Goncharov coproduct on motivic MZV's

§0 Multiple zeta values, iterated integrals, background / definitions

§1 Formal & Motivic integrals, and the coproduct

§2 Applications to transcendence questions

⌜ §3 Extensions by Brown ⌟

References: • Goncharov "Galois symmetries of fundamental groupoids"

⌜ • Brown "Mixed Tate motives over $\mathbb{Z}$"
"Decomposition of motivic MZVs"

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§0 MZV's & iterated integrals

Recall

$$\zeta(k_1, \dots, k_d) := \sum_{0 < n_1 < \dots < n_d} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}}$$

Analytic definition, conveys transcendence problems:

$$\zeta(2) = \frac{\pi^2}{6} \notin \mathbb{Q}, \notin \overline{\mathbb{Q}}$$

Euler 1735; Legendre 1794; Lindemann 1882

$$\zeta(3) = 1.202\ldots \notin \mathbb{Q}, \overset{??}{\in} \overline{\mathbb{Q}}$$

Apéry 1978

Expect $\notin$???

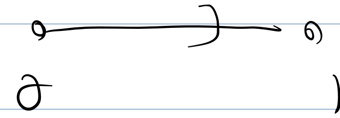
$$\zeta(5) = 1.036\ldots \overset{??}{\in} \mathbb{Q}$$

Expect $\notin \overline{\mathbb{Q}}$,

Expression via integral:

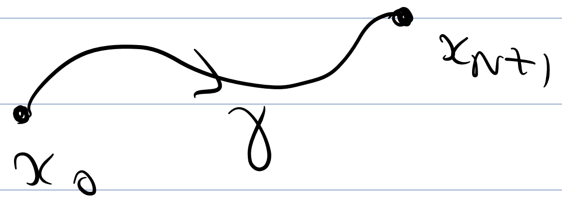
$$\underline{\zeta(k_1,\dots,k_d)} = (-1)^d I_{id}(0; 1\{0\}^{k_1-1} \cdots 1\{0\}^{k_d-1}; 1)$$

where



$$I_\gamma(x_0; x_1,\dots,x_N; x_{N+1}) = \int_{0<t_1<\dots<t_N<1} \omega_{x_1}(\gamma(t_1))\wedge\dots\wedge\omega_{x_N}(\gamma(t_N))$$

for $\gamma:[0,1]\to\mathbb{C}$, $\gamma(0)=x_0$, $\gamma(1)=x_{N+1}$, $\omega_{x_i}(t)=\frac{dt}{t-x_i}$



Proof: Expand geometric series & integrate term-by-term.

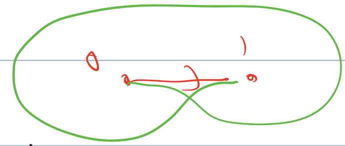
Generalisation:

$$\mathrm{Li}_{k_1 \cdots k_d}\left(\frac{x_2}{x_1}, \frac{x_3}{x_2}, \dots, \frac{1}{x_d}\right)$$
$$= (-1)^d I_\gamma\left(0;\, x_1\{0\}^{k_1-1} \cdots x_d\{0\}^{k_d-1};\, 1\right)$$

where $\mathrm{Li}_{k_1 \cdots k_d}$ is multiple polylogarithm function

$$\mathrm{Li}_{k_1 \cdots k_d}(z_1 \cdots z_d) = \sum_{0<n_1<\cdots<n_d} \frac{z_1^{n_1} \cdots z_d^{n_d}}{n_1^{k_1} \cdots n_d^{k_d}}$$

(plus analytic continuation along path γ.) if $\gamma \neq \mathrm{id}$.



Properties of $I_{(\gamma)}$:

$\int_a^b \emptyset = 1$ (Convention)

i) $I_\gamma(a;b) = 1$

ii) Shuffle product $\int_{0<t<1} \omega_1 \int_{0<s<1} \omega_2 = \int_{0<t<s<1} \omega_1\omega_2 + \int_{0<s<t<1} \omega_1\omega_2$

$$I_\gamma(a;x_1,\dots,x_r;b)\, I_\gamma(a;x_{r+1}\dots x_{r+s};b)$$

$$= \sum_{\underbrace{\sigma\in(r,s)\text{-shuffles}}_{\substack{\sigma\in S_{r+s},\\ \sigma(1)<\dots<\sigma(r)\\ \sigma(r+1)<\dots<\sigma(r+s)}}} I_\gamma(a;x_{\sigma(1)}\dots x_{\sigma(r+s)};b)$$

