Functional equations for Nielsen polylogarithms

Steven Charlton
Universität Hamburg

6 July 2021

JENTE

Outline

Joint work with H. Gangl & D. Radchenko

CNTP 15.2

arXiv 1908.04770

Classical polylogarithms

Definition (Polylogarithm)

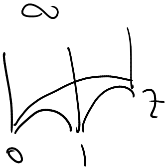
The weight n polylogarithm is

$$\mathrm{Li}_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n}, \quad |z| < 1 \qquad \mathrm{Li}_1(z) = -\log(1-z)$$

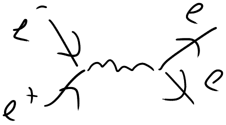
- For $n \geq 1$, derivative $\frac{\mathrm{d}}{\mathrm{d}z} \mathrm{Li}_n(z) = \frac{1}{z} \mathrm{Li}_{n-1}(z)$
- Analytical continuation to $\mathbb{C} \setminus \{0, 1\}$

$$\mathrm{Li}_n(z) = \int_0^z \frac{1}{t} \mathrm{Li}_{n-1}(t)\, dt$$

- Appears in
  - Hyperbolic geometry
  - Mathematical physics
  - Number Theory



Vol $\sim \mathrm{Li}_2(z)$



Feynman integrals

$\zeta_F(n)$

Integer.

Functional equations

Key feature

Polylogarithms satisfy interesting functional equations

Theorem (5-term relation, Abel, Spence, Kummer, ...)

For $|x| + |y| < 1$ *we have*

$\mathrm{Li}_2(1\ \ )$

$$\Big[\ \mathrm{Li}_2(x) + \mathrm{Li}_2(y) - \mathrm{Li}_2\left(\cancel{\frac{x}{1-y}}\right) - \mathrm{Li}_2\left(\cancel{\frac{y}{1-x}}\right) + \mathrm{Li}_2\left(\cancel{\frac{xy}{(1-x)(1-y)}}\right)$$

$$= -\log(1-x)\log(1-y)$$

$=$ products

- Proof on the level of power series
- Or by differentiation

The arguments arise from cross-ratios $\mathrm{cr}(a,b,c,d) = \frac{a-c}{a-d}\Big/\frac{b-c}{b-d}$ from 5 points in $\mathbb{P}^1(\mathbb{C})$

$\{\infty, 0, 1, x, y\}$

- Simpler 2-term relation when $y = 1 - x$.
- Does every Li_2 relation follows from the 5-term?

$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) =$ elementary

Multiple polylogarithms and iterated integrals

Definition (Multiple polylogarithm)

Depth d, weight $n_1 + \cdots + n_d$ multiple polylogarithm is

$$\mathrm{Li}_{n_1,\ldots,n_d}(z_1,\ldots,z_d) := \sum_{0<k_1<k_2<\cdots<k_r} \frac{z_1^{k_1}\cdots z_d^{k_d}}{k_1^{n_1}\cdots k_d^{n_d}}, \quad |z_i| < 1$$

- Multiple zeta values at $z_1 = \cdots = z_d = 1$
- Stuffle product structure

Definition (Iterated integral)

$$I(x_0; x_1, \ldots, x_N; x_{N+1}) = \int_{x_0<t_1<\cdots<t_N<x_{N+1}} \frac{\mathrm{d}t_1}{t_1 - x_1} \wedge \frac{\mathrm{d}t_2}{t_2 - x_2} \wedge \cdots \wedge \frac{\mathrm{d}t_N}{t_N - x_N}$$

$$\mathrm{Li}_{n_1,\ldots,n_d}(z_1,\ldots,z_d) = (-1)^d I(0; \tfrac{1}{z_1\cdots z_d}, \{0\}^{n_1-1}, \tfrac{1}{z_2\ldots z_d}, \{0\}^{n_2-1}, \cdots, \tfrac{1}{z_d}, \{0\}^{n_d-1}; 1)$$

