[AGZT UHH, 20/04/22 @ 12:15]

Symmetries for MZV's

[j/w Michael Hoffman]

§1 Definitions / Introduction

Define multiple zeta values (MZV's) as

$$\zeta(n_1, \dots, n_m) = \sum_{k_1 < \dots < k_m} \frac{1}{k_1^{n_1} \cdots k_m^{n_m}}$$

Reasons for interest?

↳ Includes $\zeta(n)$ into algebra structure
→ new insights about Riemann zeta vals?

↳ Easy sounding questions, v. difficult to answer
⇝ $\zeta(3) \notin \mathbb{Q}$ by Apéry 1978
⇝ $\zeta(3) \notin \overline{\mathbb{Q}}$??? Open

↳ Connections to HEP, Knot Theory,
Mixed Tate Motives.

Variants of MZV's

Allow signs / characters $\varepsilon_1, \dots, \varepsilon_m \in \{|z| = 1\}$

$$\zeta\begin{pmatrix} \varepsilon_1 & \cdots & \varepsilon_m \\ n_1 & \cdots & n_m \end{pmatrix} = \sum_{1 \leq k_1 < \dots < k_m} \frac{\varepsilon_1^{k_1} \cdots \varepsilon_m^{k_m}}{k_1^{n_1} \cdots k_m^{n_m}}$$

Special case of MPL's.

Allow linear forms in denominator, e.g.

$$\sum_{1 \le k_1 < \cdots < k_m} \frac{1}{(3k_1-1)^{n_1}(2k_2-1)^{n_2}(5k_3+2)^{n_3}}$$

⌜ Terasoma showed such things reduce to MZV's at higher roots of unity $\varepsilon_1, \ldots, \varepsilon_m \in \{|z|=1\}$ ⌟

We focus on MtV's

$$\underbrace{t\begin{pmatrix} \varepsilon_1 & \cdots & \varepsilon_m \\ n_1 & \cdots & n_m \end{pmatrix}}_{\substack{\text{write } t(n_1,\ldots,n_m) \\ \text{if all } \varepsilon_1 \ldots \varepsilon_m = 1}} = \sum_{0 \le k_1 < \cdots < k_m} \frac{\varepsilon_1^{k_1} \cdots \varepsilon_m^{k_m}}{(2k_1+1)^{n_1} \cdots (2k_m+1)^{n_m}}$$

Here all denominators odd, somehow the 'simplest' "higher level" variant. ⌜(Re)introduced by Hoffman after earlier investigation by Nielsen.⌟

<u>Compare & Contrast</u>

1) MZV's & MtV's satisfy stuffle product

$$t(a)t(b) = t(a,b) + t(b,a) + t(a+b)$$

also $t \to \zeta$, and w/ signs.

2) MZV's satisfy duality $\zeta(1,2) = \zeta(3)$.

MtV's do not seem to ...

$$t(3) = \frac{7}{8}\zeta(3), \quad t(1,2) = -\frac{7}{16}\zeta(3) + \frac{3}{4}\zeta(2)\log 2$$

no relation.

3) MZV's satisfy shuffle product, have integral rep

$$\zeta(1,2) = -\int_{0<s_1<s_2<s_3<1} \frac{ds_1}{s_1-1}\,\frac{ds_2}{s_2-1}\,\frac{ds_3}{s_3}$$

MtV's have no (nice) shuffle product, and asymmetric integral rep

$$t(1,2) = -\int_{0<s_1<s_2<s_3<1} \frac{ds_1}{s_1^2-1}\,\frac{s_2\,ds_2}{s_2^2-1}\,\frac{ds_3}{s_3}$$

not the same.

4) MtV's have conjectural non-zero derivation ∂ wrt $\log 2$.