Encoded via $x_1\dots x_r ⧢ x_{r+1}\dots x_{r+s}$ with formal words, where

$$aw ⧢ bv = a(w ⧢ bv) + b(aw ⧢ v)$$

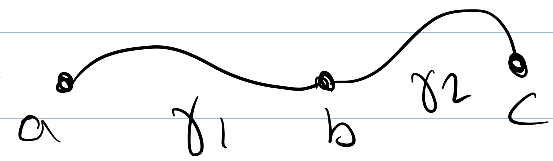
$\ulcorner$ $\rightsquigarrow$ Shuffle regularisation to define $I(x_0;x_1,\dots,x_N;x_{N+1})$ when $x_0=x_1$ OR $x_N=x_{N+1}$

$$I(\overset{x_0}{0};\overset{x_1}{0};\overset{x_{N+1}}{1}) := 0$$

$$I(\underset{x_0}{0};\underset{x_{N+1}}{1};\underset{x_{N+1}}{1}) := 0 \quad \lrcorner$$

iii) Path composition: $\int_a^b \omega + \int_b^c \omega = \int_a^c \omega$

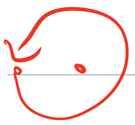
$$I_\gamma(a; x_1 \dots x_N; c) = \sum_{i=0}^{N} I_{\gamma_1}(a; x_1 \dots x_i; b)\, I_{\gamma_2}(b; x_{i+1} \dots x_N; c)$$

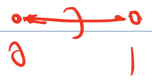
$\gamma =$ 

iv) $I(\underline{a}; x_1 \dots x_N; \underline{a}) = 0 \,, \quad \forall N \geq 1$

based

$\int_a^a \omega = 0$





§ 1 Formal & motivic integrals

Define a formal version by symbols

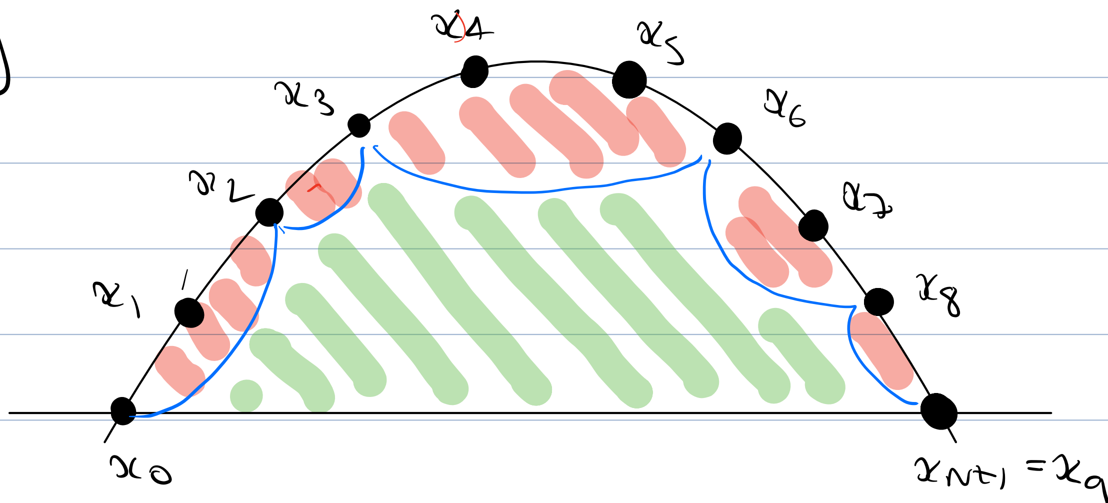
$$\mathbb{I}(x_0; x_1, \dots, x_N; x_{N+1})$$

satisfying i) – iv) above. Make into Hopf algebra by <u>defining</u>

$$\Delta \mathbb{I}(x_0; x_1 \ldots x_N; x_{N+1})$$

$$= \sum_{\substack{0=s_0<s_1<\cdots \\ <s_k<s_{k+1}=N+1}} \prod_{i=0}^{k} \mathbb{I}(x_{s_i}; x_{s_i+1} \cdots x_{s_{i+1}-1}; x_{s_{i+1}}) \otimes \mathbb{I}(x_{s_0}; x_{s_1}, \ldots, x_{s_k}; x_{s_{k+1}})$$

Pictorially



$$\rightsquigarrow \mathbb{I}(x_0; x_1; x_2)\, \mathbb{I}(x_2; x_3)\, \mathbb{I}(x_3; x_4 x_5; x_6)\, \mathbb{I}(x_6; x_7; x_8)\, \mathbb{I}(x_8; x_9)$$

$$\otimes\ \mathbb{I}(x_0; x_2 x_3 x_6 x_8; x_9)$$

⌜ Count given by $\mathbb{I}(x_0; x_1 \ldots x_N; x_{N+1}) \mapsto 0$ for $N \geq 1$

Antipode recursively constructed since wt 0 is $\mathbb{Q}$. ⌟

Do need to check compatibility of Δ with $\cdot$, using i) - iv) above

$$\Delta(x \cdot y) \overset{?}{=} \Delta x \cdot \Delta y$$

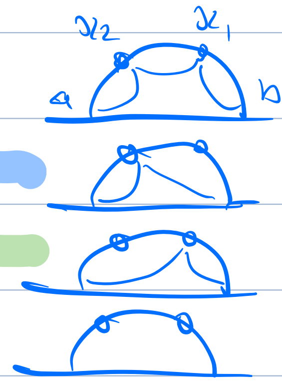
Works combinatorially:

$$\Delta \mathbb{I}(a;x_1;b)\,\mathbb{I}(a;x_2;b)$$

$$= \Delta\big(\mathbb{I}(a;x_1x_2;b) + \mathbb{I}(a;x_2x_1;b)\big)$$

$$\begin{aligned}
= \;& 1 \otimes \mathbb{I}(a;x_1x_2;b)\\
&+ \mathbb{I}(x_1;x_2;b) \otimes \mathbb{I}(a;x_1;b)\\
&+ \mathbb{I}(a;x_1;x_2) \otimes \mathbb{I}(a;x_2;b)\\
&+ \mathbb{I}(a;x_1x_2;b) \otimes 1
\end{aligned}$$