- Shuffle product structure

Nielsen polylogarithms

Reference

K. S. Kölbig. Nielsen's generalized polylogarithms. *SIAM J. Math. Anal.*, 17(5), pp.1232–1258, 1986.

Definition (Nielsen polylogarithm)

$$S_{n,p}(z) := \frac{(-1)^{n+p-1}}{(n-1)!\,p!} \int_0^1 \log^{n-1}(t) \log^p(1-zt) \frac{dt}{t} \, .$$

Equivalently

$$S_{n,p}(z) = \mathrm{Li}_{\{1\}^{p-1},n+1}(1,\ldots,1,z) = (-1)^p I(0;\{1\}^p,\{0\}^n;z)$$

- Extends to $n=0$ and $p=0$
- $S_{0,p}(z) = \frac{(-1)^p}{p!}\log^p(1-z) = \frac{1}{p!}\mathrm{Li}_1^p(z), \quad S_{n,0}(z) = \frac{1}{n!}\log^n(z)$
- For $n \geq 1$, $\frac{\mathrm{d}}{\mathrm{d}z} S_{n,p}(z) = \frac{1}{z} S_{n-1,p}(z)$
- Appear (alongside harmonic polylogarithms) in QED calculations

Motivation: Zagier's polylogarithm conjecture

Let F be a number field, $\mathcal{O}_F$ its ring of integers

Definition (Dedekind zeta function)

The Dedekind zeta function of F is

$$\zeta_F(s) = \sum_{\substack{I \neq (0) \subset \mathcal{O}_F \\ \text{ideal}}} \frac{1}{N(I)^s}, \quad \mathrm{Re}(s) > 1$$

$F = \mathbb{Q}$

$I \leftrightarrow n \in \mathbb{Z}_{>0}$

$\zeta_{\mathbb{Q}}(n) = \zeta(n)$

Conjecture (Zagier, 1991, Schematic)

For $n \geq 2$,

$\pi^{*} \cdot \det(Li_n(\,)\,)$

$$\zeta_F(n) = \textit{rational} \cdot \pi^{nd_{n+1}} \sqrt{|\Delta_F|} \cdot \det\left(\textit{weight } n \textit{ polylogs}\right)_{i,j=1}^{d_n},$$

where $d_n = \underbrace{r_1 + r_2}_{d_n \neq 0}$, n odd, and $d_n = r_2$ if n even.

Remark

Strategy/idea to tackle the conjecture involves *reducing* higher depth MPL's to depth 1 Li_n

Basic properties of Nielsen polylogarithms

Special values $z = \pm 1, \frac{1}{2}$

At $z = 1$, Riemann zeta values

$$S_{n,p}(1) = \zeta(\{1\}^{p-1}, n+1) \in \mathbb{Q}[\mathrm{RZV}]$$

At $z = -1$, alternating MZV's

$$S_{n,p}(-1) = \zeta(\{1\}^{p-1}, \overline{n+1})$$

- Appear to be irreducible (for n even)
- $S_{4,2}(-1) = \zeta(1, \overline{5}) = \mathtt{h51}$ in MZV Datamine basis (irreducible)
- $S_{3,3}(-1) = \frac{3}{2}\mathtt{h51} + \frac{1}{14}\zeta(2)^3 - \frac{1}{4}\zeta(3)^2$

At $z = \frac{1}{2}$, also alternating MZV's

Apply transformation $z \mapsto 1 - 2z$:

= alternating MZV's

$$S_{n,p}(\tfrac{1}{2}) = \pm I(0, \{1\}^p, \{0\}^n; \tfrac{1}{2}) = \pm I(1, \{-1\}^p, \{1\}^n; 0)$$

Inversion

Proposition (§5.3, Kölbig)

For $z \in \mathbb{C} \setminus [0, \infty)$, *and* $n, p \geq 1$

$$S_{n,p}\left(\frac{1}{z}\right) = (-1)^n \sum_{k=0}^{p-1} (-1)^k \sum_{m=0}^{k} \cancel{\frac{\log^m(-z^{-1})}{m!}} \binom{n+k-m-1}{k-m} S_{n+k-m,p-k}(z)$$
$$+ (-1)^p \left(\cancel{\frac{\log^{n+p}(-z^{-1})}{(n+p)!}} + \sum_{j=0}^{n-1} \cancel{\frac{\log^j(-z^{-1})}{j!}} \underline{C_{n-j,p}} \right),$$

Here $C_{n,p}$ *is an explicit polynomial in* $S_{a,b}(1) = \zeta(\{1\}^{a-1}, b+1)$, *of homogeneous weight* $n+p$.

