Applications of the Symmetry thm for MtV's

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Theorem 2.21 (Symmetry Theorem, generating series form). *Let $y_{i,j} := y_i - y_j$, and $\underline{e}(x) := \exp(2\pi \mathrm{i} x)$. Then for every choice of m and every choice of $\underline{\phi} = (\phi_1, \ldots, \phi_m)$, following generating series identity holds:*

$$\begin{aligned}
&\sum_{j=0}^{m} (-1)^j \underline{e}(\phi_1 + \cdots + \phi_j) \mathcal{L}\mathrm{i}^t_{T=\log 2}(\phi_{j+1}, \ldots, \phi_m \mid y_{j+1}, \ldots, y_m) \cdot \\
&\qquad\qquad \cdot \mathcal{L}\mathrm{i}^t_{T=\log 2}(-\phi_j, \ldots, -\phi_1 \mid -y_j, \ldots, -y_1) \\
&- \frac{1}{2^{m-1}} \sum_{j=1}^{m} (-1)^{j-1} \mathcal{L}\mathrm{i}_{T=0}(-\phi_{j-1}, \ldots, -\phi_1 \mid \tfrac{1}{2} y_{j,j-1}, \ldots, \tfrac{1}{2} y_{j,1}) \cdot \\
&\qquad\qquad \cdot B^t_{T=\log 2}(\phi_1 + \cdots + \phi_m \mid y_j) \, \mathcal{L}\mathrm{i}_{T=0}(\phi_{j+1}, \ldots, \phi_m \mid \tfrac{1}{2} y_{j+1,j}, \ldots, \tfrac{1}{2} y_{m,j}) \\
&= \begin{cases} \dfrac{1}{m!} \Big(\dfrac{\mathrm{i}\pi}{2}\Big)^m & \underline{\phi} = \underline{0} \text{ and } m \text{ even}, \\ 0 & \text{otherwise}. \end{cases}
\end{aligned}$$

where

$$\mathcal{L}\mathrm{i}(\phi_1, \ldots, \phi_m \mid y_1, \ldots, y_m) = \sum_{n_1, \ldots, n_m \geq 1} \zeta\binom{\underline{e}(\phi_1), \ldots, \underline{e}(\phi_m)}{n_1, \ldots, n_m} y_1^{n_1 - 1} \cdots y_m^{n_m - 1}.$$

$$\mathcal{L}\mathrm{i}^t(\phi_1, \ldots, \phi_m \mid y_1, \ldots, y_m) = \sum_{n_1, \ldots, n_m \geq 1} t\binom{\underline{e}(\phi_1), \ldots, \underline{e}(\phi_m)}{n_1, \ldots, n_m} y_1^{n_1 - 1} \cdots y_m^{n_m - 1}.$$

and

$$\begin{aligned}
B^t_M(\phi_1 \mid y_1) &:= \sum_{-M \leq k_1 \leq M} \frac{\underline{e}(\phi_1 k_1)}{2k_1 - 1 - y_1} \\
&= \mathcal{L}\mathrm{i}^t_M(\phi_1 \mid y_1) - \underline{e}(\phi_1) \, \mathcal{L}\mathrm{i}^t_{M+1}(-\phi_1 \mid -y_1)
\end{aligned}$$

Application 1: $t(3, \{2\}^n, 3)$

Theorem 3.1 (Evaluation of $t(3, \{2\}^n, 3)$). *We have the following evaluation for $t(3, \{2\}^n, 3)$ as a polynomial in Riemann zeta values.*

$$\begin{aligned}
t(3, \{2\}^n, 3) = \Big(-\frac{1}{4}\Big)^{n+2} \Big\{ &-\frac{9 + 6n}{2} \zeta(2) \zeta(2n+4) \\
&+ \sum_{\substack{q+r+s=n+2 \\ q,r,s \geq 1}} 2rs(2 - 2^{-2r} - 2^{-2s}) \zeta(2r+1) \zeta(2s+1) \cdot (-1)^q \frac{\pi^{2q}}{(2q)!} \\
&+ \sum_{\substack{r+s=n+2 \\ r,s \geq 1}} 2rs(2 - 2^{-2r})(2 - 2^{-2s}) \zeta(2r+1) \zeta(2s+1) \Big\}.
\end{aligned}$$

