(I)^s}, \quad \mathrm{Re}(s) > 1 \to$$

Can continue to $\mathbb{C}$ w/ pole at $s=1$

$$N(I) = \#(\mathcal{O}_F / I)$$

$$F = \mathbb{Q} \rightsquigarrow I = (n) \to N(I) = n$$

Example ($F = \mathbb{Q}(\sqrt{-2})$)

± 1

$$\zeta_{\mathbb{Q}(\sqrt{-2})}(s) = \boxed{\frac{1}{2}} \sum_{k \geq 1} \frac{\#\{x, y \in \mathbb{Z} \mid x^2 + 2y^2 = k\}}{k^s}$$

Analytic class number formula

Theorem (Analytic class number formula)

Residue of $\zeta_F(s)$ at $s = 1$ is equal up to a rational. "$\zeta_F(1)$"

$$\operatorname{Res}_{s=1} \zeta_F(s) \boxed{\sim_{\mathbb{Q}^\times}} \sqrt{|\Delta_F|}\pi^{r_2} \cdot \underbrace{\det\Big(\log|\sigma_i(\varepsilon_j)|\Big)}_{\mathrm{Reg}_F}{}_{i,j=1}^{r_1+r_2-1}$$

- Δ_F = discriminant (size of $\mathcal{O}_F$)
- r_1, r_2 # real / complex embeddings $\sigma_i : F \to \mathbb{R}, \mathbb{C}$
- $\varepsilon_j \in \mathcal{O}_F^\times$ (fundamental) units

$\sim_{\mathbb{Q}^\times}$ $\longrightarrow$ $\dfrac{2^{r_1+r_2} h}{w \, |\Delta_F|}$ h is class number

roots of 1 $\to$ w

Zagier's polylogarithm conjecture

Conjecture (Zagier, 1991)

For $n \geq 2$, there exists $y_1, \ldots, y_{d_n} \in \mathbb{Z}[F^\times]$ so that $\leftarrow \mathbb{Z}[F^\times]$ = formal combinations.

$$\zeta_F(n) \sim_{\mathbb{Q}^\times} \pi^{nd_{n+1}} \sqrt{|\Delta_F|} \cdot \det\Big(\mathscr{L}_n(\sigma_i(y_j))\Big)_{i,j=1}^{d_n},$$

where $d_n = r_1 + r_2$, n odd, and $d_n = r_2$ if n even

$\mathcal{L}_n$ = single-valued version of Li_n extended by linearity

$\mathcal{L}_2(z) = \mathrm{Im}(\mathrm{Li}_2(z) + \log(1-z)\log|z|)$

$\mathcal{L}_n(\sum_i \lambda_i [x_i]) = \sum_i \lambda_i \mathcal{L}_n(x_i)$

Example

$d_n = 1$

$$\zeta_{\mathbb{Q}(\sqrt{-2})}(2) \stackrel{?}{=} \frac{\pi^2\sqrt{2}}{48}\Big(\mathscr{L}_2(1+2\sqrt{-2}) + 6\,\mathscr{L}_2(1+\tfrac{1}{2}\sqrt{-2})\Big) \approx 1.75141751\ldots$$

Status of ZPC

"$n = 1$": $\mathbb{Z}[F(\sqrt{*})]$ ✓

$n = 2$: Zagier (weak version), Bloch-Suslin ∼1981

generalization of cross-ratio.

$n = 3$: ∼1993 by Goncharov after introducing the triple-ratio of 6 points in $\mathbb{P}^2(\mathbb{C})$, and a 22-term functional equation for Li_3

$n = 4$: 2018 by Goncharov-Rudenko after introducing the new geometric identity $\mathbf{Q}_4$ for $\mathrm{Li}_{3,1}$

Also known for special classes of number fields: cyclotomic, abelian(?)

Status of ZPC

$n = 2$: Zagier (weak version), Bloch-Suslin $\sim$1981

$n = 3$: $\sim$1993 by Goncharov after introducing the triple-ratio of 6 points in $\mathbb{P}^2(\mathbb{C})$, and a 22-term functional equation for Li_3

$n = 4$: 2018 by Goncharov-Rudenko after introducing the new geometric identity $\mathbf{Q}_4$ for $\mathrm{Li}_{3,1}$

Also known for special classes of number fields: cyclotomic, abelian(?)

Remark

Goncharov has a strategy/program to prove the weight n case. Requires input from highly non-trivial (currently unknown) identities for multiple polylogarithms.

Functional equations for Li_n

Logarithm and dilogarithm

Key property of logs

$$\log(xy) = \log(x) + \log(y) \quad \rightsquigarrow \quad \mathrm{Li}_1(1-xy) = \mathrm{Li}_1(1-x) + \mathrm{Li}_1(1-y)$$

Theorem (5-term relation, Abel, Spence, Kummer, ...)

For $|x| + |y| < 1$ *we have*

all arguments are cs $(0, 1, \infty, \hat{x}, y)$

$$\cancel{\mathrm{Li}_2}(x) + \cancel{\mathrm{Li}_2}(y) - \cancel{\mathrm{Li}_2}\left(\frac{x}{1-y}\right) - \cancel{\mathrm{Li}_2}\left(\frac{y}{1-x}\right) + \cancel{\mathrm{Li}_2}\left(\frac{xy}{(1-x)(1-y)}\right)$$

$\mathcal{L}_2 \quad \mathcal{L}_2 \quad \mathcal{L}_2 \quad \mathcal{L}_2 \quad \mathcal{L}_2$

$$\cancel{= -\log(1-x)\log(1-y)} = 0$$

- $\underline{\text{fundamental}}$

∴ clean ∴

- holds for power series.
- or by differentiating $x = 0$

$y = 1 - x$. $\quad \mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \frac{\pi^2}{6} + \log \cdot \log$.