The depth p combination

$$S_{n,p}\left(\frac{1}{z}\right) - (-1)^n S_{n,p}(z)$$

reduces to lower depth and products.

Parity

Compare with Panzer's general parity theorem

Reflection

Proposition (§5.1, Kölbig)

For $z \in \mathbb{C} \setminus \big((-\infty, 0] \cup [1, \infty)\big)$, *and* $n, p \geq 1$

$$S_{n,p}(1-z) = \frac{(-1)^p}{n!\,p!} \cancel{\log^n(1-z)} \log^p(z) + \sum_{j=0}^{n-1} \cancel{\frac{\log^j(1-z)}{j!}} \left(S_{n-j,p}(1) - \sum_{k=0}^{p-1} \cancel{\frac{(-1)^k \log^k(z)}{k!}} S_{p-k,n-j}(z) \right).$$

After neglecting products, one has

$$S_{p,n}(z) = -S_{n,p}(1-z) + S_{n,p}(1) \quad (\text{mod products})\,.$$

Path deconcatenation

Follows from path deconcatenation and shuffle product of iterated integrals

$z \mapsto 1-z$ $\quad I(0, \ldots, z) \quad I(0, \ldots, 1)$

Simple consequences

Proposition ($S_{2,2}$ reduction)

For $z \in \mathbb{C} \setminus \big((-\infty, 0] \cup [1, \infty)\big)$, express $S_{2,2}(z)$ via Li_4 and products of lower weight.

$$\begin{aligned}
S_{2,2}(z) = \; & -\mathrm{Li}_4(1-z) + \mathrm{Li}_4(z) + \mathrm{Li}_4\left(\frac{z}{z-1}\right) - \mathrm{Li}_3(z)\log(1-z) \\
& + \frac{1}{4!}\log^4(1-z) - \frac{1}{3!}\log(z)\log^3(1-z) \\
& + \frac{1}{2!}\zeta(2)\log^2(1-z) + \zeta(3)\log(1-z) + \zeta(4)\,.
\end{aligned}$$

Proof.

Inversion of $S_{1,3}$ gives

$$S_{1,3}(z^{-1}) = -S_{1,3}(z) + \boxed{S_{2,2}(z)} - S_{3,1}(z) \quad (\text{mod products, constants})$$

Li_4

Then apply reflection. □

$\rightsquigarrow S_{3,1}$

Interlude: $\zeta_F(4)$ after Goncharov-Rudenko

Substituting (29) to the relation $\mathbf{Q}_3$ we get the 22-term relation for trilogarithm from [G91a]. Therefore the map $\{x\}_3 \longmapsto \{x\}_3$ induces an isomorphism

$$\mathrm{B}_3(\mathrm{F}) \xrightarrow{\sim} \mathbb{L}_3(\mathrm{F}). \tag{30}$$

The relation (29) has the following geometric interpretation. Take five points $(\infty; 0, x, 1, y)$ on $\mathbb{P}^1$, where the last four points are ordered cyclically. Then

$$\begin{aligned}&\{x,y\}_{2,1} = \\ &\{[\infty,0,x,1]\}_3 + \{[\infty,1,y,0]\}_3 + \{[\infty,y,0,x]\}_3 + \{[\infty,x,1,y\}_3 - \{[0,x,1,y]\}_3 - \{1\}_3.\end{aligned} \tag{31}$$

Definition 1.8. *The $\mathbb{Q}$-vector space $\mathbb{L}_4(\mathrm{F})$ is generated by elements $\{x\}_4$, where $x \in \mathbb{P}^1(\mathrm{F})$, and $\{x,y\}_{3,1}$ where $x, y \in \mathrm{F}^\times$, obeying the following relations:*

