Tate's Thesis Student Seminar 5

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Continue briefly on from last time when Jens was trying to give some link between the idele class group and the ideal class group, and between characters of each.

The idele group of the number field K is

$$\mathbb{I} = \mathbb{A}_K^\times \coloneqq \prod_\nu{}' K_\nu^\times ,$$

where ν runs through the finite and infinite places of K. The diagonal embedding $k \hookrightarrow \mathbb{A}$, and $k^\times \hookrightarrow \mathbb{A}^*$ sending $\alpha \mapsto (\ldots, \alpha, \ldots)$.

The idele class group is the quotient

$$\mathbb{A}^\times / K^\times .$$

Last time Jens was trying to relate this to the class group, and so link the characters of each.

In Neukirch, the correspondence is between Hecke and Grossencharacters modulo $\mathfrak{m}$.

Given a Hecke character χ of the Idele class group $A^*/K^* \to \mathbb{C}^*$, but the image of the unit group $\prod_{\mathfrak{p} \nmid \infty} U_\mathfrak{p} \to S^1$ is compact and totally disconnected. Here $U_\mathfrak{p} = \{x \in K_\mathfrak{p} \mid v(x) = 0\} = \mathbb{Z}_\mathfrak{p}$, and $U_\mathfrak{p}^(n) = 1 + \mathfrak{p}^n$. So the kernel is of the form $\prod_{\mathfrak{p} \nmid \infty} U_\mathfrak{p}^(n_\mathfrak{p})$ with $n_\mathfrak{p} = 0$ for almost all $\mathfrak{p}$. Then $\mathfrak{m} = \prod_\mathfrak{p} \mathfrak{p}^{n_\mathfrak{p}}$ is the conductors.

For each $\mathfrak{p} \nmid \infty$, chose a fixed prime $\pi_\mathfrak{p}$ of $K_\mathfrak{p}$. Then we get a homomorphism

$$\begin{aligned} I^{\text{mod } \mathfrak{m}}(K) &\to \mathbb{A}^\times / \text{units group} \cdot K^\times \\ \mathfrak{p} &\mapsto \langle (\ldots, 1, \underbrace{\pi_\mathfrak{p}}_{\mathfrak{p}}, 1, \ldots) \rangle , \end{aligned}$$

and extend multiplicatively. This doesn't depend on the choice of $\pi_\mathfrak{p}$, since the idele we are killing any units which may appear.

Another way to see a link between the idele and ideal class groups is as follows. The ideal class group is $I(K)/P(K)$, the fractional ideals mod the principal fractional ideals. By unique factorisation of fractional ideals, $I(K)$ is the free abelian group on the prime ideals.

Consider the following map:

$$\begin{aligned} \varphi \colon \mathbb{I} &\to I(K) \\ (\ldots, a_\mathfrak{p}, \ldots) &\mapsto \prod_\mathfrak{p} \mathfrak{p}^{\operatorname{ord}_\mathfrak{p} a_\mathfrak{p}} , \end{aligned}$$

where the product is well defined since there are only finitely many not equal to 1. Those with $a_\mathfrak{p} = 1$, have ord $= 0$.

Take $\alpha \in K^\times \subset \mathbb{I}$, then this maps to the principal fractional ideal (α), since the product is over exactly those primes which appear in (α), and the powers match. So $\varphi(K^\times) \subset P(K)$ and φ descends to the quotient:

$$\begin{array}{ccc} \mathbb{I} & \xrightarrow{\varphi} & I(K) \\ \downarrow & & \downarrow \\ \mathbb{I}/K^\times & \longrightarrow & I(K)/P(K) \end{array}$$

(And this is the modern way to do class field theory…)
Then characters of the ideal class group pull back to characters of the idele class group.

Now I'd like to go back and fill in some of the gaps that have developed as we've jumped around a bit. This is mainly going to be Section 2.4 and 2.5 of Tate.

Firstly recall some definitions and notation:

- Here k denotes the completion of a number field at some place
- Quasicharacter: continuous homomorphism of $k^* \to \mathbb{C}^*$, character $k^* \to S^1$.
- Any quasicharacter is of the form $c(\alpha) = \widetilde{c}(\widetilde{\alpha}) \left|\alpha\right|^s$
- If k is $\mathfrak{p}$-adic, the conductor of quasicharacter is $\mathfrak{p}^\nu$ with minimal ν such that $\widetilde{c}(1+\mathfrak{p}) = 1$.