Set $\partial \log 2 = 1$, $\partial\, t(\overbrace{1,\dots,1}^{\alpha}, k_1,\dots,k_m) = t(\underbrace{1,\dots,1}_{\alpha-1}, k_1,\dots,k_m)$

$$\partial \zeta(k_1,\dots,k_{m-1},\overline{k_m}) = 0$$

$\overline{k} = \binom{-1}{k}$ w/ sgns.

Conjecturally: $\Lambda = 0 \Rightarrow \partial\Lambda = 0$

$$(*) \quad t(1,3,2) = -\frac{2}{21}t(6) - \frac{3}{196}t(3)^2 - \frac{1}{2}t(2)\zeta(1,\bar{3}) + \frac{1}{4}\zeta(1,\bar{5}) - \frac{1}{2}t(5)\log 2 + \frac{4}{7}t(2)t(3)\log 2$$

Then $\partial(*) \rightarrow t(3,2) \overset{?}{=} -\frac{1}{2}t(5) + \frac{4}{7}t(2)t(3)$

which is true

[I formalized this somewhat, by connecting it to the motivic derivation D_1 in Gencherov/Brown coproduct]

MtV's would only have trivial derivations.
[Hoffman proved this, but also see $D_1 \equiv 0$ as no weight 1 MtV's.]

5) Symmetry

Hoffman conjectured that MtV's have a symmetry mod products

$$t(n_1, \dots n_L) \equiv (-1)^{\text{weight}-1}\, t(n_L, \dots, n_1) \quad (\text{mod } t \times t)$$

Not true for MZV's.

$$\zeta(2,3) - \zeta(3,2) = -10\zeta(5) + 5\zeta(3)\zeta(2)$$
$$t(2,3) - t(3,2) = -\tfrac{1}{7}\, t(2)t(3)$$

Thm: For stuffle regularised MtV's w/ $t(1) = \lg 2$, the symmetry

$$t(n_1 \ldots n_L) \equiv (-1)^{\text{weight}-1}\, t(n_L, \ldots, n_1) \pmod{t \times t}$$

holds.

§2 Proof idea / main steps

0. Take inspiration from Goncharov

1. Introduce depth m MtV generating series

$$\mathcal{L}i^t(\phi_1, \ldots \phi_m \mid y_1, \ldots, y_m) = \sum_{n_1, \ldots, n_m \geq 1} t\binom{e^{2\pi i \phi_1} \ldots e^{2\pi i \phi_m}}{n_1 \quad \ldots \quad n_m} y_1^{n_1-1} \cdots y_m^{n_m-1}$$

(+ similar w/ $t \to \zeta$ for $\mathcal{L}i$)

2. Use idea from Goncharov to evaluate

$$B^t(\phi_1 \ldots \phi_m \mid y_1 \ldots y_m) := \sum_{-\infty < k_1 < \cdots < k_m < \infty} \frac{e^{2\pi i(\phi_1 k_1 + \cdots + \phi_m k_m)}}{(2k_1 - 1 - y_1) \cdots (2k_m - 1 - y_m)}$$

in 2 different ways.

3 ...... get result.

Issues: How to deal with divergent MZV's? What regularisation parameter $t(1) = T$?

Do this for truncated MZV's / MtV's first. [Sums only go to some bound M.]

$$B_M^t(\phi \mid y) = \sum_{-M < k < M} \frac{e^{2\pi i \phi}}{2k - 1 - y}$$

$$= \mathcal{L}i_M^t(\phi \mid y) - e^{2\pi i \phi} \mathcal{L}i_M^t(-\phi \mid -y)$$

using geometric series

Evaluation 1: Decompose

$$-M \leq k_1 < \cdots < k_m \leq M = \bigcup_{j=0}^{m} \{-M \leq k_1 < \cdots < k_j \leq 0 < k_{j+1} < \cdots < k_m \leq M\}$$

Then

$$B_M^t(\phi_1, \ldots, \phi_m \mid y_1, \ldots, y_m)$$

$(*) \quad = \sum_{j=0}^{m} (-1)^j e^{2\pi i(\phi_1+\dots+\phi_j)} \mathcal{L}i^t_m(\phi_{j+1} \dots \phi_m | y_{j+1} \dots y_m)$

$$\cdot \mathcal{L}i^t_{m+1}(-\phi_j \dots -\phi_1 | -y_j \dots -y_1)$$

[This is $\mathcal{L}i_m(\phi_1 \dots \phi_m | y) \pm e^{*} \mathcal{L}i_{m+1}(-\phi_m \dots -\phi_1 | y^{\text{rev}})$ mod products.]

