On MZV's and generators of alternating MZV's

Multiple zeta values:

$$\zeta(s_1, \dots, s_d) = \sum_{n_1 < \dots < n_d} \frac{1}{n_1^{s_1} \cdots n_d^{s_d}}$$

alternating: $s_i \rightsquigarrow \bar{s_i}$ insert $(-1)^{s_i}$

Call $s_1 + \dots + s_d = \text{weight}$, $d = \text{depth}$

MZV's studied by Euler, $d=2$

Resurgence w/ Zagier and Hoffman ~ 1990

Recently: Hoffman investigated an odd variant

$$t(s_1, \dots, s_d) = \sum_{1 \le n_1 < \dots < n_d} \frac{1}{(2n_1 - 1)^{s_1} \cdots (2n_d - 1)^{s_d}}$$

Originally studied by Nielsen for $d=1$?

Why?

MZV's and multiple polylogs in general appear in Feynman diagram scattering amplitude calcs.

Source of some easy-sounding v. hard problems:

what is $\zeta(2)$? $= \pi^2/6$.

is $\zeta(2)$ irrational / trans? ✓

$\zeta(3)$? irrational ~1970 Apéry

Little else known.

By introducing extra structure (via multiple variables ...) maybe we get some insight?

MZV's satisfy many relations:

Eg in weight 12, expect 2^{10} possible, reduces to only 12

(via LLL, then algebraically!)

Some of many (all?) relations:

Multiple MZV's as sums

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{m=1}^{\infty} \frac{1}{m^r} = \sum_{n<m} + \sum_{n>m} + \sum_{n=m}$$

$$= \zeta(s,r) + \zeta(r,s) + \zeta(r+s)$$

$s=r=2 \to 2\zeta(2,2) + \zeta(4).$

Multiply as integrals

$$\zeta(s_1,\dots,s_d) = (-1)^d \int_{0<t_1<\dots<t_{wt}<1} \frac{dt}{t-1} \underbrace{\frac{dt}{t} \cdots \frac{dt}{t}}_{s_1-1} \cdots \frac{dt}{t-1} \underbrace{\frac{dt}{t}\frac{dt}{t}}_{s_d-1} \qquad (*)$$

Eg $\underbrace{\zeta(2)\zeta(2)}_{\text{shuffle } 0<t_1<t_2<1 \text{ and } 0<s_1<s_2<1.}$ $= 4\zeta(1,3) + 2\zeta(2,2)$

So $4\,3(1,3) = 5(4)$.

[Regularisation to handle $s_1 = 1$.]

Some experimentation (LLL to find numerical guesses for dimensions)

$d_k = \dim($ weight k MZV's $)$

$= d_{k-2} + d_{k-3}$

$d_0 = 1,\ d_1 = 0,\ d_2 = 1.$

Conjecture (Hoffman)

$\zeta(k_1, \ldots, k_d)$, $k_i \in \{2, 3\}$

are a basis for MZV's.

Thm (Brown '12).

$\zeta^{(m)}(k_1, \dots, k_d)$ $k_i \in \{2,3\}$

span the space of MZV's.

are a basis for space of motivic MZVs

Recently:

Thm (Murakami '21)

$t^{(m)}(k_1, \dots, k_d)$ $k_i \in \{2,3\}$

span the space of MZV's

basis for space of motivic MZVs.

Thm (C)

i) $t^{m}(k_1, \dots, k_{d-1}, k_d + 1)$ $k_i \in \{1,2\}$

are linearly independent

(partial answer to Saha basis conjecture)

ii) $t^{neg}(k_1, \dots, k_d)$ $\quad k_i \in \{1, 2\}$

span space of (extended) mtv's and all alternating mtv's.

Proof strategy (in Brown's case for simplicity).