Extract coefficient of $y_1^2 y_2^1 \cdots y_n^1 y_{n+1}^2$; with $\underline{\phi} = \underline{0}$

$$\begin{aligned}
2t(3, \{2\}^n, 3) = &\sum_{i=0}^{n} t(\{2\}^i, 3) t(\{2\}^{n-i}, 3) - \frac{1}{2^{2n+3}} t(2) \sum_{i=0}^{n-1} \zeta(\{2\}^i, 3) \zeta(\{2\}^{n-1-i}, 3) \\
&- \frac{1}{2^{2n+2}} t(2) \Big\{ 3\zeta(\{2\}^n, 4) + 2 \sum_{i=0}^{n-1} \zeta(\{2\}^i, 3, \{2\}^{n-1-i}, 3) \Big\}.
\end{aligned}$$

By duality & shuffle regularisation

$$3\zeta(\{2\}^n,4) + 2\sum_{i=0}^{n-1}\zeta(\{2\}^i,3,\{2\}^{n-1-i},3) = -(-1)^{n+1}\,\mathrm{reg}^{ш}_{T=0}\,I(0;0;\{1,0\}^n,1,0,0;1)$$
$$= -\,\mathrm{reg}^{ш}_{T=0}\,\zeta(1,\{2\}^{n+1},1)\,.$$

So :

$$t(3,\{2\}^n,3) = \frac{1}{2}\sum_{i=0}^{n} t(\{2\}^i,3)t(\{2\}^{n-i},3) - \frac{t(2)}{2^{2n+4}}\sum_{i=0}^{n-1}\zeta(\{2\}^i,3)\zeta(\{2\}^{n-1-i},3)$$
$$+ \frac{t(2)}{2^{2n+3}}\,\mathrm{reg}^{ш}_{T=0}\,\zeta(1,\{2\}^{n+1},1)\,.$$

Shuffle antipode :

$$\sum_{i=0}^{m}(-1)^i I(0;w_1,w_2,\ldots,w_i;1)I(0;w_m,w_{m-1},\ldots,w_{i+1};1) = 0\,,$$

So identity

$$\mathrm{reg}^{ш}_{T=0}\Big(\sum_{i=0}^{n}\zeta(1,\{2\}^i,1)\zeta(\{2\}^{n-i}) + \zeta(\{2\}^n,1,1) - \sum_{i=0}^{n}\zeta(\{2\}^i,1)\zeta(1,\{2\}^{n-i})\Big) = 0\,,$$

Better

Some general notation:

$$G_{\alpha\{\beta\}\gamma}(u) := \sum_{n=0}^{\infty}\mathrm{reg}^{ш}_{T=0}\,\zeta(\alpha,\{\beta\}^n,\gamma)u^{|\alpha|+n|\beta|+\gamma}\,,$$
$$G^{t,\bullet}_{\alpha\{\beta\}\gamma}(u) := \sum_{n=0}^{\infty}\mathrm{reg}^{*}_{T=\log 2}\,t^{\bullet}(\alpha,\{\beta\}^n,\gamma)u^{|\alpha|+n|\beta|+\gamma}\,,$$

So

$$G^t_{3\{2\}3}(u) = \frac{1}{2}G^t_{\{2\}3}(u)^2 - t(2)u^2G_{\{2\}3}(\tfrac{u}{2})^2 + 2t(2)u^2G_{1\{2\}1}(\tfrac{u}{2})\,.$$

$$G_{1\{2\}1}(u)G_{\{2\}}(u) + G_{\{2\}11}(u) - G_{\{2\}1}G_{1\{2\}}(u) = 0\,,$$

$\Downarrow$

$$G_{1\{2\}1}(u) = \Big(G_{\{2\}1}^{(u)}G_{1\{2\}}(u) - G_{\{2\}11}(u)\Big)G_{\{2\}}(u)^{-1}\,.$$

Now

$$\mathrm{reg}^{ш}_{T=0}\,\zeta(\{2\}^n,1) = \mathrm{reg}^{ш}_{T=0}(-1)^{n+1}I(0;0,\{1,0\}^n;1) = -2\sum_{i=0}^{n-1}\zeta(\{2\}^i,3,\{2\}^{n-1-i})$$
$$\mathrm{reg}^{ш}_{T=0}\,\zeta(\{2\}^n,1,1) = \mathrm{reg}^{ш}_{T=0}(-1)^{n+2}I(0;0,0,\{1,0\}^n;1)$$
$$= -3\sum_{i=0}^{n-1}\zeta(\{2\}^i,4,\{2\}^{n-1-i}) - 4\sum_{i+j+k=n-2}\zeta(\{2\}^i,3,\{2\}^j,3,\{2\}^k)$$