Eval 2: Partial fractions gives

$$\frac{1}{(2k_1-1-y_1)\dots(2k_m-1-y_m)} = \sum_{j=1}^{m} \frac{1}{(2k_j-1-y_j)\prod_{i\neq j}(2k_i-2k_j-y_i+y_j)}$$

So (after some work)

$B^t_m(\phi_1, \dots, \phi_m | y_1, \dots, y_m)$

$$= \frac{1}{2^{m-1}} \sum_{j=1}^{m} (-1)^{j-1} \sum_{-m\le k_j \le m} \mathcal{L}i_{m+k_j}\left(-\phi_{j-1} \dots -\phi_1 \,\Big|\, \frac{y_j-y_{j-1}}{2} \dots \frac{y_j-y_1}{2}\right)$$

$(**) \quad \times \dfrac{e^{2\pi i(\phi_1+\dots+\phi_m)k_j}}{2k_j-1-y_j} \mathcal{L}i_{m-k_j}\left(\phi_{j+1} \dots \phi_m \,\Big|\, \frac{y_{j+1}-y_j}{2} \dots \frac{y_m-y_j}{2}\right)$

Equating $(*) = (**)$ gives a result on truncated MtV's. Want to take $\lim_{m\to\infty}$.

[Note: $t_m\begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_m \\ n_1 & \dots & n_m \end{pmatrix}$ converges if $(\varepsilon_m, n_m) \neq (1,1)$]

Assuming $\phi_1, \phi_m, \phi_1+\dots+\phi_m \in \mathbb{R}\setminus\mathbb{Z}$, all MZV's, MtV's in $\mathcal{L}i^t$, $\mathcal{L}i$ converge as $m\to\infty$

(*) = (**) gives

Thm:

$$\sum_{j=0}^{m} (-1)^j e^{2\pi i(\phi_1+\dots+\phi_j)} \mathcal{L}_i^t(\phi_{j+1}\dots\phi_m \mid y_{j+1}\dots y_m) \cdot \mathcal{L}_i^t(-\phi_j \dots -\phi_1 \mid y_j \dots y_1)$$

(*)

$$= \frac{1}{2^{m-1}} \sum_{j=1}^{m} (-1)^{j-1} \mathcal{L}_i\left(-\phi_{j-1}\dots -\phi_1 \,\middle|\, \frac{y_j - y_{j-1}}{2} \dots \frac{y_j - y_1}{2}\right) B^t(\phi_1+\dots+\phi_m \mid y_j) \, \mathcal{L}_i\left(\phi_{j+1}\dots\phi_m \,\middle|\, \frac{y_{j+1}-y_j}{2} \dots \frac{y_m - y_j}{2}\right)$$

[Need some analysis exercise to check

$$\lim_{m\to\infty} \sum_{k=1}^{m} f_{m+k}\, g_{m-k}\, s_k \longrightarrow FGS$$

for 'nice' sequences $f_k \to F$, $g_k \to G$, and s_k w/ $\sum_{k=1}^{\infty} s_k = S$.]

Now we have result for speed / generic $\phi_1, \dots, \phi_m$. How to send $\phi_i \to 0$?