Single-valued polylogs

Polylogs are multivalued, useful to introduce a single-valued version

Definition (Bloch-Wigner-Ramakrishnan-Zagier polylogarithm)

A single-valued polylogarithm is defined by

$$\mathscr{L}_n(z) = \mathrm{Re}_n \left(\sum_{k=0}^{n-1} \frac{2^k B_k}{k!} \mathrm{Li}_{n-k}(z) \log^k(z) \right), \quad n \geq 2$$

$\frac{t}{e^t+1}$

$k=0$ $\mathrm{Li}_n(z)$

- $\mathrm{Re}_n = \mathrm{Re}$ if n odd, $= \mathrm{Im}$ if n even.
- B_k the m-th Bernoulli number

Single-valued polylogs

Polylogs are multivalued, useful to introduce a single-valued version

Definition (Bloch-Wigner-Ramakrishnan-Zagier polylogarithm)

A single-valued polylogarithm is defined by

$$\mathscr{L}_n(z) = \mathrm{Re}_n\left(\sum_{k=0}^{n-1} \frac{2^k B_k}{k!} \mathrm{Li}_{n-k}(z) \log^k(z)\right), \quad n \geq 2$$

- $\mathrm{Re}_n = \mathrm{Re}$ if n odd, $= \mathrm{Im}$ if n even.
- B_k the m-th Bernoulli number

- $\mathscr{L}_1(z) = -\log|1-z|$ ↪ ACNF
- $\mathscr{L}_2(z) = \mathrm{Im}(\mathrm{Li}_2(z) + \log(1-z)\log|z|)$
- $\mathscr{L}_3(z) = \mathrm{Re}(\mathrm{Li}_3(z) - \mathrm{Li}_2(z)\log|z| - \frac{1}{3}\log(1-z)\log^2|z|)$

Single valued, real analytic on $\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\} \to \mathbb{P}^1(\mathbb{C})$

Functional equations of $\mathscr{L}_n$

Key properties

- $\mathscr{L}_n$ satisfies functional equations *without* products
- Functional equations characterised by algebraic criterion

If $\sum_i \lambda_i \underline{(1 - f_i(z)) \wedge f_i(z) \otimes (f_i(z))}^{\otimes n-2} = 0$ in $(\mathbb{C}(z)^\times)^{\otimes n} \otimes_{\mathbb{Z}} \mathbb{Q}$, then

$$\sum_i \lambda_i \mathscr{L}_n(\underline{f_i(z)}) = \underline{\text{constant}}.$$

- related to the symbol of MPL $Li_{n_1,\dots,n_r}$

$B_2(F) = \dfrac{\mathbb{Z}[F^\times]}{\text{dilog relations}}$

$[[1-x]\wedge[x]] \to$

Criterion $\to$ encodes derivative of $\mathcal{L}_n$.

Example ($\mathscr{L}_n(\frac{1}{z}) = (-1)^{n-1}\mathscr{L}_n(z)$)

$$(1-\tfrac{1}{z}) \wedge (\tfrac{1}{z}) \otimes (\tfrac{1}{z})^{\otimes n-2} = (-1)^{n-1}(\overset{[z-1]}{[1-z]} - [z]) \wedge (z) \otimes (z)^{\otimes n-2}$$

$$= (-1)^{n-1}(1-z) \wedge (z) \otimes (z)^{\otimes n-2}$$

$1 - \frac{1}{z} = \frac{z-1}{z}$

Trilogarithms

$$\mathcal{L}_2(z) + \mathcal{L}_2(1-z) = 0$$

(Next) simplest functional equation

$$\mathscr{L}_3(z) + \mathscr{L}_3(1-z) + \mathscr{L}_3(1-z^{-1}) = \mathscr{L}_3(1)$$

Trilogarithms

(Next) simplest functional equation

$$\mathscr{L}_3(z) + \mathscr{L}_3(1-z) + \mathscr{L}_3(1-z^{-1}) = \mathscr{L}_3(1)$$

Theorem (9-term relation, Spence 1809, Kummer 1840)

9(+1)

$$\mathscr{L}_3\left(\left[\frac{x(1-y)^2}{y(1-x)^2}\right] - 2\left[\frac{1-y}{1-x}\right] - 2\left[\frac{1-y^{-1}}{1-x}\right] - 2\left[\frac{1-y}{1-x^{-1}}\right] - 2\left[\frac{1-y^{-1}}{1-x^{-1}}\right] \right.$$

Li$_3$

$$\left. + [xy] + \left[\frac{x}{y}\right] - 2[x] - 2[y] + \underline{2[1]}\right) = 0$$

$\Uparrow$ $\Downarrow$?

+ products

There is a more fundamental 22-term relation in 3 variables (Goncharov 1993).

Trilogarithm - 22-term

Theorem (22-term relation, Goncharov 1993)

22(+1)

$$\mathscr{L}_3\left(\mathrm{Cyc}\left([z]+\left[-\frac{x(yz-z+1)}{xz-x+1}\right]+\left[\frac{yz-z+1}{y(xz-x+1)}\right]-\left[\frac{yz-z+1}{yz(xz-x+1)}\right]\right.\right.$$
$$\left.\left.+[xz-x+1]-\left[\frac{xz-x+1}{z}\right]+\left[\frac{xz-x+1}{xz}\right]\right)+[-xyz]\right)=3\,\mathscr{L}_3(1)$$

Trilogarithm - 22-term

Theorem (22-term relation, Goncharov 1993)

$(\cdot\ \cdot)\quad (x\to y,\ y\to z,\ z\to x) + (\cdot\ \cdot)$

$$\mathscr{L}_3\left(\mathrm{Cyc}\left([z] + \left[-\frac{x(yz-z+1)}{xz-x+1}\right] + \left[\frac{yz-z+1}{y(xz-x+1)}\right] - \left[\frac{yz-z+1}{yz(xz-x+1)}\right] + [xz-x+1] - \left[\frac{xz-x+1}{z}\right] + \left[\frac{xz-x+1}{xz}\right]\right) + [-xyz]\right) = 3\,\mathscr{L}_3(1)$$

The arguments arise from a so-called triple-ratio of 6 points in $\mathbb{P}^2(\mathbb{C})$ $\quad \mathbb{P}^{n-1}(\mathbb{C})$

$$\mathrm{cr}_3(v_1,\dots,v_6) = \frac{\langle 124\rangle\langle 235\rangle\langle 316\rangle}{\langle 125\rangle\langle 236\rangle\langle 314\rangle} \qquad \langle ijk\rangle = \det(v_i v_j v_k)$$

$\to$ 7-term + S_6 = 840 term

- Used to prove Zagier's conjecture for $\zeta_F(3)$

Functional equation leaderboard

Distribution

$$\mathrm{Li}_n(x^p) = p^{1-n} \sum_{\lambda^p = 1} \mathrm{Li}_n(\lambda x)$$

Li_2: Abel-Spence, 5-term, 2 variable – 'fundamental'

Li_3: Landen (1780), 3-term, 1 variable
Spence-Kummer (1809/1840), 9-term, 2 variable
Goncharov (1993), 22-term $= 7 \times S_6$-orbit, 3 variable – 'fundamental'

Li_4: Kummer (1840), 20-term, 2 variable 9-term in 1 variable
Gangl (2012), 931-term $= 9 \times S_5 \times S_5 \times \mathbb{Z}/2$-orbit, 4 variable
C-Gangl-Radchenko (2019), $9 \times 368 \times S_8$-orbit, 4 variable – 'fundamental'??