Symmetric sums $\implies$ polynomials in RZV's

[Ohno-Zagier thm to evaluate]

$$G_{\{2\}11}(u) = \frac{1}{2}\frac{\pi u}{\sinh(\pi u)} - \frac{1}{2}\frac{\sinh(\pi u)}{\pi u} + \zeta(2)u^2\frac{\sinh(\pi u)}{\pi u} + 2u^2A(\mathrm{i}u)^2\frac{\sinh(\pi u)}{\pi u}.$$

$$\begin{aligned} G_{1\{2\}1}(u) &= \Big(G_{\{2\}11}(u) - G_{\{2\}1}(u)G_{1\{2\}}(u)\Big)\cdot G_{\{2\}}(u)^{-1} \\ &= \frac{1}{2} - \zeta(2)u^2 + 2u^2A(\mathrm{i}u)^2 - 3\zeta(2)\frac{u^2}{\sinh^2(\pi u)} - 4\mathrm{i}u^3A(\mathrm{i}u)B'(\mathrm{i}u)\frac{\sinh(\pi u)}{\pi u}. \end{aligned}$$

where $A(z) = \sum_{r=1}^\infty \zeta(2r{+}1)z^{2r}$ and $B(z) = \sum_{r=1}^\infty (1{-}2^{-2r})\zeta(2r{+}1)z^{2r}$.

Zagier
$$\begin{aligned} F(u,v) &= \sum_{a,b\geq 0}^\infty (-1)^{a+b}\zeta(\{2\}^a,3,\{2\}^b)u^{2a}v^{2b} \\ &= \frac{\sin(\pi v)}{\pi u^2 v}\Big[A(u+v)+A(u-v)-2A(v)\Big] - \frac{\sin(\pi u)}{\pi u^2 v}\Big[B(u+v)-B(u-v)\Big], \end{aligned}$$

Murakami
$$\begin{aligned} F^t(u,v) &= \sum_{a,b\geq 0}^\infty (-1)^{a+b}t(\{2\}^a,3,\{2\}^b)u^{2a}v^{2b} \\ &= \frac{1}{2uv}\cos\Big(\frac{\pi v}{2}\Big)\Big[A\Big(\frac{u+v}{2}\Big) - A\Big(\frac{u-v}{2}\Big)\Big] + \frac{1}{2uv}\cos\Big(\frac{\pi u}{2}\Big)\Big[B\Big(\frac{u+v}{2}\Big) - B\Big(\frac{u-v}{2}\Big)\Big]. \end{aligned}$$

where $A(z) = \sum_{r=1}^\infty \zeta(2r{+}1)z^{2r}$ and $B(z) = \sum_{r=1}^\infty (1{-}2^{-2r})\zeta(2r{+}1)z^{2r}$.

So can get $G^t_{2\{3\}}$ & $G_{2\{3\}}$, and find

$$\begin{aligned} G^t_{3\{2\}3}(u) = t(2)u^2 - t(4)u^4 - \frac{u^4}{8}A'\Big(\frac{\mathrm{i}u}{2}\Big)^2 - \frac{u^4}{8}B'\Big(\frac{\mathrm{i}u}{2}\Big)^2 \\ - \frac{u^4}{4}A'\Big(\frac{\mathrm{i}u}{2}\Big)B'\Big(\frac{\mathrm{i}u}{2}\Big)\cosh\Big(\frac{\pi u}{2}\Big) - 2t(2)^2u^4\operatorname{csch}^2\Big(\frac{\pi u}{2}\Big). \end{aligned}$$

$\implies$ thm above.