§ Asymptotic expansion

Refines $Li_{k_1 \dots k_m 1 \dots 1}(x_1 \dots x_m \mid 1-\varepsilon \dots)$

Behaviour of $\zeta\begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_n & 1-\varepsilon & \dots & 1-\varepsilon \\ k_1 & \dots & k_m & 1 & \dots & 1 \end{pmatrix}$ as $\varepsilon \to 0$ is some polynomial in $\log \varepsilon$

$\left[\mathrm{Li}_{abc}(x\,y\,z) = \sum_{k_1<k_2<k_3} \frac{x^{k_1} y^{k_2} z^{k_3}}{k_1^a k_2^b k_3^c}\right]$

Eg.

$$\mathrm{Li}_{21}(x \;\; 1-\varepsilon)$$

$$= \underbrace{\mathrm{Li}_1(1-\varepsilon)}_{\sim \log \varepsilon} \underbrace{\mathrm{Li}_2(x)}_{\sim \mathrm{Li}_2(x)} - \underbrace{\mathrm{Li}_3((1-\varepsilon)x)}_{\sim \mathrm{Li}_3(x)}$$

$$- \underbrace{\mathrm{Li}_{12}(1-\varepsilon, x)}_{\sim \mathrm{Li}_{12}(1, x)}$$

Constant term wrt $\log \varepsilon$ is related to Stuffle regularisation (at least for Li w/ same argument $f(\varepsilon)$ for indices 1). [Regularisation parameter is constant term in $\mathrm{Li}_1(f(\varepsilon))$.]

[Can use this to understand how regularisation parameter changes. Eg

$$\mathrm{Li}_{21}(x^2 \; y^2) =$$

$$\frac{1}{2^2}\Big(\mathrm{Li}_{21}(x \;\; y) + \mathrm{Li}_{21}(x \; -y) + \mathrm{Li}_{21}(-x \;\; y) + \mathrm{Li}_{21}(-x \; -y)\Big)$$

With $y = 1-\varepsilon$, get

$$\mathrm{Li}_1((1-\varepsilon)^2) \sim -\log \varepsilon - \log 2$$

$$\mathrm{Li}_1(1-\varepsilon) \sim -\log \varepsilon + 0$$

So need $Li_1(1) = -\log 2$ on LHS, and $Li_1(1) = 0$ on RHS

$$\operatorname{reg}_{-\log 2} Li_{21}(x^2, 1)$$
$$= \operatorname{reg}_0\Big(Li_{21}(x,1) + Li_{21}(-x,1) + Li_{21}(x,-1) + Li_{21}(-x,-1)\Big)$$

§ Conclusions

The asymptotic expansion of [illegible] to get result holding for regularized t's & ζ's.

Thm: (Symmetry)

$$\sum_{j=0}^{m} (-1)^j e^{2\pi i(\phi_1+\dots+\phi_j)} \operatorname{reg}_{\log 2} \mathcal{L}_i^t(\phi_{j+1}\dots\phi_m \mid y_{j+1}\dots y_m)$$
$$\operatorname{reg}_{\log 2} \mathcal{L}_i^t(-\phi_j \dots -\phi_1 \mid y_j \dots y_1)$$

$$-\frac{1}{2^{m-1}} \sum_{j=1}^{m} (-1)^{j-1} \operatorname{reg}_0 \mathcal{L}_i\Big(-\phi_{j-1}\dots-\phi_1 \,\Big|\, \frac{y_j - y_{j-1}}{2} \dots \frac{y_j - y_1}{2}\Big)$$
$$\operatorname{reg}_{\log 2} B^t(\phi_1+\dots+\phi_m \mid y_j)$$
$$\operatorname{reg}_0 \mathcal{L}_i\Big(\phi_{j+1}\dots\phi_m \,\Big|\, \frac{y_{j+1}-y_j}{2} \dots \frac{y_m - y_j}{2}\Big)$$

$$= \begin{cases} \frac{1}{m!}\left(\frac{i\sqrt{2}}{2}\right)^m & m \text{ even}, \ \phi = 0 \\ 0 & \text{o/w} \end{cases}$$

Cor: Looking at coefficient of $y_1^{n_1-1} \cdots y_m^{n_m-1}$ find when $\Phi = \underline{0}$:

$$\operatorname{neg}_{\log 2}\{t(n_1 \ldots n_m) + (-1)^{\text{weight}} t(n_m \ldots n_1)\}$$
$$= (\text{products } t \times \zeta \times \zeta + t \times t)$$

How to get rid of ζ in terms of t??