Li_5: Kummer (1840), 33-term, 2 variable ← here Kummer's approach stops.
Radchenko (2016), $3 \times S_6$-orbit, 3-variable

→ reduction

Li_6: Gangl (1990), 167-term, 2 variable

Li_7: Gangl (1990), 274-term, 2 variable

Li_8: Unknown, except for 'trivial relations'

Expect $(2n+1) \times S_{2n}$ - orbit

Depth reduction and the Lie coalgebra

To be continued.

Simplicity in higher depth?

$$\mathrm{Li}_{ab}(x,y) = \sum_{n_1<n_2} \frac{x^{n_1} y^{n_2}}{n_1^a n_2^b}$$

Proposition

For $|xy| < 1, |y| < 1$

$$\mathrm{Li}_{1,1}(x,y) = \mathrm{Li}_1(x)\,\mathrm{Li}_1(y) + \mathrm{Li}_2\left(\frac{-x}{1-x}\right) - \mathrm{Li}_2\left(\frac{x(y-1)}{1-x}\right)$$

↳ Directly check on power series

↳ Differentiate. ⇒ 5-term relation.

$$\underset{\overset{\downarrow}{n_1}}{\mathrm{Li}_1}(x)\,\underset{\overset{\downarrow}{n_2}}{\mathrm{Li}_1}(y) = \overset{<}{\mathrm{Li}_{11}(x,y)} + \overset{>}{\mathrm{Li}_{11}(y,x)} + \overset{=}{\mathrm{Li}_2(xy)}$$

Reducing $\mathrm{Li}_{2,1}$ to Li_3

Goncharov-Rudenko give identity $\mathbf{Q}_3$ (of cluster-geometric nature):

$I_{21}(x,y) = Li_{21}\left(\frac{y}{x}, \frac{1}{y}\right)$

$$\mathrm{Cyc}_6\Big(I_{2,1}([x_1,x_2,x_3,x_4],[x_4,x_5,x_6,x_1]) + I_3([x_1,x_2,x_4,x_5]) - 2I_3([x_1,x_3,x_4,x_5])$$
$$-\frac{4}{3}I_3([x_1,x_2,x_3,x_4,x_5,x_6])\Big) = 0 \quad (\text{mod products})$$

$I_3(x) = Li_3\left(\frac{1}{x}\right)$

where

$$[x_1,x_2,\ldots,x_{2k}] = (-1)^k\frac{(x_1-x_2)(x_3-x_4)\cdots(x_{2k-1}-x_{2k})}{(x_2-x_3)(x_4-x_5)\cdots(x_{2k}-x_1)}$$

$[x_1 \ldots x_4] = 1 - cr(x_1, x_3, x_2, x_4)$

Reducing $\mathrm{Li}_{2,1}$ to Li_3

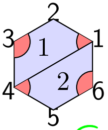
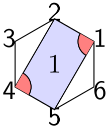
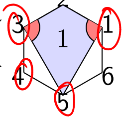
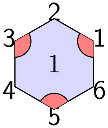
Goncharov-Rudenko give identity $\mathbf{Q}_3$ (of cluster-geometric nature):

$$\mathrm{Cyc}_6\Big(I_{2,1}([x_1,x_2,x_3,x_4],[x_4,x_5,x_6,x_1]) + I_3([x_1,x_2,x_4,x_5]) - 2I_3([x_1,x_3,x_4,x_5])$$
$$-\frac{4}{3}I_3([x_1,x_2,x_3,x_4,x_5,x_6])\Big) = 0 \quad \text{(mod products)}$$

Wonder if $I_{3,1}$ reduces to I_4 ?

where

$$[x_1,x_2,\ldots,x_{2k}] = (-1)^k\frac{(x_1-x_2)(x_3-x_4)\cdots(x_{2k-1}-x_{2k})}{(x_2-x_3)(x_4-x_5)\cdots(x_{2k}-x_1)}$$

$\mathrm{Cyc}_6\Big(I_{2,1}\Big($  $\Big) + I_3\Big($  $\Big) - 2I_3\Big($  $\Big) - \frac{4}{3}I_3\Big($  $\Big)\Big) = 0$

$$x_2 = x_5 \longrightarrow I_{21}(X,Y) + \cancel{I_{21}(X,1)}^{=\mathrm{Li}_3} + \cdots = I_3\text{'s}$$

$\Rightarrow$ 22-term by substituting back in!

Hopf algebra of MPL's

Motivic iterated integrals (Goncharov, Brown, ...)

Purely algebraic version

Iterated integrals $I(x_0; x_1, \dots, x_n; x_{n+1})$ can be upgraded to framed mixed Tate motives, to define

$$I^{\mathscr{M}}(x_0; x_1, \dots, x_n; x_{n+1}),$$

$\zeta(s) \in \mathcal{H}_s, \quad \mathcal{H}_0 = \mathbb{Q}$.

$$I = \int_{x_0}^{x_{n+1}} \frac{dt_1}{t_1 - x_1} \wedge \dots \wedge \frac{dt_n}{t_n - x_n}$$

simplex $\rightsquigarrow$ variety

$\leftarrow$ de Rham (cohomology)

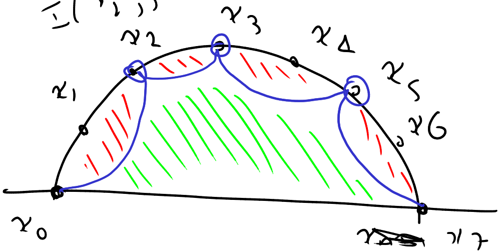
elements of a graded Hopf algebra $\mathcal{H}$

by weight.

coproduct $\Delta I^{\mathscr{M}}(x_0; x_1, \dots, x_n; x_{n+1}) =$

$$\sum_{\substack{0=i_0<i_1<\cdots\\<i_k<i_{k+1}=n}} I^{\mathscr{M}}(x_0; x_{i_1}, \dots, x_{i_k}, x_{n+1}) \otimes \prod_{p=0}^{k} I^{\mathscr{M}}(x_{i_p}; x_{i_p+1}, \dots, x_{i_{p+1}-1}; x_{i_{p+1}})$$

$I(x_1; \dots; x_3) = 1$



$$I^m(x_0; x_1, x_3, x_5; x_7)$$

$$\otimes \ I^m(x_0; x_1; x_2) \, I^m(x_3; x_4; x_5) \, I^m(x_5; x_6; x_7)$$

Lie coalgebra

$\delta^2 = 0$

A graded Hopf algebra induces a Lie coalgebra $\mathcal{L} = \mathcal{H}/\mathcal{H}_{>0}\cdot\mathcal{H}_{>0}$, with $\delta = \Delta - \Delta^{\mathrm{op}}$