App 2a: $t(1,\{2\}^n,1)$

$$\operatorname{reg}^*_{T=\log 2} 2t(1,\{2\}^n,1) = \frac{\delta_{n=0}}{2!}\Big(\frac{\mathrm{i}\pi}{2}\Big)^2 + \sum_{i=0}^{n} \operatorname{reg}^*_{T=\log 2}\big(t(\{2\}^i,1)t(\{2\}^{n-i},1)\big)$$
$$- \frac{1}{2^{2n-1}} t(2) \sum_{i=0}^{n-1} \operatorname{reg}^*_{T=0}\big(\zeta(\{2\}^i,1)\zeta(\{2\}^{n-1-i},1)\big) .$$

$$\Rightarrow \quad G^t_{1\{2\}1}(u) = -\frac{\pi^2}{8} + \big(G^t_{\{2\}1}(u)\big)^2 - 2u^2 t(2)\big(G_{\{2\}1}(\tfrac{u}{2})\big)^2 .$$

C

$$\sum_{a,b\geq 0} (-1)^{a+b} \operatorname{reg}_{T=\log 2} t(\{2\}^a,1,\{2\}^b)\cdot(2x)^{2a}(2y)^{2b} =$$
$$\frac{1}{2}\cos(\pi x)(A(x-y)+A(x+y)) + \frac{1}{2}\cos(\pi y)(B(x-y)+B(x+y)+2\log 2) ,$$

So can get $G^t_{\{2\}1}$, and $G_{\{2\}1}$ from before, so

$$G^t_{1\{2\}1}(u) = -\frac{\pi^2 u^2}{16} + \frac{1}{2}\Big(\log 2 + A\Big(\frac{\mathrm{i}u}{2}\Big) + B\Big(\frac{\mathrm{i}u}{2}\Big)\cosh\Big(\frac{\pi u}{2}\Big)\Big)^2 - \frac{1}{2}A\Big(\frac{\mathrm{i}u}{2}\Big)^2 \sinh\Big(\frac{\pi u}{2}\Big) .$$

App 2b: $t(1\{\overline{1}\}^n,1)$

$$\operatorname{reg}_{T=\log 2} 2t(1,\{\overline{1}\}^n,1) = -\delta_{n=0}t(2) + \sum_{i=0}^{n} \operatorname{reg}_{T=\log 2} t(\{\overline{1}\}^i,1)t(\{\overline{1}\}^{n-i},1)$$
$$- \operatorname{reg}_{T=0} \frac{\pi}{2^{n+1}}\Big((1-(-1)^n)\zeta(\{\overline{1}\}^n,1) - \sum_{i=0}^{n-1}(-1)^i\zeta(\{\overline{1}\}^i,1)\zeta(\{\overline{1}\}^{n-1-i},1)\Big) .$$

C also: generally $t(\overline{1}^a,1,\overline{1}^b)$

$$G^t_{\{\overline{1}\}1}(u) = \frac{u}{2}\Big(\cos\Big(\frac{\pi u}{4}\Big) - \sin\Big(\frac{\pi u}{4}\Big)\Big)\Big(2A\Big(\frac{u}{2}\Big) + \log 2\Big)$$
$$+ \frac{u}{2}\Big(-A\Big(\frac{u}{8}\Big) + A\Big(\frac{u}{4}\Big) + 2C\Big(\frac{u}{2}\Big) + \log 2\Big) ,$$

w/ $C(z) = \frac{1}{8}(\psi(\frac{1}{4}+\frac{z}{4}) - \psi(\frac{1}{4}-\frac{z}{4}) - \psi(\frac{3}{4}+\frac{z}{4}) + \psi(\frac{3}{4}-\frac{z}{4})) = \sum_{r=1}^{\infty}\beta(2r)z^{2r-1} .$

Need also following eval, shown via asymptotic formula for ${}_2F_1$

Proposition 3.6 ($\mathrm{reg}_{T=0}\,\zeta(\{\overline{1}\}^m,1)$ evaluation)**.** *The following regularized generating series evaluation holds.*

$$\begin{aligned} L(x) &:= \sum_{r=0}^{\infty} \mathrm{reg}_{T=0}(-1)^r \zeta(\{\overline{1}\}^r,1)x^r \\ &= \frac{1}{x} - \frac{\Gamma(\frac{1}{2})}{\Gamma(1-\frac{x}{2})\Gamma(\frac{1+x}{2})}\Big(-\log 2 - 2A(x) + \pi\Big(\cot\Big(\frac{\pi x}{2}\Big) - \cot(\pi x)\Big)\Big), \end{aligned} \tag{28}$$

where

$$A(z) = \psi(1) - \frac{1}{2}(\psi(1+z)+\psi(1-z)) = \sum_{r=1}^{\infty}\zeta(2r+1)z^{2r},$$

with $\psi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\log\Gamma(x)$ *the logarithmic derivative of the gamma function* $\Gamma(x)$.