1. *The generators $\{x\}_4$ satisfy the 4-logarithmic relations $\mathcal{R}_4(\mathrm{F})$;*
2. *Specialization relations:*[9]

$$\begin{aligned}\{x,0\}_{3,1} &:= \mathrm{Sp}_{t\to 0}\{x,t\}_{3,1} = -\{x\}_4,\\ \{x,1\}_{3,1} &= -\left\{1-x^{-1}\right\}_4 - \{1-x\}_4 + \{x\}_4.\end{aligned} \tag{32}$$

3. $\mathbf{Q}_4$ *: For any configuration $(x_1, x_2, \ldots, x_7) \in \mathcal{M}_{0,7}(\mathrm{F})$ the following cyclic sum is zero:*

$$\begin{aligned}\mathrm{Cyc}_7\Big(&-\{[x_1,x_2,x_3,x_4],[x_4,x_6,x_7,x_1]\}_{3,1}\\ &+\{[x_1,x_2,x_3,x_4],[x_4,x_5,x_7,x_1]\}_{3,1}\\ &-\{[x_1,x_2,x_3,x_4],[x_4,x_5,x_6,x_1]\}_{3,1}\\ &+\{[x_1,x_2,x_4,x_6]\}_4 + \{[x_1,x_2,x_3,x_4,x_5,x_6]\}_4\Big) = 0.\end{aligned} \tag{33}$$

Conjecture 1.9. *Relation $\mathbf{Q}_4$ and its specializations imply the tetralogarithm relations $\mathcal{R}_4(\mathrm{F})$.*

Let us define the coproduct maps

$$\begin{aligned}\delta &: \mathbb{L}_2(\mathrm{F}) \longrightarrow \mathrm{F}^\times \wedge \mathrm{F}^\times,\\ \delta &: \mathbb{L}_3(\mathrm{F}) \longrightarrow \mathbb{L}_2(\mathrm{F}) \otimes \mathrm{F}^\times,\\ \delta &: \mathbb{L}_4(\mathrm{F}) \longrightarrow \mathbb{L}_3(\mathrm{F}) \otimes \mathrm{F}^\times \bigoplus \mathbb{L}_2(\mathrm{F}) \wedge \mathbb{L}_2(\mathrm{F}).\end{aligned} \tag{34}$$

First, we define them on the generators: the coproduct $\delta\{x\}_k$ is given by formula (6), and[10]

$$\begin{aligned}\delta\{x,y\}_{2,1} &= \left\{\frac{1-y}{1-x}\right\}_2 \otimes \frac{y}{x} + \left\{\frac{y}{x}\right\}_2 \otimes \frac{1-y}{1-x} + \{x\}_2 \otimes (1-y^{-1}) + \{y\}_2 \otimes (1-x^{-1}),\\ \delta\{x,y\}_{3,1} &= \{x,y\}_{2,1} \otimes \frac{x}{y} + \left\{\frac{x}{y}\right\}_3 \otimes \frac{1-x}{1-y} + \{x\}_3 \otimes (1-y^{-1}) - \{y\}_3 \otimes (1-x^{-1})\\ &\quad - \{x\}_2 \wedge \{y\}_2.\end{aligned} \tag{35}$$

Let us give a motivic interpretation of elements $\{x,y\}_{m-1,1}$ and their coproduct formula (35).

[9] Specialisation relations (32) could be deduced from relation $\mathbf{Q}_4$, but this would require long calculations.
[10] Formulas (35) coincide with the map $\kappa(x,y)$ given by formulas (5) and (6) in [G91a].

10

2. *Specialization relations:*[9]

$$\begin{aligned}\{x,0\}_{3,1} &:= \mathrm{Sp}_{t\to 0}\{x,t\}_{3,1} = -\{x\}_4,\\ \{x,1\}_{3,1} &= -\left\{1-x^{-1}\right\}_4 - \{1-x\}_4 + \{x\}_4.\end{aligned}$$

Li4's

$$\{x,y\}_{3,1} \cong I_{3,1}(x,y) - 3\,\mathrm{Li}_4\!\left(\tfrac{x}{y}\right)$$

$$\begin{aligned}\{x,1\}_{3,1} &\cong I_{3,1}(x,1) - 3\,\mathrm{Li}_4(x)\\ &= S_{2,2}(x) \quad (\text{mod products})\end{aligned}$$

Therefore, understanding $S_{n,p}$ is necessary

Simple consequences

Proposition ($S_{3,2}$ two-term)

The following two-term functional equation holds

$$S_{3,2}(1-z)+S_{3,2}(z)=\mathrm{Li}_5(1-z)+\mathrm{Li}_5(1-z^{-1})+\mathrm{Li}_5(z) \quad (\text{mod products, constants})$$

Proof.