- $I^{\mathcal{M}}(x_0; x_1, \dots, x_n; x_{n+1})$ becomes $I^{\mathfrak{L}}(x_0; x_1, \dots, x_n; x_{n+1}) \quad (\text{mod products})$

Example

- $\delta \log^{\mathfrak{L}}(x) = 0 \quad \longleftarrow \quad \Delta \log x = 1 \otimes \log x + \log x \otimes 1.$
- $\delta \, \mathrm{Li}_n^{\mathfrak{L}}(x) = -\underline{\mathrm{Li}_{n-1}^{\mathfrak{L}}}(x) \wedge \log^{\mathfrak{L}}(x) \in \boxed{\mathcal{L}_{n-1} \wedge \mathcal{L}_1}$
- $\delta_{2,2} \, I_{3,1}^{\mathfrak{L}}(x, y) = \cancel{\mathrm{Li}_3 * \wedge \lg *} + \mathrm{Li}_2(x) \wedge \mathrm{Li}_2(y) \neq 0$ $\quad \in L_2 \wedge L_2.$

$\Rightarrow I_{3,1} \neq \sum I_4$

Conjecture (Goncharov, Freeness)

The kernel of $\delta_{\geq 2}$ *is generated by classical polylogarithms* $\mathrm{Li}_n^{\mathfrak{L}}(x)$

Expectation $\delta_{2,2} I_{3,1}\left((x) - (y) + \left(\frac{x}{y}\right) - \left(\frac{1-x}{1-y}\right) + \left(\frac{x(1-y)}{y(1-x)}\right), z \right) = 0$

$I_{3,1}^{\mathfrak{L}}(\text{5-term relation}, z) = \sum \mathrm{Li}_4^{\mathfrak{L}}$'s

Reductions of $I_{3,1}$

mod products, using the 'symbol'

Theorem (Gangl, 2012)

$I_{3,1}$(5-term relation, z) is a sum of 122 Li_4 terms

Corollary

There exists a 931-term, 4 variable functional equation for Li_4

$$I_{3,1}(5\text{-term}(x,y), 5\text{-term}(z,w))$$

$$= 122\text{-term}(x,y;*) \neq 122\text{-term}(z,w;*)$$

$\Rightarrow$ Li_4 functional equation.

Reductions of $I_{3,1}$

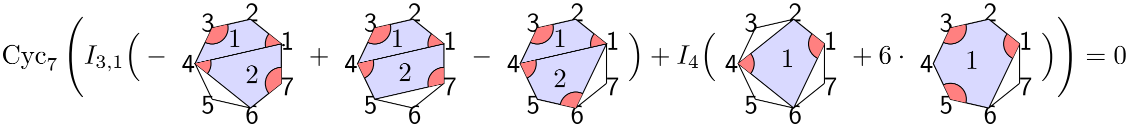
Theorem (Gangl, 2012)

$I_{3,1}$(5-term relation, z) is a sum of 122 Li_4 terms

Corollary

There exists a 931-term, 4 variable functional equation for Li_4

Goncharov-Rudenko give a conceptual derivation of the 122-term relation from $\mathbf{Q}_4$:



↳ can lead $I_{3,1}([x]+[1-x], y) = I_4$'s

$I_{3,1}(5\text{-term}, z) = I_4$'s.

Expectation in weight 5

- $\delta_{\geq 2, \geq 2} I^{L}_{4,1}(x, y) = -\operatorname{Li}^{L}_2(x) \wedge \operatorname{Li}^{L}_3(y) + \operatorname{Li}^{L}_3(x) \wedge \operatorname{Li}^{L}_2(y)$

$$I^{+}_{4,1}(x,y) = I_{4,1}(x,y) + I_{4,1}\left(x, \frac{1}{y}\right)$$

$$\delta_{\geq 2, \geq 2} I^{+}_{4,1}(x,y) = -2 \operatorname{Li}_2(x) \wedge \operatorname{Li}_3(y).$$

Hopeful conjecture

- $I^{+}_{4,1}(\text{5-term}, z) = \sum \operatorname{Li}_5$'s
- $I^{+}_{4,1}(x, \text{22-term}) = \sum \operatorname{Li}_5$'s

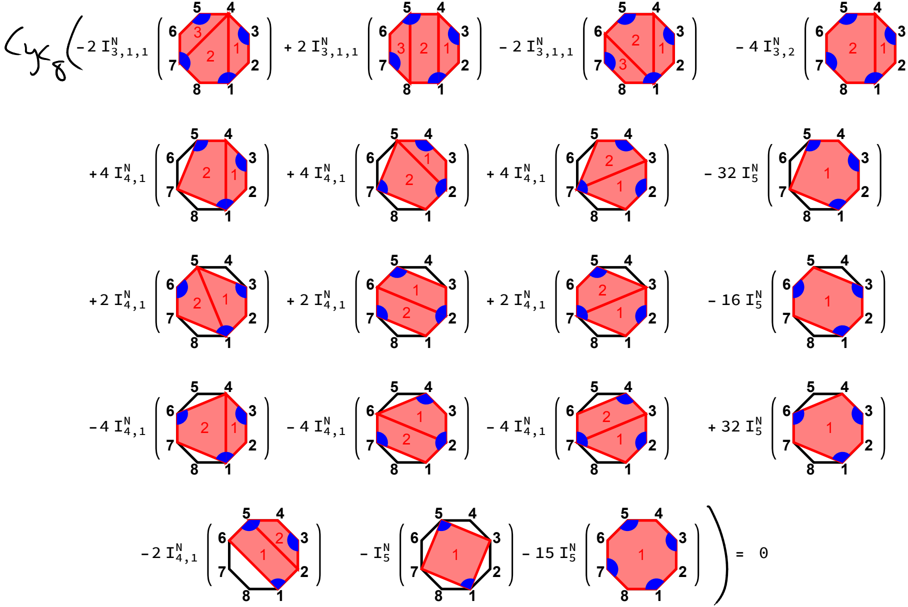
Some identity $\mathbf{Q}_5$ exists, and implies this?

Results in weight 5 - $\mathbf{Q}_5$

$\ker\delta = Li_n \rightsquigarrow \delta^2 = 0$

$(L_2 \wedge L_2) \wedge L_2 \cong L_2 \wedge (L_2 \wedge L_2)$

$\delta^2 = 0 \Rightarrow$ depth 2.