Recall: Murakami showed every ζ is a $\mathbb{Q}$-linear combination of t's (of the form $t(k_1, \ldots, k_n)$, $k_i \in \{2,3\}$).

⌜Uses motivic framework.⌟

So $t \times \zeta \times \zeta \rightsquigarrow t \times t \times t$ ✓

⌜Case $t(1,1)$ has irreducible $t(2)$ term $\frac{1}{2}\left(\frac{\sqrt{-1}}{2}\right)^2$ constant.⌟

Cor: For $\phi_i \in \{\frac{1}{2}, 0\}$, get for any $\varepsilon_i \in \{\pm 1\}$

$$\mathrm{neg}\ \log 2 \left\{ t\begin{pmatrix} \varepsilon_1 \cdots \varepsilon_n \\ n_1 \cdots n_m \end{pmatrix} + (-1)^{\text{weight} + \#\{\varepsilon_i = -1\}} t\begin{pmatrix} \varepsilon_m \cdots \varepsilon_1 \\ n_m \cdots n_1 \end{pmatrix} \right\}$$

$$= \left(\text{products } \zeta \times \zeta \times t \text{ at } \pm 1 \text{ and } t \times t \text{ at } \pm 1 \right)$$

Then I showed ζ at ± 1 can be written as a $\mathbb{Q}$ linear combination of $t(k_1 \cdots k_n)$, $k_i \in \{1, 2\}$.

So $\zeta \times \zeta \times t \rightsquigarrow t \times t \times t$ ✓

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§3 Applications

1) Murakami evaluated

$$t(\underbrace{2 \cdots 2}_{a}\ 3\ \underbrace{2 \cdots 2}_{b}) = \text{polynomial in single } t\text{'s (or } \zeta\text{'s)}$$

Zagier evaluated $\zeta(\{2\}^a\ 3\ \{2\}^b)$ = polynomial in single ζ's (or t's)

From symmetry theorem:

$$2t(3, \{2\}^n, 3) =$$

$$\sum_{i=0}^{n} t(\{2\}^i, 3)\, t(\{2\}^{n-i}\, 3) - \frac{t(2)^{n-1}}{2^{2n+3}} \sum_{i=0}^{n-1} \zeta(\{2\}^i 3)\, \zeta(\{2\}^{n-1-i}, 3)$$

$$-\frac{1}{2^{2n+2}}\, t(2) \left\{ \underbrace{3\,\zeta(\{2\}^n,4) + 2\sum_{i=0}^{n-1} \zeta(\{2\}^i 3 \{2\}^{n-1-i} 3)}_{=\ \text{shuffle reg of } \zeta(1\{2\}^{n+1}1)} \right\}$$

Evaluation of shuffle reg $\zeta(1\{2\}^{n+1}1)$:

- express via $\zeta(\{2\}11)$, $\zeta(\{2\}1)$, $\zeta(1\{2\})$, and $\zeta(\{2\})$ via shuffle product.
- Use duality on $\zeta(1\{2\}^n) = \zeta(\{2\}^{n-1}3)$
- Explicitly regularise $\zeta(\{2\}1)$
- Express $\zeta(\{2\}11)$ via sums of $\sum_{i+j=n} \zeta(\{2\}^i 4\{2\}^j)$, $\sum_{i+j+k=n-1} \zeta(\{2\}^i 3\{2\}^j 3\{2\}^k)$
- Use Symmetric sum theorem / Ohno-Zagier.

