Functional equations for Nielsen polylogarithms

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Outline

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Functional equations of classical polylogarithms

Definition (Polylogarithm)

The weight n polylogarithm is

$$\mathrm{Li}_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n}, \quad |z| < 1$$

Key feature

Polylogarithms satisfy interesting functional equations

Theorem (5-term relation, Abel, Spence, Kummer, ...)

For $|x| + |y| < 1$ *we have*

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(y) - \mathrm{Li}_2\left(\frac{x}{1-y}\right) - \mathrm{Li}_2\left(\frac{y}{1-x}\right) + \mathrm{Li}_2\left(\frac{xy}{(1-x)(1-y)}\right) = -\log(1-x)\log(1-y)$$

- The arguments are cross-ratios $\mathrm{cr}(a,b,c,d) = \frac{a-c}{a-d} \Big/ \frac{b-c}{b-d}$ from 4 of 5 points in $\mathbb{P}^1(\mathbb{C})$

Nielsen polylogarithms

Basic reference

K. S. Kölbig. Nielsen's generalized polylogarithms. *SIAM J. Math. Anal.*, 17(5), pp.1232–1258, 1986.

Definition (Nielsen polylogarithm)

The Nielsen polylogarithm of depth p, weight $n+p$ is

$$S_{n,p}(z) \coloneqq \frac{(-1)^{n+p-1}}{(n-1)!\,p!} \int_0^1 \log^{n-1}(t) \log^p(1-zt) \frac{dt}{t} \,.$$

Equivalently $\quad S_{n,p}(z) = \mathrm{Li}_{\{1\}^{p-1},n+1}(1,\ldots,1,z) = (-1)^p I(0; \underbrace{\{1\}^p}_{\text{repeat } p \text{ times}}, \{0\}^n; z)\,.$

Notation for (depth d) multiple polylogs and iterated integrals

$$\mathrm{Li}_{n_1,\ldots,n_d}(z_1,\ldots,z_d) \coloneqq \sum_{0<k_1<k_2<\cdots<k_r} \frac{z_1^{k_1}\cdots z_d^{k_d}}{k_1^{n_1}\cdots k_d^{n_d}}\,, \quad |z_1\cdots z_i| < 1$$

$$I(x_0; x_1,\ldots,x_N; x_{N+1}) \coloneqq \int_{x_0<t_1<\cdots<t_N<x_{N+1}} \frac{\mathrm{d}t_1}{t_1-x_1} \wedge \frac{\mathrm{d}t_2}{t_2-x_2} \wedge \cdots \wedge \frac{\mathrm{d}t_N}{t_N-x_N}$$

Motivation: Zagier's polylogarithm conjecture

Let F be a number field, $\mathcal{O}_F$ its ring of integers

Definition (Dedekind zeta function)

The Dedekind zeta function of F is

$$\zeta_F(s) = \sum_{\substack{\mathfrak{a} \neq (0) \subset \mathcal{O}_F \\ \text{non-zero ideal}}} \left| \mathcal{O}_F / \mathfrak{a} \right|^{-s} , \quad \mathrm{Re}(s) > 1$$

Conjecture (Zagier, 1991 – schematic version)

For $n \geq 2$,

$$\zeta_F(n) = \textit{rational} \cdot \pi^{n d_{n+1}} \sqrt{|\Delta_F|} \cdot \det \Big(\textit{sums of single-valued } \mathrm{Li}_n(x_k)\textit{'s} \Big)_{i,j=1}^{d_n} ,$$

where $x_k \in F$, $d_n = r_1 + r_2$, n odd, and $d_n = r_2$ if n even.