Ingredient 1: Identity

$$\zeta(2^a 3 2^b) = 2 \sum_{r=1}^{a+b+1} (-1)^r \left(\binom{2r}{2a+2} - (1-2^{-2r}) \binom{2r}{2b+1} \right) \zeta(2r+1)\, \zeta(\underbrace{2^{a+b+1-r}})$$

$$\zeta(2^k) = \frac{\pi^{2k}}{(2k+1)!}$$

Proven by Zagier - magic

$$\sum_{a,b\geq 0} \text{LHS}\; x^{2a} y^{2b} = \text{Hypergeometric}$$

$$\sum_{a,b\geq 0} \text{RHS}\; x^{2a} y^{2b} = \sum \text{sine \& cosine} \times \underbrace{\text{digamma}}$$

$$\psi(x) = \frac{d}{dx} \log \Gamma(x)$$

Thm: i) growth conditions

ii) agreement for (x, x), for (n, y), (x, k)
$\qquad\qquad\qquad\quad\ \ \mathbb{N}$ $\qquad\qquad\quad\ \mathbb{N}$
$\qquad\qquad\qquad\quad\ \ 2$ $\qquad\qquad\quad\ \ 2$

$\Rightarrow$ agreement everywhere.

[Similar identity for

$t(2^a 3 2^b)$ by Murakami,
similar proof

$t(2^a, 2^b)$ by $<$, needing

Some asymptotic properties of ${}_4F_3$ as

$t(2 \cdots 2 1)$ is divergent and need regularisation

includes $\delta_a 2 \log 2 \times t(2^b)$

$\delta_b \log 2 \; t(2^a)$

terms

Ingredient 2: $\boxed{\cancel{\text{Motivic Framework}}}$

Black box.

$\zeta(k_1, \ldots, k_d) \overset{\text{wt}}{\rightsquigarrow} \zeta^m(k_1, \ldots, k_d)$

Analytic Algebraic

$\zeta^m(k_1,\dots,k_d)$

$\approx \left[H^n(X \setminus A), \ \text{simplex } \Delta, \ \text{form } (*) \right]^m$

$\approx \dfrac{(\mathbb{P}^1 \setminus \{0,1,\infty\})^n}{\text{symmetry}}$ $\quad 0<t_1<\dots<t_n<1 \quad \dfrac{dt}{t-1}\left(\dfrac{dt}{t}\right)^{k_1-1}\cdots\dfrac{dt}{t-1}\left(\dfrac{dt}{t}\right)^{k_d-1}$

Idea: We keep track of Δ and form ω separately, to avoid "accidental" identities.

Recover $\zeta(k_1,\dots,k_d)$ by

$$\text{per}\,\zeta^m(k_1,\dots,k_d) = \int_\Delta \omega$$

Conjecture (Grothendieck)

per (generally) is injective, i.e. all relations are motivic.

Relations on ζ^m are geometric:

— change of variables

— linearity, etc.

Upshot: ζ^m are more rigid.

Form a graded Hopf algebra

So $\underset{\text{wt } 2}{\zeta^m(2)}$, $\underset{\text{wt } 3}{\zeta^m(3)}$, $\underset{\text{wt } 5}{\zeta^m(5)}$ are trivially isolated. ($\emptyset$ has wt 0)

Exists a coproduct / coaction

$$\Delta \zeta^m = \sum \zeta^a \otimes \zeta^m$$

$\circlearrowleft$ mod $\pi i\,(\zeta(2))$

Helpful to take the linearized version

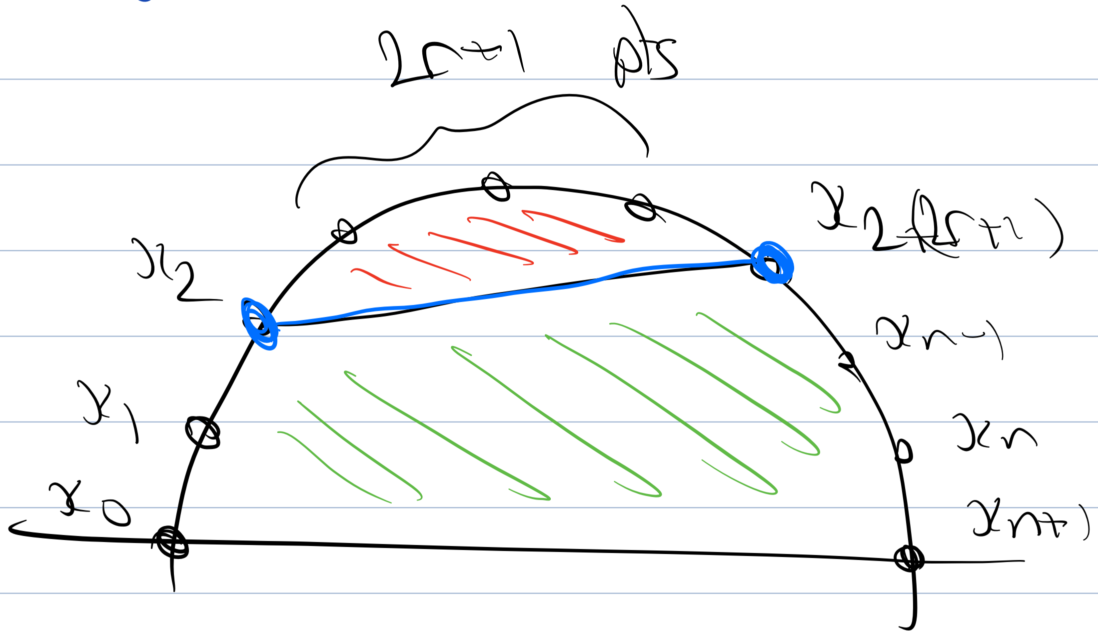
$$D_{2r+1}\zeta^m(k_1,\dots,k_d) = \sum \zeta^L \otimes \zeta^m$$