So first

$$G^t_{1\{\overline{1}\}1}(u) = -\frac{1}{2}t(2)u^2 + \frac{1}{2}G^t_{\{\overline{1}\}1}(u)^2 + \frac{\pi u^2}{16}\Big(2\Big(L\Big(\frac{u}{2}\Big) - L\Big(-\frac{u}{2}\Big)\Big) + uL\Big(\frac{u}{2}\Big)L\Big(-\frac{u}{2}\Big)\Big)$$

Or after simplification.

$$\begin{aligned} &G^t_{1\{\overline{1}\}1}(u) = \\ &-\frac{u^2\pi^2}{16} + \Big\{\frac{\pi u}{4} - \frac{\pi^2u^2}{16}\cot\Big(\frac{u\pi}{4}\Big) + \frac{u^2}{8}\Big(2A\Big(\frac{u}{2}\Big) + \log 2\Big)^2 \sin\Big(\frac{u\pi}{2}\Big) - \frac{\pi^2x^2}{16}\tan\Big(\frac{u\pi}{4}\Big)\Big\} \\ &+ \frac{u^2}{8}\Big\{\Big(\cos\Big(\frac{\pi u}{4}\Big) + \sin\Big(\frac{\pi u}{4}\Big)\Big)\Big(2A\Big(\frac{u}{2}\Big) + \log 2\Big) - A\Big(\frac{u}{8}\Big) + A\Big(\frac{u}{4}\Big) - 2C\Big(\frac{u}{2}\Big) + \log 2\Big\}^2. \end{aligned}$$

App 3:

Theorem 3.7. *The following generating series identity holds:*

$$R(u,\lambda) := \sum_{n,\ell\geq 0} t^{1/2}(\{1\}^n, 2\ell+2)u^n\lambda^{2\ell+2} = \frac{\lambda^2\pi^2\sec\left(\frac{\lambda\pi}{2}\right)\sec\left(\frac{\pi u}{4}\right)}{8\Gamma\left(1-\frac{\lambda}{2}-\frac{u}{4}\right)\Gamma\left(1+\frac{\lambda}{2}-\frac{u}{4}\right)}\frac{\Gamma\left(\frac{1}{2}-\frac{u}{4}\right)}{\Gamma\left(\frac{1}{2}+\frac{u}{4}\right)}. \tag{29}$$

In particular, $t^{1/2}(\{1\}^n, 2\ell+2)$ *is always a polynomial in* $\log 2$*, and Riemann zeta values.*

Idea

$$\mathcal{L}i^t_{T=\log 2}(\{ur\}^d) - \sum_{n=0}^{k-2}\frac{(ru)^n}{n!}\frac{\partial^n}{\partial W^n}\bigg|_{W=0}\mathcal{L}i^t_{T=\log 2}(\{ur\}^{d-1}, W)$$

$$= \sum_{n=d}^{\infty}\sum_{\substack{I=(i_1,\ldots,i_d)\\|I|=n\\i_d\geq k}} t(i_1,\ldots,i_{d-1},\underbrace{i_d}_{\geq k})(ru)^{n-d}.$$

So after summing over $\sum_{d=1}^{\infty} \cdot u^d$, $r=\frac{1}{2}$:

$$G^{t,1/2}_{\{1\}k}(u) = \sum_{d=1}^{\infty} \frac{u^d}{2^{1-k}} \Big(\mathcal{L}\mathrm{i}^t_{T=\log 2}(\{\tfrac{u}{2}\}^d) - \sum_{n=0}^{k-2} \frac{u^n}{2^n \cdot n!} \frac{\partial^n}{\partial W^n}\Big|_{W=0} \mathcal{L}\mathrm{i}^t_{T=\log 2}(\{\tfrac{u}{2}\}^{d-1}, W)\Big).$$

Sum over k even

$$\begin{aligned} R(u,\lambda) &= \sum_{\substack{k=2\\ k \text{ even}}} G_{\{1\}k}(u)\Big(\frac{\lambda}{u}\Big)^k \\ &= \sum_{d=1}^{\infty} \frac{2\lambda^2 u^d}{4\lambda^2 - u^2}\Big(- \mathcal{L}\mathrm{i}^t_{T=\log 2}(\{\tfrac{u}{2}\}^d) + \mathcal{D}\, \mathcal{L}\mathrm{i}^t_{T=\log 2}(\{\tfrac{u}{2}\}^{d-1}, W)\Big), \end{aligned}$$

w/ formal $\mathcal{D} := \cosh\big(\lambda \frac{\partial}{\partial W}\big|_{W=0}\big) + \frac{2\lambda}{u} \sinh\big(\frac{\partial}{\partial W}\big|_{W=0}\big)$.