$$\begin{aligned}
&+ 1 \otimes \mathbb{I}(a;x_2x_1;b)\\
&+ \mathbb{I}(x_2;x_1;b) \otimes \mathbb{I}(a;x_2;x_1)\\
&+ \mathbb{I}(a;x_2;x_1) \otimes \mathbb{I}(a;x_1;b)\\
&+ \mathbb{I}(a;x_1x_2;b) \otimes 1
\end{aligned}$$



$$\begin{aligned}
= \;& 1 \otimes \mathbb{I}(a;x_1;b)\,\mathbb{I}(a;x_2;b)\\
&+ \mathbb{I}(a;x_1;b)\,\mathbb{I}(a;x_2;b) \otimes 1\\
&+ \big(\mathbb{I}(\underline{a};x_2;\underline{x_1}) + \mathbb{I}(\overline{x_1};x_2;\underline{b})\big) \otimes \mathbb{I}(a;x_1;b)\\
&+ \big(\mathbb{I}(a;x_1;x_2) + \mathbb{I}(x_2;x_1;b)\big) \otimes \mathbb{I}(a;x_2;b)
\end{aligned}$$

$$\begin{aligned}
= \;& 1 \otimes \mathbb{I}(a;x_1;b)\,\mathbb{I}(a;x_2;b)\\
&+ \mathbb{I}(a;x_1;b)\,\mathbb{I}(a;x_2;b) \otimes 1\\
&+ \mathbb{I}(a;x_2;b) \otimes \mathbb{I}(a;x_1;b)\\
&+ \mathbb{I}(a;x_1;b) \otimes \mathbb{I}(a;x_2;b)
\end{aligned}$$

$$= \left(1 \otimes \mathbb{I}(a; x_1; b) + \mathbb{I}(a; x_1; b) \otimes 1\right)$$
$$\cdot \left(1 \otimes \mathbb{I}(a; x_2; b) + \mathbb{I}(a; x_2; b) \otimes 1\right)$$

$$= \Delta \mathbb{I}(a; x_1; b) \cdot \Delta \mathbb{I}(a; x_2; b)$$

Generalises to any $\Delta \mathbb{I} \cdot \mathbb{I}$ via combinatorial means. [In recursive definition of $a w \sqcup b v'$, consider whether a, b in $s_1', \dots, s_k$ or not.]

So far this is only formal; what is the connection with I?

Technical: Integral $I_\gamma(x_0; x_1, \dots, x_N; x_{N+1})$ is a period of a framed Hodge-Tate structure

[Idea: Hodge structure generalises/abstracts the result that

$$\Rightarrow \underbrace{H^n(M, \mathbb{Q}) \otimes \mathbb{C}}_{\text{singular}} \cong \bigoplus_{p+q=n} \underbrace{H^{p,q}(M)}_{(p,q)\text{-forms}}$$

$dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{i_{p+1}} \wedge \dots \wedge d\bar{z}_{i_{p+q}}$

Hodge-Tate structure is a mixed Hodge structure (combination of Hodge structures), with $h_{p,q} = 0$ if $\underline{p \neq q}$.

$h_{p,q}$: dim $H^{p,q}$

<u>Framing</u> picks out a particular period:

$$\begin{array}{c|cc} & \boxed{\gamma_1} & \gamma_2 \\ \omega_1 & \int_{\gamma_1}\omega_1 & \int_{\gamma_2}\omega_1 \\ \boxed{\omega_2} & \boxed{\int_{\gamma_1}\omega_2} & \int_{\gamma_2}\omega_2 \end{array}$$

$\int_{\gamma_1}\omega_2$: "<u>the</u> period"

Equivalence classes of Hodge-Tate structures (with the same period) form a Hopf algebra, graded by weight

$\rightsquigarrow$ leads to motivic integrals

$$I^m(x_0; x_1, \dots, x_N; x_{N+1})$$

N = weight

Properties:
- graded by weight
- coproduct
- independent of path γ.

Since I^m independent of γ, the period $\overline{I}$ can't depend on γ, i.e. we should work "mod branch cuts", equivalently "mod $i\pi$".

$$\mathrm{per}\, I^m(x_0; x_1 \dots x_N; x_{N+1})$$

$$= \overline{I}_\gamma(x_0; x_1 \dots x_N; x_{N+1}) \quad (\text{"mod } i\pi\text{"})$$

some $\gamma: x_0 \to x_{N+1}$

+ technicalities

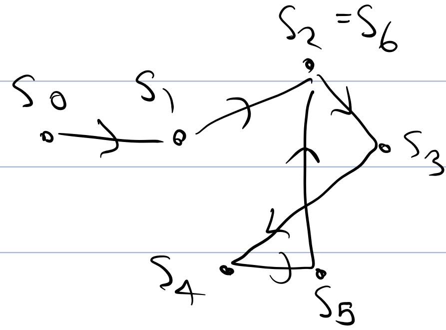
As $\pi^2 \equiv 0$ ("mod $i\pi$"), have that

$$\zeta^m(2) := I^m(0; 10; 1) = 0.$$

Since I^m's form a Hopf algebra (via Hodge structure considerations), there is an actual coproduct ΔI^m. We need to relate it to $\Delta \overline{I}$ above.