$\circlearrowleft$ mod products

Mnemonic: $\downarrow \frac{dt}{t-x_i}$

$$D_{2r+1} I(x_0; x_1,\dots,x_n; x_{n+1})$$

$$= \sum_{\text{segments}} I^L(\quad)\otimes I^m(\quad)$$



Key property:

$$\operatorname{Ker} D_{<N} = \zeta^{m}(N)\,\mathbb{Q}$$

$$\parallel$$

$$D_3 \oplus D_5 \oplus \cdots \oplus D_{N-1}$$

↳ gives a recursive way to understand identities.

Brown's proof: $\zeta(h_1 \ldots h_d)$, $h_i \in \{2,3\}$

level $=$ # $h_i = 3$

defines filtration.

double zetas or worse

$\parallel$

$\zeta(2^{\alpha} 3\, 2^{\beta} 3\, 2^{\gamma})$

Show:

$$gr_L D_{2r+1}\, \zeta(\text{level } L)$$

$$= \sum \boxed{\circledast\, \zeta(2s+1)} \otimes \zeta(\text{level } L-1)$$

$$+ \cancel{\boxed{\text{complicated}} \otimes \zeta(\text{level} \leq L-2)}$$

Comes from $\zeta(2^\alpha 3 2^\beta)$

$$= \zeta(2\alpha + 2\beta + 3)$$

mod products.

Pass to associated graded quotient.

Then project $\zeta(2s+1) \longmapsto 1$

(as we can never get it from $\otimes \zeta(\cdots)$)

So $\partial_{N,L}\, \zeta^{m}(\text{level } L,\ \text{weight } N)$

$$= \sum \underbrace{\circledast}_{\text{coefficient from } \zeta(2^\alpha 3 2^\beta)} \zeta^{m}(\text{level} < L,\ \text{weight} < N)$$

Idea: $\partial_{N,L}$ would convert relation in level L to relation in level $< L$.

If $\partial_{N,L}$ is <u>injective</u>, the level $< L$ relation is non-trivial.

By induction no relation exists in level $< L$.

Starts w/ $\zeta(2), \zeta(2,2), \zeta(2,2,2), \dots$ in level 0

all different weight!

Why is $\partial_{N,L}$ injective? Relies on properties of $\zeta(2^a 3 2^b)$ coefficients.

Eg: $N = 8, L = 2$

$$
\begin{array}{c|cccccc}
 & 223 & 232 & 23 & 322 & 32 & 3 \\
\hline
2233 & 8 & & -7/2 & & & 28 \\
2323 & & 3 & -11/2 & & & \\
2332 & & & 9/2 & & & \\
3223 & & & 12 & 3 & & -291/16 \\
3232 & & & & & 9/2 & 75/8 \\
3322 & & & & -2 & 12 & -291/16
\end{array}
$$

integer

odd

even

$$\times 2 \qquad\qquad \times 2 \qquad \times 16$$

So $\det = \dfrac{\text{odd}}{2^k} \neq 0$.

Murakami: filtration $L = \#3$'s

$\hookrightarrow t(\text{level } L) = \sum S(\text{level} \leq L)$ and vice versa.

C: i) filtration $L = \#1$'s $+ \#3$'s.

ii) filtration $L = \#1$'s.

$\longrightarrow$ Maps $\partial_{N,L}$ injective so get independence.