The Fourier transform $\widehat{f}$ of f on k^+ is given by:

$$\widehat{f}(\eta) = \int f(\xi) \exp(-2\pi \mathrm{i} \Lambda(\eta\xi)) \mathrm{d}\xi$$

and satisfies the inversion formula

$$\widehat{\widehat{f}}(-\xi) = f(\xi)$$

Let $f(\xi)$ denote a complex valued function on k^+, where k is the completion of a number field at some place, and $f(\alpha)$ is the restriction to k^*. Restrict to those functions for which:

i) $f(\xi)$ and $\widehat{f}(\xi)$ are continuous, $\in L_1(k^+)$, and

ii) $f(\alpha) \left|\alpha\right|^\sigma$ and $\widehat{f}(\alpha) \left|\alpha\right|^\sigma \in L_1(k^\times)$ for $\sigma > 0$.

(L_1 being absolutely integrable function. Other people take slightly different requirements…)

The set of these functions is going to be written $\mathfrak{z}$. Introduce a ζ-function of k as follows.

Definition 1 (Def 2.4.1)**.** For each $f \in \mathfrak{z}$, introduce a function $\zeta(f, c)$ of quasi–characters c defined for all quasi-characters of exponent grater than 0 by:

$$\zeta(f, c) = \int f(\alpha) c(\alpha) \mathrm{d}\alpha \,.$$

Such a function is a ζ-function of k. (I presume we're integrating over all of k.)

(Recall that a quasi-character is in general given by $c(\alpha) = \widetilde{c}(\widetilde{\alpha}) \left|\alpha\right|^s$, for some s. It's exponent is $\sigma := \operatorname{Re} s$.)

Two quasi-characters are called equivalent if their quotient is unramified. By Lemma 2.3.1 such an equivalence class is all characters of the form $c_0(\alpha) \left|\alpha\right|^s$, for some fixed c_0. By introducing this complex parameter s we can view such an equivalence class as a Riemann surface.

For $\mathfrak{p}$ Archimidean/infinite s is determined uniquely by c, so this surface is just $\mathbb{C}$.

For $\mathfrak{p}$ non/finite/discrete s is determined mod $2\pi \mathrm{i}/\log \mathrm{N}\,\mathfrak{p}$. So the surface is $\mathbb{C}/(2\pi \mathrm{i}/\log \mathrm{N}\,\mathfrak{p})\mathbb{Z}$, i.e. a cylinder.

With this we can sensibly talk about functions of quasi-characters being at a point, in a region. We can talk about singulatiries, etc. Also we can consider the question of analytic continuation outside the region of exponent > 0. But of course this is for each equivalence class of quasi-characters/Riemann surface separately.

Lemma 2 (Lem 2.4.1)**.** *A ζ-function is regular in the domain of all quasi-charactes of exponent greater than 0. I.e. in* $\operatorname{Re} s > 0$.

Proof. Apparently this is a routine thing. The integral converges absolutely near $s = 0$, and so one can show it has a derivative there, and that the derivative is given by differentiating under the integral sign. (*** HOW ***?) □

Now we'll move on to the analytic continuation and functionanl equations of these local ζ-functions.

Define $\widehat{c}(\alpha) = |\alpha|\, c\alpha^{-1}$. This is a sort of inverse character, not the Fourier transform of the character.

Lemma 3 (Lem 2.4.1)**.** *In the domain* $0 <$ *exponent* < 1, *we have:*

$$Z := \zeta(f, c)\zeta(\widehat{g}, \widehat{c}) = \zeta(\widehat{f}, \widehat{c})\zeta(g, c)$$

i.e. this is symmetric in f and g, for any two functions $f, g \in \mathfrak{z}$.