$I_5^N = I_5$

$I_{3,1,1}^N(x,y,z)$

$= I_{3,1,1}\left(x, \frac{1}{yz}, \frac{1}{y}\right)$

$= Li_{3,1,1}\left(\frac{1}{xyz}, z, y\right)$

$I_{4,1}^N(x,y)$

$= I_{4,1}\left(x, \frac{1}{y}\right)$

$= Li_{4,1}\left(\frac{1}{xy}, y\right)$

Consequences of $\mathbf{Q}_5$

Degenerating $\mathbf{Q}_5$ to $x_1 = x_3 = x_5$ isolates a single $I_{3,1,1}$ term

Proposition (CGR, 2019-?)

$I_{3,1,1}$ can be expressed via $I_{3,2}$, $I_{4,1}$ and I_5

Further degenerating to $x_2 = x_7$ isolates a single $I_{3,2}$ term

Proposition (C 2016, CGR 2019-?)

$I_{3,2}$ can be expressed via $I_{4,1}$ and I_5. ← Find tiles w/ much more complicated I_5 terms.

$\delta I_{4,1}^{+} = Li_2(x) \wedge Li_3(y)$

Substituting back into $\mathbf{Q}_5$

Pre-Theorem (CGR 2019-?, 22-term)

$I^+_{4,1}(x, \text{22-term})$ can be expressed via a sum of $I^+_{4,1}(\text{5-term}, z_i)$, and I_5's

$T(x_i, y_j), z_k$. thousand.

500 terms

Reduction of $I_{3,2}$

$$
\begin{aligned}
& I_{3,2}(x,y) = \\
& - I_{4,1}\left(1-x, -\tfrac{x-y}{y}\right) + I_{4,1}\left(\tfrac{1}{x}, \tfrac{1}{y}\right) - I_{4,1}\left(-\tfrac{1-x}{x}, \tfrac{y}{x}\right) - I_{4,1}\left(x, -\tfrac{x-y}{y}\right) \\
& - I_{4,1}\left(\tfrac{1}{1-y}, \tfrac{x-y}{x(1-y)}\right) - I_{4,1}\left(\tfrac{1-x}{1-y}, \tfrac{1}{1-y}\right) - I_{4,1}\left(\tfrac{1-x}{1-y}, -\tfrac{y}{1-y}\right) + I_{4,1}\left(\tfrac{x(1-y)}{x-y}, -\tfrac{1-y}{y}\right) \\
& - I_{4,1}\left(\tfrac{x}{y}, \tfrac{1}{y}\right) - I_{4,1}\left(1, \tfrac{1}{1-x}\right) + I_{4,1}\left(\tfrac{1}{1-y}, \tfrac{1}{1-y}\right) - I_{4,1}\left(1-y, -\tfrac{1-y}{y}\right) \\
& + I_5\left(-\tfrac{1-x}{x-y}\right) + 3I_5\left(\tfrac{x}{x-y}\right) - I_5\left(\tfrac{x(1-y)}{x-y}\right) + 8I_5\left(-\tfrac{1-x}{y}\right) - 5I_5\left(\tfrac{x}{y}\right) \\
& + 3I_5\left(-\tfrac{x-y}{(1-x)y}\right) + 4I_5\left(-\tfrac{x-y}{xy}\right) + 6I_5\left(\tfrac{1}{1-x}\right) + I_5\left(\tfrac{1}{x}\right) + I_5\left(\tfrac{1}{1-y}\right) \\
& - 4I_5\left(-\tfrac{1}{y}\right) - 4I_5\left(\tfrac{1}{y}\right) \qquad \text{(mod products)}
\end{aligned}
$$

Reduction of $I_{3,1,1}$

$$
\begin{aligned}
I_{3,1,1}(x,y,z) \quad = \quad & I_{3,2}(x,y) \\
& - I_{4,1}\left(1-x, -\tfrac{y-z}{z}\right) - I_{4,1}\left(\tfrac{1}{x}, \tfrac{1}{y}\right) - I_{4,1}\left(\tfrac{1}{x}, \tfrac{1}{z}\right) - I_{4,1}\left(x, \tfrac{y-z}{1-z}\right) + I_{4,1}\left(x, -\tfrac{z}{1-z}\right) \\
& - I_{4,1}\left(\tfrac{1}{1-y}, -\tfrac{y-z}{(1-y)z}\right) + I_{4,1}\left(\tfrac{x}{z}, \tfrac{y}{z}\right) + I_{4,1}\left(\tfrac{1-x}{1-y}, \tfrac{y(1-z)}{(1-y)z}\right) + I_{4,1}\left(\tfrac{1-x}{1-y}, -\tfrac{y-z}{(1-y)z}\right) \\
& + I_{4,1}\left(1-y, \tfrac{(1-y)z}{y(1-z)}\right) - I_{4,1}\left(\tfrac{x(1-y)}{x-y}, \tfrac{(1-y)z}{y(1-z)}\right) - I_{4,1}\left(\tfrac{x(1-y)}{x-y}, -\tfrac{(1-y)z}{y-z}\right) + I_{4,1}\left(\tfrac{x}{y}, \tfrac{y-z}{y(1-z)}\right) \\
& - I_{4,1}\left(\tfrac{x}{y}, \tfrac{x(1-z)}{(1-x)z}\right) - I_{4,1}\left(\tfrac{1-x}{1-z}, \tfrac{y}{z}\right) - I_{4,1}\left(1-z, \tfrac{z}{y}\right) + I_{4,1}\left(\tfrac{x(1-z)}{x-z}, \tfrac{y-z}{x-z}\right) \\
& - I_{4,1}\left(\tfrac{x(1-z)}{x-z}, -\tfrac{z}{x-z}\right) + I_{4,1}\left(\tfrac{x(1-z)}{x-z}, \tfrac{(x-y)z}{y(x-z)}\right) - I_{4,1}\left(\tfrac{x-z}{y-z}, -\tfrac{x-z}{(1-x)z}\right) + I_{4,1}\left(\tfrac{y}{z}, \tfrac{1}{z}\right) \\
& + I_{4,1}\left(-\tfrac{x-z}{z}, -\tfrac{x-z}{(1-x)z}\right) - I_{4,1}\left(\tfrac{y(x-z)}{(x-y)z}, -\tfrac{x-z}{(1-x)z}\right) \\
& - I_5\left(\tfrac{1}{1-x}\right) - I_5\left(\tfrac{1}{1-y}\right) + 2I_5\left(\tfrac{x(1-y)}{x-y}\right) + 2I_5\left(\tfrac{x}{y}\right) + I_5\left(-\tfrac{x-y}{(1-x)y}\right) \\
& + I_5\left(\tfrac{1-x}{1-z}\right) + I_5\left(\tfrac{1-z}{y-z}\right) - 4I_5\left(\tfrac{x(1-z)}{y-z}\right) - I_5\left(-\tfrac{(x-y)(1-z)}{(1-x)(y-z)}\right) + I_5\left(\tfrac{x-z}{y-z}\right) \\
& + 3I_5\left(\tfrac{x}{z}\right) - I_5\left(\tfrac{y}{z}\right) - I_5\left(-\tfrac{1-z}{z}\right) + 4I_5\left(-\tfrac{x(1-z)}{z}\right) + I_5\left(\tfrac{x(1-z)}{(1-x)z}\right) + 4I_5\left(\tfrac{y(1-z)}{z}\right) \\
& - 4I_5\left(\tfrac{xy(1-z)}{(x-y)z}\right) - I_5\left(-\tfrac{x-z}{z}\right) + I_5\left(-\tfrac{x-z}{(1-x)z}\right) + I_5\left(\tfrac{y(x-z)}{(x-y)z}\right) + 3I_5\left(-\tfrac{y-z}{z}\right) \\
& + I_5\left(-\tfrac{y-z}{(1-y)z}\right) + I_5\left(-\tfrac{x(y-z)}{(x-y)z}\right) \qquad \text{(mod products)}
\end{aligned}
$$