Inversion of $S_{1,4}$ gives

$-S_{3,2}(1-z)$

$$S_{1,4}(z^{-1})=-S_{1,4}(z)+S_{2,3}(z)-S_{3,2}(z)+S_{4,1}(z) \quad (\text{mod products, constants})$$

Then apply reflection. □

Li_5

$\mathrm{Li}_2(1-z)+\mathrm{Li}_2(z) = \text{elementary}.$

Interlude: Motivic framework

Hopf algebra of MPL's

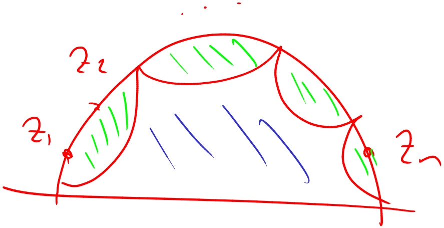
Motivic iterated integrals (Goncharov, Brown, ...)

Iterated integrals $I(x_0; x_1, \ldots, x_n; x_{n+1})$ can be upgraded to framed mixed Tate motives, to define

$$I^{\mathfrak{u}}(x_0; x_1, \ldots, x_n; x_{n+1}),$$

elements of a graded Hopf algebra $\mathcal{H}$ (grading is by weight)

$$\Delta I^{\mathfrak{u}}(x_0; x_1, \ldots, x_n; x_{n+1}) = \sum_{\substack{0 = i_0 < i_1 < \cdots \\ < i_k < i_{k+1} = n}} I^{\mathfrak{u}}(x_0; x_{i_1}, \ldots, x_{i_k}, x_{n+1}) \otimes \prod_{p=0}^{k} I^{\mathfrak{u}}(x_{i_p}; x_{i_p+1}, \ldots, x_{i_{p+1}-1}; x_{i_{p+1}})$$



Lie coalgebra

A graded Hopf algebra induces a Lie coalgebra $\mathcal{L} = \mathcal{H}/\mathcal{H}_{>0}\cdot\mathcal{H}_{>0}$, with $\delta = \Delta - \Delta^{\mathrm{op}}$

- $I^{\mathfrak{u}}(x_0; x_1, \ldots, x_n; x_{n+1})$ becomes $I^{\mathfrak{m}}(x_0; x_1, \ldots, x_n; x_{n+1}) \quad (\text{mod products})$

Example

- $\delta \log^{\mathfrak{m}}(x) = 0$
- $\delta\, \mathrm{Li}_n^{\mathfrak{m}}(x) = \mathrm{Li}_{n-1}^{\mathfrak{m}}(x) \wedge \log^{\mathfrak{m}}(x) \in \mathcal{L}_{n-1} \wedge \mathcal{L}_1$
- $\delta^{\geq 2} S_{3,2}(x) = -\mathrm{Li}_2^{\mathfrak{m}}(x) \wedge \zeta^{\mathfrak{m}}(3) + \cancel{\zeta^{\mathfrak{m}}(2) \wedge \mathrm{Li}_3^{\mathfrak{m}}(x)}$ 0

weight 1 ∧ weight 4

$\ker \delta = \sum \lambda_i \mathrm{Li}_n(x_i)$

Conjecture (Goncharov, Freeness)

The kernel of $\delta^{\geq 2}$ is generated by classical polylogarithms $\mathrm{Li}_n^{\mathfrak{m}}(x)$

Expectation

$$S_{3,2}^{\mathfrak{m}}(\text{dilogarithm relations}) = \sum \mathrm{Li}_5^{\mathfrak{m}}\text{'s}$$

Symbols of MPL's

- Algebraic invariant of MPL's
- Captures structure of "main-terms" of identities