Proof. Write down this as the product of two integrals, and then as a double integral

$$\begin{aligned} Z &= \int f(\alpha)c(\alpha)\mathrm{d}\alpha \cdot \int \widehat{g}(\beta)c(\beta)^{-1}\,|\beta|\,\mathrm{d}\beta \\ &= \iint_{k^* \times k^*} f(\alpha)\widehat{g}(\beta)c(\alpha\beta^{-1})\,|\beta|\,\mathrm{d}(\alpha,\beta) \end{aligned}$$

Set $\beta' = \alpha\beta$. Then $\mathrm{d}\beta' = \mathrm{d}\beta$, by the left-invariance of Haar measure. So:

$$= \iint_{k^* \times k^*} f(\alpha)\widehat{g}(\alpha\beta)c(\beta^{-1})\,|\alpha\beta|\,\mathrm{d}(\alpha,\beta)$$

And write this as an integrated integral:

$$= \int \underbrace{\left(\int f(\alpha)\widehat{g}(\alpha\beta)\,|\alpha|\,\mathrm{d}\alpha\right)}_{\text{symmetric in } f \text{ and } g?} c(\beta^{-1})\,|\beta|\,\mathrm{d}\beta$$

To show the symmetry, consider:

$$\iint f(\xi)g(\eta)\exp(-2\pi\mathrm{i}\Lambda(\xi\beta\eta))\mathrm{d}(\xi,\eta)$$

which by Fubini is:

$$\int f(\xi)\left(\int g(\eta)\exp(-2\pi\mathrm{i}\Lambda(\xi\beta\eta))\mathrm{d}\eta\right)\mathrm{d}\xi = \int_{k^+} f(\xi)\widehat{g}(\xi\beta)\mathrm{d}\xi$$

But by Lemma 2.3.2:

$$\int_{k^+} f(\xi)\widehat{g}(\xi\beta)\mathrm{d}\xi = \int_{k^+-0} f(\xi)\widehat{g}(\xi\beta)\mathrm{d}\xi$$
$$= \int_{k^\times} f(\alpha)\widehat{g}(\alpha\beta)\,|\alpha|\,\mathrm{d}_1\alpha$$

where $\mathrm{d}_1\alpha$ is a Haar measure on $k^\times$, so differs by a multiplicative constant from $\mathrm{d}\alpha$. So the integral above is symmetric in f and g, and so is Z. □

So now we come to the 'Main Theorem', the analytic continuation and functional equation of the ζ-functions.

Theorem 4 (Thm 2.4.1)**.** *A ζ-function has an analytic contuation to the domain of all quasi-characters given by a functional equation of the type:*

$$\zeta(f, c) = \rho(c)\zeta(\widehat{f}, \widehat{c})$$

The factor $\rho(c)$ is independent of the function a f, is memomorphic on quasi-charcters $0 <$ Exponent < 1 by the functional equation, and has an analytic continuation to all quasi-characters.

Proof. Laster, for each class C of quasi-characters We'll find particular functions $f_C \in \mathfrak{z}$ such that:

$$\rho(c) = \zeta(f_C, c)/\zeta(\widehat{f_C}, \widehat{c})$$

is defined, so meromorphic, in $0 <$ exponent < 1. These functions will be familiar meromorphic functions of s so will be analytically continuable, and so the result will follow directly.

By the previous we have for any function $f \in \mathfrak{z}$:

$$\zeta(f, c)\zeta(\widehat{f_C}, \widehat{c}) = f(\widehat{f}, \widehat{c})\zeta(f_C, c)\,,$$

so on dividing through, we get:

$$\zeta(f, c) = \rho(c)\zeta(\widehat{f}, \widehat{c})$$

□

Maybe look at Lemma 2.4.3 in Tate, which shows some simple properties of $\rho(c)$ directly from the functional equation.

Now for real and complex quasicharacters, some just bog standard integration. For $\mathbb{R}$, $u = \{\pm 1\}$, so two classes of quasicharacters. For $\mathbb{C}$, the classes of quasicharacters are $c_n(\alpha)\,|\alpha|^s$, with $c_n(\alpha) = \alpha^n$ish. Integrate.

[Some of the equalities that appear in Tate come from functional equations for the Γ function. The duplication formula:

$$\Gamma(z)\Gamma\left(z + \tfrac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)\,,$$

and the duplication formula:

$$\Gamma(1 - z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}\ ]$$