Further results for $I^{+}_{4,1}$

With computer search

Proposition (CGR 2019-?)

Certain degenerate versions of 5-term can be reduced

Nielsen polylogs.

$$I^{+}_{4,1}\left([x]-[y]+\left[\frac{y}{x}\right]-\left[\frac{1-y}{1-x}\right]+\left[\frac{x(1-y)}{(1-x)y}\right],z\right)=\sum \mathrm{Li}_5\text{'s}$$

~302 terms.

when $z=1$ *or* $z=y$ *or* $z=\frac{1-y}{x}$.

Older results

Proposition (C 2016)

Explicit reductions for the 2-term relations in Li_2, *or the 3-term relation in* Li_3 *as follows:*

$$I^{+}_{4,1}([x]+[\tfrac{1}{x}],y)\,,\, I^{+}_{4,1}([x]+[1-x],y)\,,\ \text{and } I^{+}_{4,1}(x,[y]+[1-y]+[1-\tfrac{1}{y}]-[1])$$

Still trying to find $I^{+}_{4,1}(\text{5-term},z)$.

Strategy to prove Zagier's conjecture

Strategy to prove ZPC - K-Theory background

Classifying space

- Quillen defines algebraic K-theory of a field $K_m(F) = \pi_m(\mathrm{BGL}^+F)$

Theorem (Borel, 1974/1977)

- *For each $n \geq 2$, there is a canonical class $\underline{c_{2n-1}} \in H^{2n-1}(\mathrm{GL}_n(\mathbb{C}), \mathbb{R})$*
- *The canonical class induces a regulator*

$$R_n^{\mathrm{bo}} \colon \underline{K_{2n-1}(F)} \to \mathbb{R}^{d_n}$$

identifying $K_{2n-1}(F)/\text{torsion}$ with a lattice in $\mathbb{R}^{d_n}$ (of full rank)

ZPC

- *One has*

$$\zeta_F(n) \sim_{\mathbb{Q}^\times} \boxed{\pi^{nd_{n+1}} \sqrt{|\Delta_F|}} \, \underline{\mathrm{covol}(R_n^{\mathrm{bo}})}$$

So try to express $\underline{R_n^{\mathrm{bo}} \text{ via } \mathscr{L}_n}$.

Strategy to prove ZPC - Grassmannian polylogs

Key tool

Definition (Grassmannian polylog)

Grassmannian polylog Gr_m is a multivalued analytic function defined by

$$d\,\mathrm{Gr}_m(\underbrace{v_1,\ldots,v_{2m}}_{2}) = \mathrm{Alt}_{2m}\,\mathcal{A}(v_1,\ldots,v_m \mid v_{m+1},\ldots,v_{2m})\cdot d\log\langle m+1,\ldots,2m\rangle$$

vectors in $\mathbb{C}^m$

where $\mathcal{A}(v_1,\ldots,v_m \mid w_1,\ldots,w_m)$ is geometrically defined Aomoto polylogarithm.

Theorem (Goncharov)

A single-valued version of Gr_n *represents* c_{2n-1}

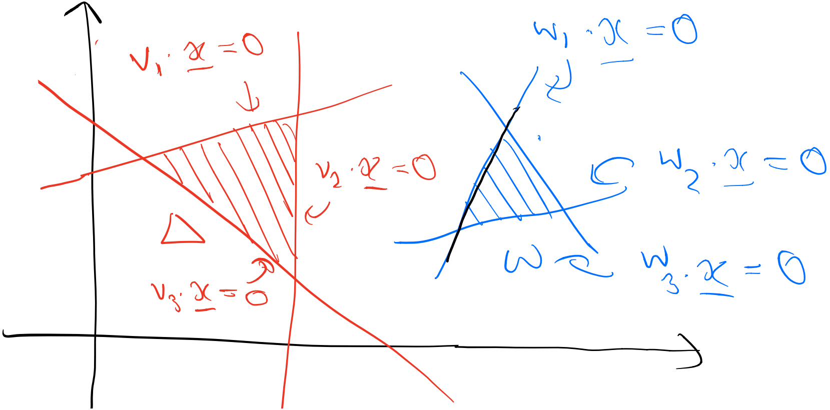
So try to reduce Gr_n (up to a cohomological coboundary) to $\mathscr{L}_n$

$$\delta \mathrm{Gr}_n \neq 0. \qquad \delta(\mathrm{Gr}_n + \text{correction}) = 0$$

Aomoto polylogs

$n=3$

2-d



$$\mathcal{A}(v_1, \ldots, v_n \mid w_1, \ldots, w_n) = \int_\Delta d\log\left(\frac{w_2 \cdot \mathbf{x}}{w_1 \cdot \mathbf{x}}\right) \wedge \cdots \wedge d\log\left(\frac{w_n \cdot \mathbf{x}}{w_1 \cdot \mathbf{x}}\right)$$

Construction of $c_{2\cancel{m}-1}$

$2n-1$

Construction of c_1

Represent $c \in H^{m-1}_{\text{cts}}(G, \mathbb{R})$ via cochain $\phi \colon G^m \to \mathbb{R}$.