↳ Connection goes via a "path algebra" $P(S)$, $\underline{S}$ a set of points.

Vector space basis: $p_{s_0, \dots, s_n}$ correspond to paths



Structure(s) from composition of paths. One shows that

$\longrightarrow$ $\mathrm{Aut}\, P(S)$ is a group scheme

Fact: The regular functions $\mathcal{O}(G)$ on G form a Hopf algebra

Then one describes (details!)

$$\mathcal{O}(\underbrace{\mathrm{Aut}\, P(S)}_{\mathbb{I}}) = \text{polynomial algebra in variables } \underline{I_{s_0, \dots, s_{N+1}}}$$

So get an isomorphism (details!)

$$\mathcal{O}(\underline{\mathrm{Aut}\, P(S)}) \cong \boxed{\text{formal integrals } \mathbb{I}}$$

BUT: $P(S)$ is connected to the motivic fundamental group $\pi_1^{m}(\underline{\mathbb{A}^1} \setminus \underline{S})$, hence(!) to motivic iterated integrals I^{m}

§ 2 Applications to transcendence

In $\mathbb{R}$, we don't know whether

i) $\zeta(5) \in \mathbb{Q}$ expect $\notin \mathbb{Q}$

ii) $\zeta(3), \zeta(5) \in \overline{\mathbb{Q}}$ expect $\notin \overline{\mathbb{Q}}$

iii) $\zeta(3)$ & $\zeta(5)$ linearly / algebraically dependent
expect alg. ind.

iv) $\zeta(3,5)$ reduces to polynomial in $\zeta(n)$
expect irreducible

All these expectations hold for ζ^m,

$$\zeta^m(k_1,\dots,k_d) := (-1)^d I^m(0; 1\{0\}^{k_1-1} \cdots 1\{0\}^{k_d-1}; 1).$$

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i) $\zeta^m(2n+1) \notin \mathbb{Q}$:

$\Rightarrow$ Weight is a grading for ζ^m. Since $\zeta^m(2n+1)$ has weight $2n+1$, $\mathbb{Q}$ has weight 0, $[\zeta^m(2n+1)] \notin \mathbb{Q}$
$\hookrightarrow \neq 0$ $\qquad \sim \frac{\zeta(3)}{(i\sqrt{\pi})^3} \notin \mathbb{Q}$

ii) $\zeta^m(2n+1)$ transcendental

Say $p(x) = \sum \underbrace{\lambda_i}_{\in \mathbb{Q}} x^i$, and

$$p(\zeta^m(2n+1)) = 0$$

$$\|$$

$$\sum \lambda_i \zeta^m(2n+1)^i = 0 \qquad (*)$$

Since $\zeta^m(2n+1)$ weight $2n+1$, and $\zeta^m(2n+1)^i$ has weight $(2n+1)i$, the $(2n+1)i$-part of $(*)$

$$\lambda_i \zeta^m(2n+1)^i = 0$$

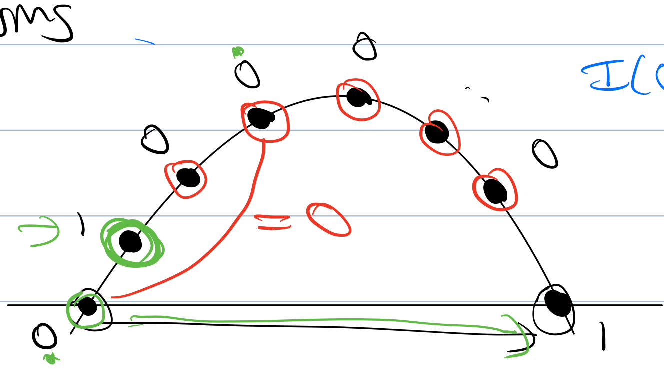
So $\lambda_i = 0 \ \forall i \Rightarrow p(x) \equiv 0$.

"primitive"

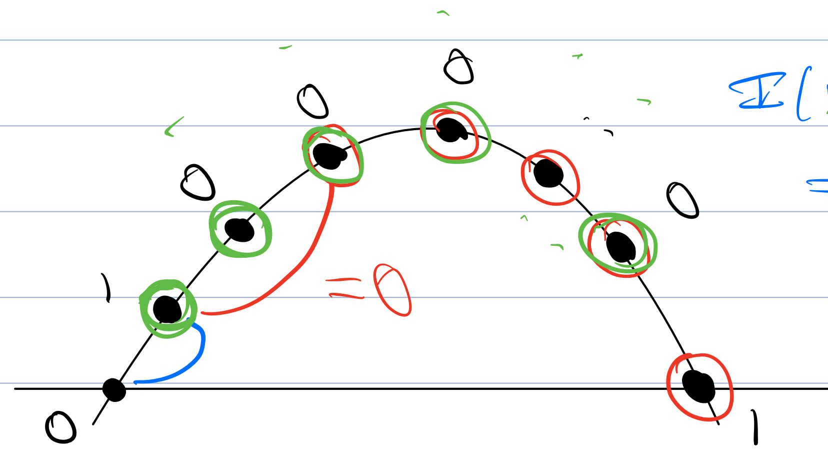
Prop: $\Delta \underbrace{\zeta^m(2n+1)}_{= -I^m(0;1\{0\}^{2n};1)} = 1 \otimes \zeta^m(2n+1) + \zeta^m(2n+1) \otimes 1$