Iterated coproduct ($\Delta^{[m]}$)

$$\mathcal{H}_m \xrightarrow{\Delta^{(m-1,1)}} \mathcal{H}_{m-1} \otimes \mathcal{H}_1 \xrightarrow{\Delta^{(m-2,1)} \otimes \mathrm{id}} \mathcal{H}_{m-2} \otimes \mathcal{H}_1^{\otimes 2} \xrightarrow{\Delta^{(m-3,1)} \otimes \mathrm{id}^{\otimes 2}} \cdots \xrightarrow{\Delta^{(1,1)} \otimes \mathrm{id}^{\otimes m-1}} \mathcal{H}_1^{\otimes m}$$

Definition (Symbol)

Symbol of $I^{\mathfrak{u}}(x_0; x_1, \ldots, x_N; x_{N+1})$ is

$$\mathcal{S} I^{\mathfrak{u}}(x_0; x_1, \ldots, x_N; x_{N+1}) = \Delta^{[N]} I^{\mathfrak{u}}(x_0; x_1, \ldots, x_N; x_{N+1}) \,.$$

Identify $I^{\mathfrak{u}}(a; b; c) = \log\left(\frac{b-c}{b-a}\right)$ with rational function $\frac{b-c}{b-a}$ and use multiplicative tensors.

- Dynkin operator also kills products (= mod-products symbol)

$$\mathcal{S}\, \mathrm{Li}_n^{\mathfrak{u}}(z) = (1-z) \otimes z \otimes z^{\otimes n-2} \quad \rightsquigarrow \quad (1-z) \wedge z \otimes z^{\otimes n-2}$$

Five-term relation for $S_{3,2}$

Five-term precursor

Recall (Inversion and One minus)

Recall we have:

$$S_{3,2}(1-z) + S_{3,2}(z) = \mathrm{Li}_5(1-z) + \mathrm{Li}_5(1-z^{-1}) + \mathrm{Li}_5(z) \quad (\text{mod products, constants})$$

$$S_{3,2}(z^{-1}) + S_{3,2}(z) = 3\,\mathrm{Li}_5(z) \quad (\text{mod products, constants})$$

Five term relation:

$$\sum_{i=1}^{5} (-1)^i \mathrm{Li}_2(\mathrm{cr}(x_1, \ldots, \widehat{x_i}, \ldots, x_5)) = 0 \quad (\text{mod products})$$

Note

- $1 - \mathrm{cr}(x_1, x_2, x_3, x_4) = \mathrm{cr}(x_1, x_3, x_2, x_4)$, and $\mathrm{Li}_2(1-x) + \mathrm{Li}_2(x) = 0 \quad (\text{mod products})$
- $1 /\ \mathrm{cr}(x_1, x_2, x_3, x_4) = \mathrm{cr}(x_2, x_1, x_3, x_4)$, and $\mathrm{Li}_2(x^{-1}) + \mathrm{Li}_2(x) = 0 \quad (\text{mod products})$

So antisymmetrise:

$$\mathrm{Alt}_5 \left\{ \mathrm{Li}_2(\mathrm{cr}(x_1, x_2, x_3, x_4)) \right\} = 0 \quad (\text{mod products})$$

Five-term

Theorem ($S_{3,2}$ of the five-term relation)

Following identity holds between the mod-products symbols of $S_{3,2}$ *and* Li_5

$$\mathrm{Alt}_5 \Big\{ 11 S_{3,2}\big(\mathrm{cr}(x_1, x_2, x_3, x_4)\big) \\ + 15\,\mathrm{Li}_5\big(\mathrm{r}_1(x_1, \dots, x_5)\big) - 9\,\mathrm{Li}_5\big(\mathrm{r}_2(x_1, \dots, x_5)\big) + \mathrm{Li}_5\big(\mathrm{r}_3(x_1, \dots, x_5)\big) \Big\} \overset{\text{Ш}}{=} 0\,.$$