Also

$$\begin{aligned} S(u,\lambda) &:= \sum_{\substack{k=2\\ k \text{ even}}} G_{k\{1\}}(u)\Big(\frac{\lambda}{u}\Big)^k \\ &= \sum_{d=1}^{\infty} \frac{2\lambda^2 u^d}{4\lambda^2 - u^2}\Big(- \mathcal{L}\mathrm{i}^t_{T=\log 2}(\{\tfrac{u}{2}\}^d) + \mathcal{D}\, \mathcal{L}\mathrm{i}^t_{T=\log 2}(W, \{\tfrac{u}{2}\}^{d-1})\Big). \end{aligned}$$

Sum w/ $(\{\frac{u}{2}\}^{d-1}, W)$:

$$\begin{aligned} &\sum_{i=0}^{d-1} (-1)^{d-1-i}\, \mathcal{L}\mathrm{i}^t_{T=\log 2}(\{\tfrac{u}{2}\}^i, W)\, \mathcal{L}\mathrm{i}^t_{T=\log 2}(\{-\tfrac{u}{2}\}^{d-1-i}) - (-1)^{d-1}\, \mathcal{L}\mathrm{i}^t_{T=\log 2}(-W, \{\tfrac{u}{2}\}^{d-1}) \\ &- \frac{1}{2^{d-1}} \sum_{i=0}^{d-2} (-1)^i \big(\mathcal{L}\mathrm{i}^t_{T=\log 2}(\tfrac{u}{2}) - \mathcal{L}\mathrm{i}^t_{T=\log 2}(-\tfrac{u}{2})\big)\, \mathcal{L}\mathrm{i}_{T=0}(\{0\}^i)\, \mathrm{Li}_{T=0}(\{0\}^{d-2-i}, \tfrac{W-u/2}{2}) \\ &- (-\tfrac{1}{2})^{d-1} (\mathcal{L}\mathrm{i}^t_{T=\log 2}(W) - \mathcal{L}\mathrm{i}^t_{T=\log 2}(-W))\, \mathcal{L}\mathrm{i}_{T=0}\,(\{\tfrac{W-u/2}{2}\}^{d-1}) = \delta_{d \text{ even}} \frac{1}{d!}\Big(\frac{\mathrm{i}\pi}{2}\Big)^d. \end{aligned}$$

Sum over $\sum_{d=1}^{\infty} \cdot u^d$

$$\begin{aligned} &\sum_{i=0}^{\infty} \mathcal{L}\mathrm{i}^t_{T=\log 2}(\{\tfrac{u}{2}\}, W) u^{i+1} \cdot e^{\gamma u/2} \frac{\Gamma(\frac{1}{2} + \frac{u}{4})}{\Gamma(\frac{1}{2} + \frac{u}{4})} + \sum_{i=0}^{\infty} (-1)^i\, \mathcal{L}\mathrm{i}^t_{T=\log 2}(-W, \{-\tfrac{u}{2}\}) u^{i+1} \\ &= \frac{\pi u}{2} e^{\gamma u/2} \frac{\Gamma(1 + \frac{u}{4} - \frac{W}{2})}{\Gamma(1 - \frac{u}{4} - \frac{W}{2})}\Big(\tan\Big(\frac{\pi u}{4}\Big) - \tan\Big(\frac{\pi W}{2}\Big)\Big). \end{aligned}$$