Proof: Δ terms

$I(0;\cdots;0) = 0$

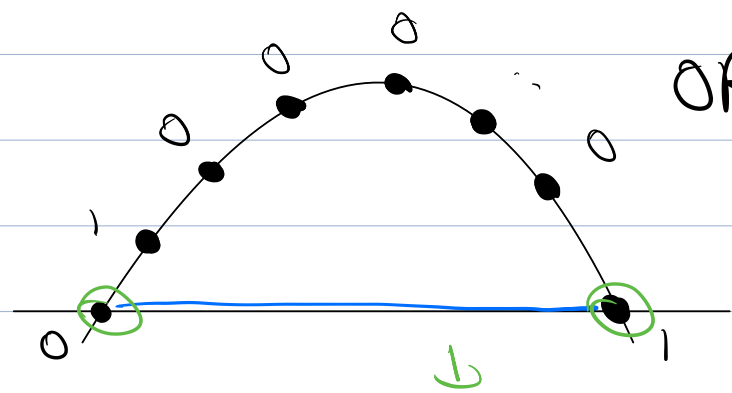


So $s_1 = 1$

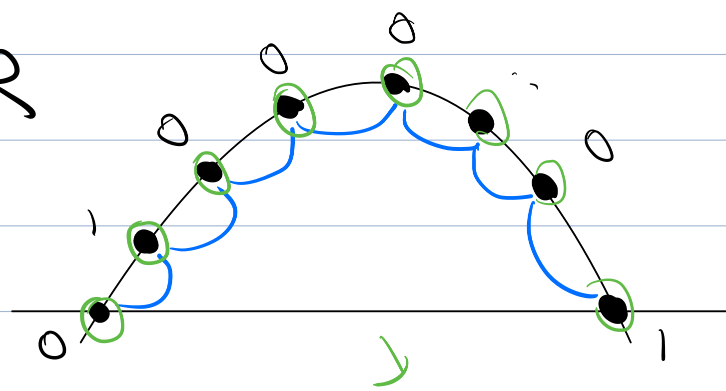


$$I(1;\overbrace{0\cdots0}^{n};0) = \frac{1}{n!}\underbrace{I(1;0;0)^n}_{=0}$$

So $s_2 = 2$



OR



$$\Delta = \zeta^m(2n+1)\otimes 1 + 1\otimes \zeta^m(2n+1)$$

iii) $\zeta^m(3)$ & $\zeta^m(5)$ alg. indep.

More interesting as eg.

$$\zeta^m(3)^{10},\ \zeta^m(3)^5\zeta^m(5)^3 \text{ and } \zeta^m(5)^6$$

have weight 30, so many coefficients to show vanish ...

Use coproduct to decompose:

Say $p(x,y) = \sum \lambda_{ij} x^i y^j$ of minimal degree w/

$$p(\zeta^m(3), \zeta^m(5)) = 0$$

[Can assume polynomial is weight-homogeneous.]

Apply Δ with $\Delta x \cdot y = \Delta x \cdot \Delta y$ to see

$$\underline{0} = \Delta p(\zeta^m(3), \zeta^m(5))$$

$$= \sum \lambda_{ij} (1 \otimes \zeta^m(3) + \zeta^m(3) \otimes 1)^i \cdot (1 \otimes \zeta^m(5) + \zeta^m(5) \otimes 1)^j$$

Project to wt 3 $\otimes$ wt $N-3$ component, (given by $\zeta^m(3) \otimes (\ldots)$.)

$$\Rightarrow 0 = \underbrace{\zeta^m(3)}_{\neq 0} \otimes \Big\{ \underbrace{\sum \lambda_{ij} \binom{i}{1} \zeta^m(3)^{i-1} \zeta^m(5)^j}_{\Rightarrow\ = 0} \Big\}$$

RHS is smaller degree polynomial vanishing at $x = \zeta^m(3)$, $y = \zeta^m(5)$.

$$\Rightarrow \lambda_{ij}\binom{i}{1} = 0 \quad \forall i \geq 1$$

$$\Rightarrow \lambda_{ij} = 0 \quad \forall i \geq 1$$

Reduces to $0 = p(\zeta^m(3), \zeta^m(5))$
$= \sum \lambda_{0j}\, \zeta^m(5)^j \Rightarrow \lambda_{0j} = 0$
as $\zeta^m(5)$ is transcendental.