Conclusion: $t(3\{2\}^n 3) =$ polynomial in single t's (or single ζ's)

2) Similar results for $t(12\cdots 21)$ and $t(1\bar{1}\cdots\bar{1}1)$ via my evaluations.

[Need to understand $\zeta(\{\bar{1}\}^n 1)$ via some hypergeometric limit for $t(1\bar{1}\cdots\bar{1}1)$.]

3) Introduce interpolated MZV's:

$$t^r(a,b,c) = t(a,b,c) + r\{t(a+b,c) + t(a,b+c)\} + r^2\, t(a+b+c)$$

Goal is to evaluate $t^{\frac{1}{2}}(1 \cdots 1\, 2)$.

Trick:

Symmetry gives $t^{\frac{1}{2}}(\{1\}^n 2) = (-1)^{n+1}\, t^{\frac{1}{2}}(2\{1\}^n)$ (mod prod)

shuffle product $t^{\frac{1}{2}}(\{1\}^n 2) = (-1)^n\, t^{\frac{1}{2}}(2\{1\}^n)$ (mod prod)

Different signs, so $t^{\frac{1}{2}}(\{1\}^n 2) = 0$ (mod prod)

Express $\mathcal{L}i^t_{\cdot}(\underline{0} \mid \{u r\}^d) = \sum_{n=d}^{\infty} \sum_{\substack{I=(i_1 \cdots i_d) \\ |I|=n}} r^{n-d}\, t(i_1 \cdots i_d)\, u^{n-d}$

$$\mathcal{L}i^t_{\cdot}(\underline{0} \mid \{u r\}^{d-1} 0) = \sum_{n=d}^{\infty} \sum_{\substack{I=(i_1 \cdots i_d) \\ |I|=n,\ i_d > 1}} r^{n-d}\, \underbrace{t(i_1 \cdots i_d)}_{>1} u^{n-d}$$

So $\sum t^{\frac{1}{2}}(\{1\}^i 2)\, u^{i+2} = \underbrace{\sum_{d=0}^{\infty} \mathcal{L}i^t_{\cdot}(\underline{0} \mid \{\tfrac{u}{2}\}^d)\, u^d}_{\text{all } t\text{'s}} - \underbrace{\sum_{d=1}^{\infty} \mathcal{L}i^t_{\cdot}(\underline{0} \mid \{\tfrac{u}{2}\}^{d-1} 0)\, u^d}_{t\text{'s ending} > 1}$

So can find generating series evaluation using symmetry theorem

$$\sum t^{\frac{1}{2}}(\{1\}^i 2) u^{i+2} + (*) \sum t^{\frac{1}{2}}(2\{1\}^i) u^{i+2}$$

$$= \text{explicit thing} + \Gamma \text{ fcns}$$

From shuffle product get

$$\sum t^{\frac{1}{2}}(\{1\}^i 2) u^{i+2} - (*) \sum t^{\frac{1}{2}}(2\{1\}^i) u^{i+2}$$

$$= \text{explicit thing.}$$

Solve simultaneously to find

$$\sum t^{\frac{1}{2}}(\{1\}^i 2) u^{i+2} = e^{u\log 2} \frac{\Gamma(1-\frac{u}{2})\Gamma(1+\frac{u}{4})^2}{\Gamma(1+\frac{u}{2})\Gamma(1-\frac{u}{4})^2} \frac{\sqrt{\pi} u}{2} \tan\left(\frac{\sqrt{\pi} u}{4}\right)$$

[Since $\Gamma(1+u) = \exp\left(-\gamma u + \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k} \zeta(k) u^k\right)$

get $t^{\frac{1}{2}}(\{1\}^i 2)$ is polynomial in $\log 2$ and single t (or ζ) vals.]

Rmk: Can generalise to evaluate $t^{\frac{1}{2}}(\{1\}^i, \underbrace{2k+2}_{\text{even}})$]