Fact

$$\phi_1 \colon \mathrm{GL}_1(\mathbb{C})^2 \to \mathbb{R}$$
$$\phi_1(g_1, g_2) = \log(\det(g_1^{-1} g_2))$$

defines 1-cocycle, and represents c_1. $\rightsquigarrow$ „$\zeta_F(1)$"

Cocycle condition corresponds to log functional equation.

$$\log(x) - \log(y) = \log\left(\frac{x}{y}\right)$$

Construction of c_3

Introduce coordinates

$$\mathrm{Conf}_m(n) = \{(v_1 \ldots, v_n) | v_i \in \mathbb{C}^m\} \big/ \mathrm{GL}_m$$

Write

$$\langle i_1, \ldots, i_m \rangle = \det(v_{i_1} \cdots v_{i_m})$$

Classical cross-ratio is a function on $\mathrm{Conf}_4(2)$

$$v_1 = \begin{pmatrix} 1 \\ z_1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ z_2 \end{pmatrix}$$

$$\begin{aligned} \mathrm{cr}(v_1, \ldots, v_4) &= \frac{\langle 13 \rangle \langle 24 \rangle}{\langle 14 \rangle \langle 23 \rangle} \\ &= \frac{z_1 - z_3}{z_1 - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_3} \end{aligned}$$

where $z_i \in \mathbb{P}^1(\mathbb{C}) \leftrightarrow v_i \in \mathbb{C}^2$

Construction of c_3

Theorem (Bloch)

$$\phi_2 : \mathrm{GL}_2(\mathbb{C})^4 \to \mathbb{R}$$
$$\phi_2(g_1, \dots, g_4) = \mathscr{L}_2(\mathrm{cr}(g_1 v, \dots, g_4 v))$$

defines 3-cocycle, and represents c_3.

Get Zagier's Conjecture for $n = 2$ via Borel.

Cocycle condition corresponds to non-trivial $\mathscr{L}_2$ functional equation

$$\mathscr{L}_2\left([x] + [y] + \left[\frac{1-x}{1-xy}\right] + [1-xy] + \left[\frac{1-y}{1-xy}\right]\right) = 0$$

Famous five-term relation.

Construction of c_5

Goncharov defines a (pre-)triple-ratio

$$\begin{pmatrix} 1 & & & 1 & 1 & 1 \\ & 1 & & 1 & a & b \\ & & 1 & 1 & 1 & c \end{pmatrix}$$

$$\mathrm{cr}_3(v_1, \dots, v_6) = \frac{\langle 124\rangle\langle 235\rangle\langle 316\rangle}{\langle 125\rangle\langle 236\rangle\langle 314\rangle}$$

$$\frac{124 \cdot 235 \cdot 316}{125 \cdot 236 \;\; 314}$$

Theorem (Goncharov)

$$\phi_3 : \mathrm{GL}_3(\mathbb{C})^6 \to \mathbb{R}$$
$$\phi_3(g_1, \dots, g_6) = \mathrm{Alt}_6\, \mathscr{L}_3(\mathrm{cr}_3(g_1 v, \dots, g_6 v))$$

defines 5-cocycle, and represents c_5.

Get Zagier's Conjecture for $n = 3$ via Borel.

Cocycle condition corresponds to 840-term Li_3 functional equation. Related 22-term functional equation.

Higher ratios

Naive generalisations of cr

How to generalise the cross-ratio and triple-ratio?

$$\mathrm{cr}(v_1,\ldots,v_4) = \frac{\langle 13\rangle\langle 24\rangle}{\langle 14\rangle\langle 23\rangle}$$

$$\mathrm{cr}_3(v_1,\ldots,v_6) = \frac{\langle 124\rangle\langle 235\rangle\langle 316\rangle}{\langle 125\rangle\langle 236\rangle\langle 314\rangle}$$

1-x 'factors nicely'

$(1-x)\wedge x \otimes x \otimes x$

Naive guesses like

$$\mathrm{cr}_4(v_1,\ldots,v_8) = \frac{\langle 1235\rangle\langle 2346\rangle\langle 3457\rangle\langle 4518\rangle}{\langle 1238\rangle\langle 2345\rangle\langle 3456\rangle\langle 4517\rangle}$$

fail.

- Not Li_4 functional equations, or
- Don't define cocycles

↳ canonical.

$\mathrm{Alt}_8\, \mathcal{L}i_4(\mathrm{cr}_4(v_1,\ldots,v_8))$

↳ cocycle condition fails.

m-ratio

Despite failure of naive generalisation, Goncharov conjectures some generalisation exists

Conjecture

For $m \geq 2$, there exists

$$\sum_i \lambda_i [r_i] , \quad r_i \in \mathbb{Q}(\mathrm{Conf}_{2m}(m)) \tag{1}$$

such that

$$\phi_m(g_1, \ldots, g_{2m}) = \mathrm{Alt}_{2m} \sum_i \lambda_i \mathscr{L}_m(r_i(g_1 v, \ldots, g_{2m} v))$$

is a $(2m-1)$-cocycle and represents c_{2m-1}.

Formal linear combination (1) is called an m-ratio

Goncharov-Rudenko show 4-ratio exists, but do not construct it.

Reduction of Gr_m

Goal: Rewrite Gr_m in terms of $\mathscr{L}_m$

Problem: An obstruction exists, meaning this is impossible

Fix: Can modify Gr_m by trivial coboundary terms depending on $\leq 2m-1$ points. Find trivial coboundary correction which kills obstruction.

Goncharov-Rudenko already do this in weight 4.

$\hookrightarrow \sim 1993$.

$\delta(I_3, + cb) = 0$

Explicit reduction of Gr_4 and a 4-ratio

$I_{3,1}$ and ρ-coordinates

Definition ($I_{3,1}$)

$I_{3,1}$ multiple polylog is defined by

$$I_{3,1}(x,y) = \mathrm{Li}_{3,1}\left(\frac{y}{x}, \frac{1}{y}\right) = \sum_{0<n<m} \frac{y^{n-m}x^{-m}}{n^3 m}$$

Definition (ρ-coordinates)

Coordinates on $\mathrm{Conf}_8(4)$

Conf$_4$(2)

$$\rho_i = \underbrace{\rho_{i,i+1,i+2}}_{\text{mod } 6} = \frac{\langle i, i+1, i+2, 7\rangle}{\langle i, i+1, i+2, 8\rangle}$$

Shorthand $\rho_{i,j} = \rho_i - \rho_j$

Reduction of Gr_4

Theorem (CGR, 2019)

Modulo products

$$\frac{7}{144}\,\mathrm{Gr}_4 = \mathrm{Alt}_8\Big[I_{3,1}\Big(\frac{\rho_{1,2}\rho_{3,4}}{\rho_{3,2}\rho_{1,4}}, \frac{\rho_1}{\rho_{1,4}}\Big) + 2I_{3,1}\Big(\frac{\rho_{1,2}}{\rho_1}, \frac{\rho_{3,2}}{\rho_{3,4}}\Big) + 6\,\mathrm{Li}_4\Big(\frac{\rho_1\rho_{3,2}}{\rho_{1,2}\rho_{3,4}}\Big)\Big].$$

Proof.