⌜ Generalises to $\{\zeta^m(n_1), \dots, \zeta^m(n_r)\}$
for $n_1, \dots, n_r$ odd by induction. ⌟

$= I^m(0; 10010000; 1)$

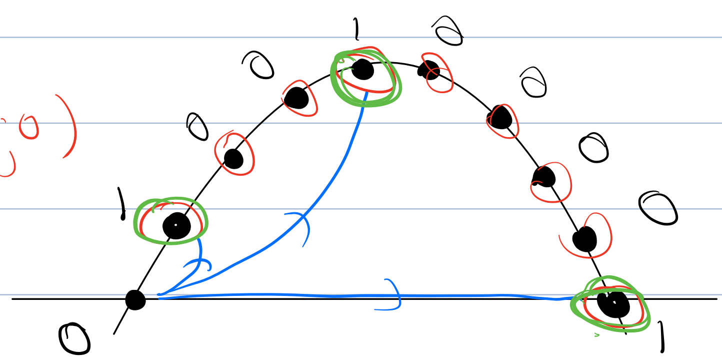
Prop: $\Delta \zeta^m(3,5)$

$$= 1 \otimes \zeta^m(3,5) + \zeta^m(3,5) \otimes 1$$

$$5\, \zeta^m(5) \otimes \zeta^m(3)$$

Proof: Terms in $\Delta\, \zeta^m(3,5)$
$= I^m(0; 10010000; 1)$

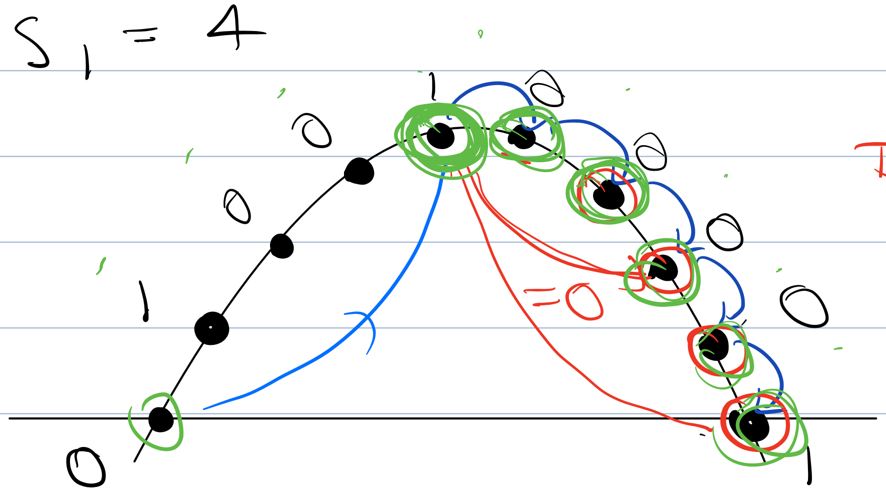
$I(0; \dots; 0)$
$= 0$



Have $s_1 = 1$ or 4 or 8.

If $s_1 = 8$ $\rightsquigarrow$ $\underline{\zeta^m(3,5)} \otimes 1$

If $s_1 = 4$

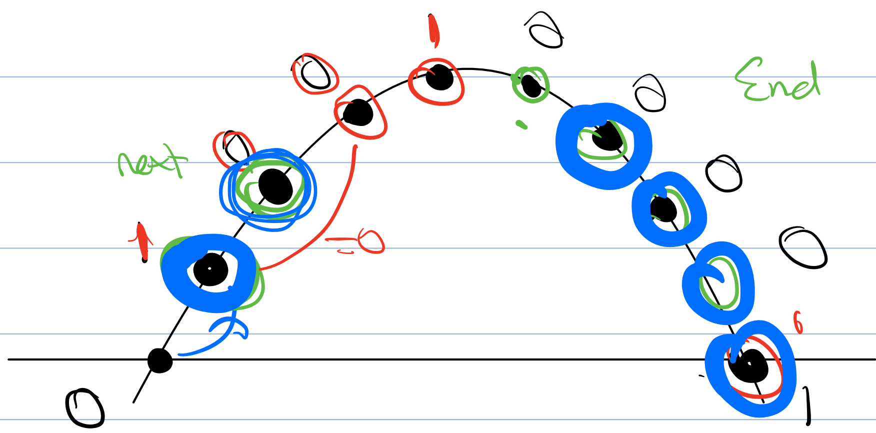


$I(1;0\cdots0;0)$
$=0$

$\rightsquigarrow$ $I^m(0;100;1) \otimes I^m(0;10000;1)$

$= \zeta^m(3) \otimes \zeta^m(5)$.

If $s_1 = 1$:



$\zeta^m(3)$ $\qquad\qquad$ $-\zeta^m(5)$

End : $I^m(1;001;0) \otimes I^m(0;10000;1)$

$-\zeta^m(4)$

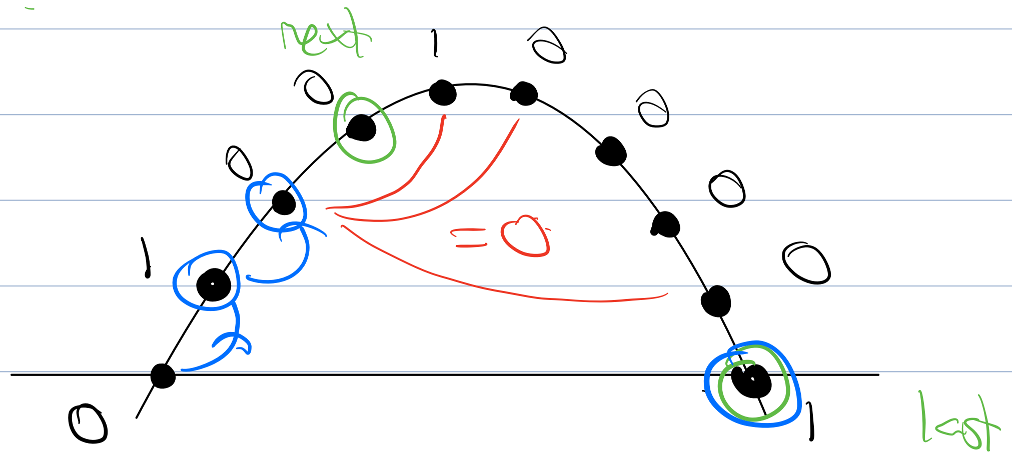
$\cancel{I^m(1;0010;0) \otimes I^m(0;1000;1)}$

$$\overset{\binom{4}{2}\zeta^m(5)}{I^m(1;00100;0)} \otimes \overset{-\zeta^m(3)}{I^m(0;100;1)}$$

$$\cancel{I^m(1;001000;0) \otimes I^m(0;10;1)} \quad -\zeta^m(2)$$

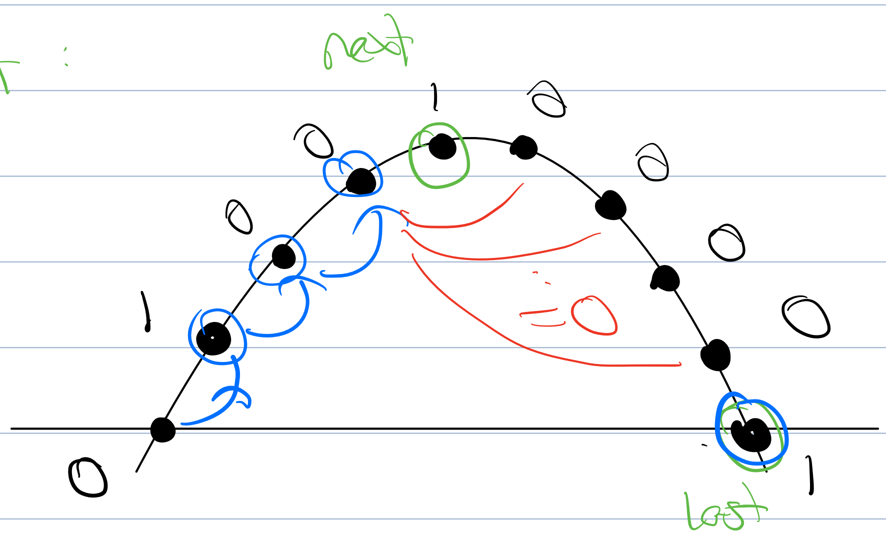
$$\rightsquigarrow \quad -\zeta^m(3) \otimes \zeta^m(5) - 6\zeta^m(5) \otimes \zeta^m(3)$$

Next :



last : $\cancel{I^m(0;010000;1) \otimes I^m(0;10;1)} \quad -\zeta^m(2)$

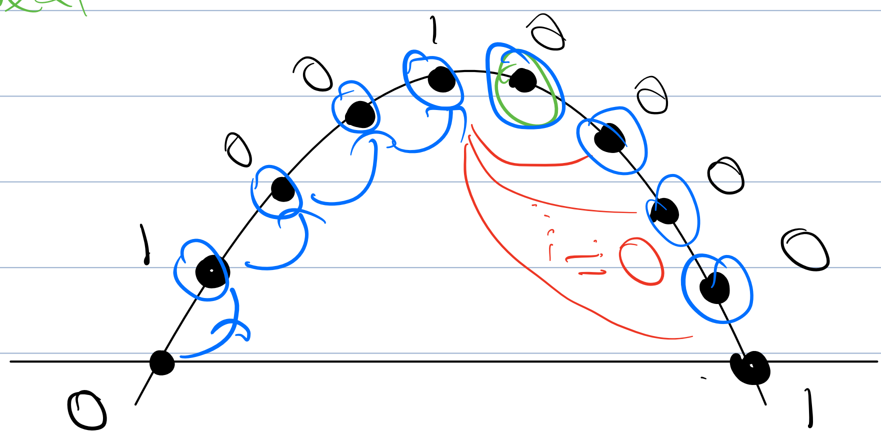
Next :



last : $\underset{}{\overset{-\zeta^m(5)}{I^m(0;10000;1)}} \otimes \overset{-\zeta^m(3)}{I^m(0;100;1)}$

$\rightsquigarrow$ $\zeta^m(5)\otimes\zeta^m(3)$

Next:



$\rightsquigarrow$ all

$\rightsquigarrow$ $1\otimes\zeta^m(3,5)$

Total Δ: $1\otimes\zeta^m(3,5)+\zeta^m(3,5)\otimes 1$

$\cancel{+\zeta^m(3)\otimes\zeta^m(5)}$

$\cancel{-\zeta^m(3)\otimes\zeta^m(5)}$

$-5\cancel{6}\,\zeta^m(5)\otimes\zeta^m(3)$

$\cancel{+\zeta^m(5)\otimes\zeta^m(3)}$

$$= 1\otimes\zeta^m(3,5)+\zeta^m(3,5)\otimes 1 - 5\,\zeta^m(5)\otimes\zeta^m(3)$$

iv) $\zeta^m(3,5)$ is irreducible

Suppose $\overset{m}{\zeta}(3,5) = \sum \lambda_{i_1\cdots i_r}\, \overset{m}{\zeta}(n_1)^{i_1} \cdots \overset{m}{\zeta}(n_r)^{i_r}$

some $n_1, \dots, n_r \in \mathbb{Z}$,

$\lambda_{i_1 \cdots i_r} \in \mathbb{Q}$

Obs 1: $$\Delta \overset{m}{\zeta}(n_1) = 1 \otimes \overset{m}{\zeta}(n_1) + \overset{m}{\zeta}(n_1) \otimes 1$$
$$= \Delta^{op} \overset{m}{\zeta}(n_1)$$

where $\Delta^{op} = \tau \circ \Delta$, $\tau(x \otimes y) = y \otimes x$

opposite coproduct

Obs 2: Δ^{op} is a coproduct, so

$$\Delta^{op}(x \cdot y) = \Delta^{op} x \cdot \Delta^{op} y$$

$$\Delta^{op}(x \cdot y) = \tau \Delta x \cdot \Delta y = \tau\left(\sum c_{(1)} \otimes c_{(2)}\right)\left(\sum d_{(3)} \otimes d_{(4)}\right) = \tau\left(\sum c_{(1)} d_{(3)} \otimes c_{(2)} d_{(4)}\right)$$

$$\Delta^{op} x \cdot \Delta^{op} y = \left(\sum c_{(2)} \otimes c_{(1)}\right)\left(\sum d_{(4)} \otimes d_{(3)}\right) = \sum c_{(2)} d_{(4)} \otimes c_{(1)} d_{(3)}$$

Now: $$(\Delta - \Delta^{op}) \sum \lambda_{i_1 \cdots i_r}\, \overset{m}{\zeta}(n_1)^{i_1} \cdots \overset{m}{\zeta}(n_r)^{i_r}$$

$$= \sum \lambda_{i_1 \cdots i_r} \left((\Delta - \Delta^{op}) \overset{m}{\zeta}(n_1)\right)^{i_1} \cdots \left((\Delta - \Delta^{op}) \overset{m}{\zeta}(n_r)\right)^{i_r}$$

$\to$ $= 0$

$(\Delta - \Delta^{op}) \overset{m}{\zeta}(n_1) = 0$

BUT

$$(\Delta - \Delta^{op})\,\zeta^m(3,5)$$

$$= 1\otimes\zeta^m(3,5) - 5\zeta^m(5)\otimes\zeta^m(3) + \zeta^m(3,5)\otimes 1$$
$$- \zeta^m(3,5)\otimes 1 + 5\zeta^m(3)\otimes\zeta^m(5) - 1\otimes\zeta^m(3,5)$$

$$= 5\,\zeta^m(3)\wedge\zeta^m(5)$$
$$\neq 0$$

$\leftarrow$ $a\wedge b = a\otimes b - b\otimes a$

§3 Extensions by Brown

Brown developed a framework (a comodule $\mathcal{H}$ over Goncharov's coalgebra $\mathcal{A}$ of $\mathcal{I}^m$), where $\zeta^m(2)\neq 0$.

⌜Notation: ζ^a for Goncharov's motives (above: $m\to a$)
ζ^m for Brown's ⌟

In Brown's setting, one can find/prove identities using the coaction Δ, upto

$$\begin{cases} \zeta^m(N) & \text{for MZV's} \\ Li^m_N(e^{2\pi i p/q}) & \text{(primitives in general)} \end{cases}$$

Results in Brown's setting:

- independence of ζ^m(2's and 3's)
- generators of mzv's
- semi-numerical algorithm to write $\zeta^m(k_1, \dots, k_d)$ in an algebra basis.

$\rightarrow$ ...

$$\Delta x = \boxed{x \otimes 1 - 1 \otimes x} + (\ldots)$$

$$\boxed{\Delta' x = 0} \Leftrightarrow x = \lambda \zeta^m(N)$$

$$- \left[\begin{array}{l} \zeta(\underline{12}^a 1313) \\ + \zeta(12^a 313) \\ + \zeta(132^a 13) \\ + \ldots \\ + \zeta(1313 2^a) \end{array} \right. = \frac{\pi^{2a+8}}{(2a+8)!}$$

$$2a+2 \quad 2 \quad 2 \quad 2 \quad 2$$

$$\left[\begin{array}{l} - I(0;\underline{(10)^a}\,1|10|01|1|00;1) \\ I(0\,|\underline{(10)^a}\,1|00|11|00;1) \end{array} \right.$$

$$2 \quad 2a+2 \quad 2 \quad 2 \quad 2$$

1313

+ Bowen – Bradly – Bradnurst – Lisacek

+ Hoffman

$$3\,(3\,3\,2^{a})$$

$$-\;\ast\;3\,(3\,2^{a}\,12)$$

$$+\;(2^{a}\,12\,12)$$

$$=\;\ast\,.$$