Here

$$\mathrm{cr}(x_1, x_2, x_3, x_4) \coloneqq \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_4)(x_2 - x_3)}$$

is the classical cross-ratio, and $\mathrm{r}_1, \mathrm{r}_2, \mathrm{r}_3$ are the following 'higher ratios'

$$\mathrm{r}_1(x_1, \dots, x_5) \coloneqq -\frac{(x_1 - x_2)(x_1 - x_4)(x_3 - x_5)}{(x_1 - x_3)(x_1 - x_5)(x_2 - x_4)}\,,$$

$$\mathrm{r}_2(x_1, \dots, x_5) \coloneqq -\frac{(x_1 - x_2)^2(x_3 - x_4)(x_3 - x_5)}{(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_5)}\,,$$

$$\mathrm{r}_3(x_1, \dots, x_5) \coloneqq -\frac{(x_1 - x_2)^3(x_1 - x_5)(x_3 - x_4)^2(x_3 - x_5)}{(x_1 - x_3)^3(x_1 - x_4)(x_2 - x_4)(x_2 - x_5)^2}\,.$$

Proof strategy

Remark

Symbol captures differential properties of MPLs. Can be computed/manipulated algorithmically, so can check using computer assistance.

$$S_{3,2}(z) \rightsquigarrow -(1-z) \wedge z \otimes ((1-z) \sqcup\!\sqcup\, z^{\otimes 2})$$
$$\mathrm{Li}_5(z) \rightsquigarrow -(1-z) \wedge z \otimes z^{\otimes 3}$$

$\rightsquigarrow$ Compare with Rogers's criterion.

Step 1 Show complicated irreducibles cancel

Step 2 Consider representation on $\bigwedge^2 V \otimes \mathrm{Sym}^3(V) \cong W_4 \oplus W_6$, $V = \mathrm{span}_{S_5}\{\mathrm{cr}(x_i, x_{i+1}, x_{i+2}, x_{i+3})\}$

Step 3 Show 6-dimensional part is trivial

Step 4 Check identity in 4-dimensional part by polynomial calculations

Refinements and corollaries

Corollary (Radchenko 2016)

Alternating over 6 points gives a Li_5 functional equation

$$\mathrm{Alt}_6 \left\{ \mathrm{Li}_5 \Big(15[\mathrm{r}_1(x_1, \ldots, x_5)] - 9[\mathrm{r}_2(x_1, \ldots, x_5)] + [\mathrm{r}_3(x_1, \ldots, x_5)] \Big) \right\} \overset{Ш}{=} 0$$

Refinement

Numerically checkable $S_{3,2}$(five-term) identity holds

- Using clean single-valued polylogarithms
- Directly by analysing the coproduct

$S_{32} + Li_5 = 0$ (Ш)

Duhr, Gang, $\to \tilde{S}_{32} + \tilde{Li}_5 = 0$ constant

Corollary

Any "accessible" dilogarithm identity/functional equation lifts to an $S_{3,2}$ depth reduction identity/functional equation

Evaluations for $S_{3,2}$

Example (Dilogarithm evaluation)

For $\phi = \frac{1+\sqrt{5}}{2}$ the golden ratio, then

$$\mathrm{Li}_2(\phi^{-2}) = \frac{2}{5}\zeta(2) - \log^2(\phi)$$

- Arises from specialising five-term relation at the start to $x = \phi^{-1}, y = 1$
- Simplify using $\phi^{-2} = 1 - \phi^{-1}$, and variants

Corresponding $S_{3,2}$ evaluation

$$\begin{aligned} S_{3,2}(\phi^{-2}) = {} & \frac{1}{66}\,\mathrm{Li}_5\left([\phi^{-6}] - 32[\phi^{-3}] + \frac{201}{2}[\phi^{-2}] - 48[\phi^{-1}]\right) \\ & + \mathrm{Li}_4\left(\phi^{-2}\right)\log(\phi) + \frac{1}{2}\zeta(5) - \frac{2}{11}\zeta(4)\log(\phi) \\ & - \zeta(3)\,\mathrm{Li}_2\left(\phi^{-2}\right) - \frac{20}{33}\zeta(2)\log(\phi)^3 + \frac{79}{330}\log(\phi)^5\,. \end{aligned}$$

Extra: Results and expectations in higher weight

Expectations for $S_{4,2}$

Observation

$\delta^{\geq 2} S^{\mathfrak{L}}_{4,2}(z) = -\operatorname{Li}^{\mathfrak{L}}_3(z) \wedge \zeta^{\mathfrak{L}}(3)$