⌜ Using

$$\mathcal{L}\mathrm{i}^t_{T=\log 2}(y) - \mathcal{L}\mathrm{i}^t_{T=\log 2}(-y) = \frac{\pi}{2}\tan\left(\frac{\pi y}{2}\right),$$

$$\sum_{i=0}^{\infty} \mathcal{L}\mathrm{i}^t_{T=\log 2}(\{\tfrac{u}{2}\}^i)(-u)^i = G^{t,1/2}_{\{1\}}(-u) = e^{\gamma u/2}\frac{\Gamma(\frac{1}{2}+\frac{u}{4})}{\Gamma(\frac{1}{2}-\frac{u}{4})},$$

$$\sum_{i=0}^{\infty} \mathcal{L}\mathrm{i}_{T=0}(\{0\}^i)y^i = \frac{e^{-\gamma y}}{\Gamma(1+y)},$$

$$\sum_{i=1}^{\infty} \mathcal{L}\mathrm{i}_{T=0}(\{0\}^{i-1},x)y^i = \frac{e^{-\gamma y}}{\Gamma(1+y)} - \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)}.$$

$$\sum_{i=0}^{\infty} \mathcal{L}\mathrm{i}_{T=0}(\{x\}^i)y^i = 1 + \sum_{n\geq k\geq 1}\sum_{\substack{I=(i_1,\ldots,i_k)\\|I|=n}} \mathrm{reg}_{T=0}\,\zeta(i_1,\ldots,i_n)x^{n-k}y^k$$
$$= E\Big(\Big(\frac{y}{x}-1\Big)x\Big)E(-x)^{-1} = \frac{e^{\gamma y}\Gamma(1-x)}{\Gamma(1-x+y)}.$$

$$E(t) \coloneqq \textstyle\sum_{j=0}^{\infty} \mathrm{reg}_{T=0}\,\zeta(\{1\}^j)t^j = \frac{e^{-\gamma t}}{\Gamma(1+t)}.$$ ⌟

$\Rightarrow$
$$R(u,\lambda)\cdot G^{t,1/2}_{\{1\}}(-u) + S(-u,\lambda) = \frac{4\lambda^2\pi^2}{u^2-4\lambda^2}e^{\gamma u/2}\frac{\sec\left(\frac{\lambda\pi}{2}\right)\sec\left(\frac{\pi u}{4}\right)}{\Gamma(-\frac{\lambda}{2}-\frac{u}{4})\Gamma(\frac{\lambda}{2}-\frac{u}{4})}.$$

Stuffle antipode:

$$R(u,\lambda)\cdot G^{t,1/2}_{\{1\}}(-u) - S(-u,\lambda) = 0.$$

Solve simultaneously.

$$R(u,\lambda) \coloneqq \sum_{n,\ell\geq 0} t^{1/2}(\{1\}^n, 2\ell+2)u^n\lambda^{2\ell+2} = \frac{\lambda^2\pi^2\sec\left(\frac{\lambda\pi}{2}\right)\sec\left(\frac{\pi u}{4}\right)}{8\Gamma(1-\frac{\lambda}{2}-\frac{u}{4})\Gamma(1+\frac{\lambda}{2}-\frac{u}{4})}\frac{\Gamma(\frac{1}{2}-\frac{u}{4})}{\Gamma(\frac{1}{2}+\frac{u}{4})},$$

$$S(u,\lambda) = \sum_{n,\ell\geq 0} \mathrm{reg}_{T=\log 2}\, t^{1/2}(2\ell+2,\{1\}^n)u^n\lambda^{2\ell+2} = e^{-\gamma u/2}\frac{\lambda^2\pi^2\sec\left(\frac{\lambda\pi}{2}\right)\sec\left(\frac{\pi u}{4}\right)}{8\Gamma(1-\frac{\lambda}{2}+\frac{u}{4})\Gamma(1+\frac{\lambda}{2}+\frac{u}{4})}.$$

Some further extensions:

- With stuffle antipode can find t^* from t.
- Can find $t^{1/2}(2L+1, 1^{ev}, 2L+1)$ = poly in $\mathbb{Q}[\zeta's]$.

Stuffle antipode in Hopf algebra of interpolated MtV's.

$$z_I = \sum_{I_1 \sqcup \cdots \sqcup I_k = I} (-1)^{\ell(I)-k} \Sigma^{1-2r} R(z_{I_1}) \overset{r}{*} \cdots \overset{r}{*} \Sigma^{1-2r} R(z_{I_k}),$$

$$\mathrm{reg}^*_{T=\log 2}\, t^r(I) = \sum_{I_1 \sqcup \cdots \sqcup I_k = I} (-1)^{\ell(I)-k}\, \mathrm{reg}^*_{T=\log 2}\, t^{1-r}(\bar{I}_1) \cdots \mathrm{reg}^*_{T=\log 2}\, t^{1-r}(\bar{I}_k),$$

then recombine last $k-1$ terms into

$$t^r(I_2 \ldots I_k).$$