Found with computer assistance. Explicit calculation of the symbol by hand. □

Note: some structure in this reduction.

Makes explicit first step of Goncharov-Rudenko.

$I_{1,1,1,1} = \sum' I_{3,1} + I_0$'s.

$\mathrm{Gr}_4 = \sum I_{3,1}$'s by earlier results.

Gr_4 coboundary

Goncharov gives Gr_4 coboundary as

$$\mathrm{Alt}_8\, I_{3,1}(\mathrm{cr}(34 \mid 2567), \mathrm{cr}(67 \mid 1345)) \,,$$

with projected cross-ratio

$$\mathrm{cr}(ab \mid cdef) = \frac{\langle abce \rangle}{\langle abcf \rangle} \frac{\langle abdf \rangle}{\langle abde \rangle} \,.$$

Symmetrise $I_{3,1}$ for convenience using

$$I_{3,1}(x,y) + I_{3,1}(x^{-1},y) = \mathrm{Li}_4\text{'s} \,,$$
$$I_{3,1}(x,y) + I_{3,1}(1-x,y) = \mathrm{Li}_4\text{'s} \,.$$

Write $\mathrm{Sym}_{36}(x,y)$ for these extra Li_4 terms.

Gr_4 to Li_4's

Theorem (Explicit 4-ratio, CGR, 2019)

$$\frac{7}{144}\mathrm{Gr}_4 + 2\,\mathrm{Alt}_8\, I^{\mathrm{sym}}_{3,1}(\mathrm{cr}(34|2567), \mathrm{cr}(67|1345)) =$$

$$\mathrm{Alt}_8 \Big\{ -V\Big(\tfrac{\rho_4}{\rho_1};\, [\tfrac{\rho_{4,2}}{\rho_{4,1}}; \tfrac{\rho_{4,1}}{\rho_{4,3}}] - [\mathrm{cr}(43|2685); \mathrm{cr}(48|7653)]$$

$$+ \tfrac{1}{4}[\mathrm{cr}(43|1256); \mathrm{cr}(43|1268)] - \tfrac{1}{12}[\mathrm{cr}(43|1256); \mathrm{cr}(42|1365)]\Big)$$

$$+ V(\tfrac{\rho_2}{\rho_1};\, -[\mathrm{cr}(43|2685; \mathrm{cr}(48|7653)] + [\mathrm{cr}(48|7235; \mathrm{cr}(48|7263)]$$

$$+ \tfrac{1}{2}[\mathrm{cr}(46|5238; 43|2568)])$$

$$+ \mathrm{Sym}_{36}(\tfrac{\rho_{1,2}\rho_{3,4}}{\rho_{1,4}\rho_{3,2}}, \tfrac{\rho_1}{\rho_{1,4}}) + 2\,\mathrm{Sym}_{36}(\tfrac{\rho_{1,2}}{\rho_1}, \tfrac{\rho_{3,2}}{\rho_{3,4}}) + 6\,\mathrm{Li}_4(\tfrac{\rho_1\rho_{3,2}}{\rho_{1,2}\rho_{3,4}}) \Big\}.$$

$V(x; y, z)$

$=$ 5-term

Corollary

Symmetrising over 9 *points gives a new* Li_4 *functional equation with* 1775 S_8*-orbits. Compute assistance gives* 368 S_8*-orbits.*

$0 = \mathrm{Alt}_9(*) \to$ 4-ratio!

$\mathrm{Sym}\,\mathrm{Gr} = \mathrm{Alt}_8\,(1234)\cdot(2345)\cdot(3456)\cdot(4567)$

Status of ZPC on $\zeta_F(5)$

Theorem (CGR, 2019)

Expression for Gr_5 *in terms of four* $I_{4,1}$ *terms and 2* Li_5*, under* Alt_{10}*. Coboundary correction term expressed via two* $I_{4,1}$ *terms.*

Together with reductions for $I^+_{4,1}$, have a starting point for the combinatorial step in weight 5.

Missing ingredient

Still searching for the reduction of

$$I^+_{4,1}(\text{5-term}, z) = \sum \mathrm{Li}_5\text{'s}$$

$\mathrm{Gr}_5 - \mathrm{cob} = \sum I_{4,1}(5\text{-term}, *) + \sum I^+_{4,1}(*, 22\text{-term}) = \mathrm{Li}_5\text{'s}$

Generally $\mathrm{Alt}_{11} \rightsquigarrow \mathrm{Li}_5$-free.

Theorem (CGR, 2019)

For any $m \geq 2$*, have*

$$-\frac{2m-1}{m!(m-1)!}\,\mathrm{Gr}_m = \mathrm{Alt}_{2m}\, I(0; 0, \rho_1, \rho_2, \ldots, \rho_{m-1}; \rho_m)\,,$$

with generalised ρ*-coordinates.*

Nested iterated integral.

Summary

- Definitions and key properties of multiple polylogarithms
- Motivation for studying: Zagier's polylogarithm conjecture on $\zeta_F(n)$
- Functional equations of Li_n
  - Directly
  - Through depth reduction to Li_n
- 'Key identity' $\mathbf{Q}_5$ for weight 5 MPL's
- Reductions of $I_{3,1,1}$, $I_{3,2}$ in weight 5
- Partial reductions of $I^{+}_{4,1}$ under dilog/trilog identities
- Strategy to prove Zagier's conjecture on $\zeta_F(n)$
- Explicit construction of the 4-ratio (re-proving $\zeta_F(4)$)
- Small progress towards $\zeta_F(5)$

"The remarkable dilog"

$\mathrm{Li}_2\left(\frac{1+\sqrt{5}}{2}\right) = \frac{\pi^2}{12} + \log^2 \phi$

$I_{4,1}(S.\text{-term}, y, z)$

$=$ depth 2

$\mathrm{Li}_3(\phi^2) \ldots$

$I_{3,1}\left(\frac{y+1}{y}\right) = \mathrm{Li}_2 x \wedge \mathrm{Li}_4 y$

$+ \mathrm{Li}_4 x \wedge \mathrm{Li}_2 y$

$S_{32}(\phi) =$

$I^{+}_{4,1}(S\text{-term}, 1)$

$\cong S_{32}(S\text{-term})$

$S_{42}(\phi^2)$ symbolic