- Expect $S_{4,2}$ satisfies trilog functional equations modulo Li_6

Remark

So far, only know

$$S_{4,2}(z) - S_{4,2}(z^{-1}) = 4\operatorname{Li}_6(z) \quad (\text{mod products})$$

$$\begin{aligned} & S_{4,2}(z) + S_{4,2}(1-z) + S_{4,2}(1-x^{-1}) = \\ & 2\big(\operatorname{Li}_6(z) + \operatorname{Li}_6(1-z) + \operatorname{Li}_6(1-z^{-1})\big) \quad (\text{mod products}) \end{aligned}$$

- Can still investigate special values.

$$\operatorname{Li}^{\mathfrak{u}}_3(-1) = -\tfrac{3}{4}\zeta^{\mathfrak{u}}(3) \quad \rightsquigarrow \quad \delta^{\geq 2} S^{\mathfrak{L}}_{4,2}(-1) = 0$$

So $S_{4,2}(-1) = \zeta(1,\overline{5})$ should be depth 1?

Special values of $S_{4,2}$

Claim

$$S_{4,2}(-1) = \frac{1}{13}\left(\frac{1}{3}\,\mathrm{Li}_6\left(-\frac{1}{8}\right) - 162\,\mathrm{Li}_6\left(-\frac{1}{2}\right) - 126\,\mathrm{Li}_6\left(\frac{1}{2}\right)\right) - \frac{1787}{624}\zeta(6)$$
$$+ \frac{3}{8}\zeta(3)^2 + \frac{31}{16}\zeta(5)\log(2) - \frac{15}{26}\zeta(4)\log^2(2) + \frac{3}{104}\zeta(2)\log^4(2) - \frac{1}{208}\log^6(2)\,.$$

Remark

Analytic proof via identity $\mathrm{Li}_{5,1}(-x,-1) = 9$ $S_{4,2}$ terms $+$ 117 Li_6 terms $\pmod{\text{products}}$

Motivic structure

Key points

- $\Delta' S^{\mathfrak{m}}_{4,2}(-1) = \frac{3}{4}\zeta^{\mathfrak{m}}(3) \otimes \zeta^{\mathfrak{u}}(3) + \frac{31}{16}\log^{\mathfrak{m}}(2) \otimes \zeta^{\mathfrak{u}}(5)$
- $\Delta^{(5,1)}\,\mathrm{Li}^{\mathfrak{m}}_6(x) = \mathrm{Li}^{\mathfrak{m}}_5(x) \otimes \log^{\mathfrak{u}}(x)$
- $\mathrm{Li}^{\mathfrak{m}}_5\left(-\frac{1}{8}\right) - 162\,\mathrm{Li}^{\mathfrak{m}}_5\left(-\frac{1}{2}\right) - 126\,\mathrm{Li}^{\mathfrak{m}}_5\left(\frac{1}{2}\right) = \frac{403}{16}\zeta^{\mathfrak{m}}(5) +$ products.

Summary

$$\widetilde{I}_{4,1,1}(5\text{-term}, y, z) = \boxed{\widetilde{I}^{*}_{4,1,1}(1, y_i, z_i)} + \mathrm{Li}_6\text{'s} + \text{depth } 2.$$

$$\rightarrow I_{4,2} + I_{5,1} + I_6.$$

- Definitions and key properties of Nielsen polylogarithms
- Motivic yoga to determine which reductions occur
  - Connections to Zagier's polylogarithm conjecture, and
  - Goncharov's freeness conjecture
- Main result: five-term relation for $S_{3,2}$
- Evaluations for $S_{3,2}$ at dilogarithm identities

$$I^{+}_{5,1}(\mathrm{Li}_3, y) = \mathrm{Li}_6 \rightsquigarrow S_{4,2}(\text{things}) = \mathrm{Li}_6.$$

$$\mathrm{Li}_5.$$

- Extra: Evaluations in higher weight by motivic viewpoint

$$\uparrow$$

$$I^{+}_{4,1}(5\text{-term}, y) = \mathrm{Li}_5\text{'s} + S_{3,2}(*)$$