Remark

Strategy/idea to tackle the conjecture involves *reducing* higher depth MPL's to Li_n

Motivation: $\zeta_F(4)$ after Goncharov-Rudenko

Substituting (29) to the relation $\mathbf{Q}_3$ we get the 22-term relation for trilogarithm from [G91a]. Therefore the map $\{x\}_3 \longmapsto \{x\}_3$ induces an isomorphism

$$\mathrm{B}_3(\mathrm{F}) \xrightarrow{\sim} \mathbb{L}_3(\mathrm{F}). \tag{30}$$

The relation (29) has the following geometric interpretation. Take five points $(\infty; 0, x, 1, y)$ on $\mathbb{P}^1$, where the last four points are ordered cyclically. Then

$$\begin{aligned}&\{x,y\}_{2,1} = \\ &\{[\infty,0,x,1]\}_3 + \{[\infty,1,y,0]\}_3 + \{[\infty,y,0,x]\}_3 + \{[\infty,x,1,y\}_3 - \{[0,x,1,y]\}_3 - \{1\}_3.\end{aligned} \tag{31}$$

Definition 1.8. *The $\mathbb{Q}$-vector space $\mathbb{L}_4(\mathrm{F})$ is generated by elements $\{x\}_4$, where $x \in \mathbb{P}^1(\mathrm{F})$, and $\{x,y\}_{3,1}$ where $x, y \in \mathrm{F}^\times$, obeying the following relations:*

1. *The generators $\{x\}_4$ satisfy the 4-logarithmic relations $\mathcal{R}_4(\mathrm{F})$;*
2. *Specialization relations:*[9]

$$\begin{aligned}\{x,0\}_{3,1} &:= \mathrm{Sp}_{t\to 0}\{x,t\}_{3,1} = -\{x\}_4,\\ \{x,1\}_{3,1} &= -\{1-x^{-1}\}_4 - \{1-x\}_4 + \{x\}_4.\end{aligned} \tag{32}$$

3. $\mathbf{Q}_4$ *: For any configuration $(x_1, x_2, ..., x_7) \in \mathcal{M}_{0,7}(\mathrm{F})$ the following cyclic sum is zero:*

$$\begin{aligned}\mathrm{Cyc}_7\Big(&-\{[x_1,x_2,x_3,x_4],[x_4,x_6,x_7,x_1]\}_{3,1}\\ &+\{[x_1,x_2,x_3,x_4],[x_4,x_5,x_7,x_1]\}_{3,1}\\ &-\{[x_1,x_2,x_3,x_4],[x_4,x_5,x_6,x_1]\}_{3,1}\\ &+\{[x_1,x_2,x_4,x_6]\}_4+\{[x_1,x_2,x_3,x_4,x_5,x_6]\}_4\Big)=0.\end{aligned} \tag{33}$$

Conjecture 1.9. *Relation $\mathbf{Q}_4$ and its specializations imply the tetralogarithm relations $\mathcal{R}_4(\mathrm{F})$.*

Let us define the coproduct maps

$$\begin{aligned}\delta &: \mathbb{L}_2(\mathrm{F}) \longrightarrow \mathrm{F}^\times \wedge \mathrm{F}^\times,\\ \delta &: \mathbb{L}_3(\mathrm{F}) \longrightarrow \mathbb{L}_2(\mathrm{F}) \otimes \mathrm{F}^\times,\\ \delta &: \mathbb{L}_4(\mathrm{F}) \longrightarrow \mathbb{L}_3(\mathrm{F}) \otimes \mathrm{F}^\times \bigoplus \mathbb{L}_2(\mathrm{F}) \wedge \mathbb{L}_2(\mathrm{F}).\end{aligned} \tag{34}$$

First, we define them on the generators: the coproduct $\delta\{x\}_k$ is given by formula (6), and[10]

$$\begin{aligned}\delta\{x,y\}_{2,1} &= \left\{\frac{1-y}{1-x}\right\}_2 \otimes \frac{y}{x} + \left\{\frac{y}{x}\right\}_2 \otimes \frac{1-y}{1-x} + \{x\}_2 \otimes (1-y^{-1}) + \{y\}_2 \otimes (1-x^{-1}),\\ \delta\{x,y\}_{3,1} &= \{x,y\}_{2,1} \otimes \frac{x}{y} + \left\{\frac{x}{y}\right\}_3 \otimes \frac{1-x}{1-y} + \{x\}_3 \otimes (1-y^{-1}) - \{y\}_3 \otimes (1-x^{-1})\\ &\quad - \{x\}_2 \wedge \{y\}_2.\end{aligned} \tag{35}$$

Let us give a motivic interpretation of elements $\{x,y\}_{m-1,1}$ and their coproduct formula (35).

[9] Specialisation relations (32) could be deduced from relation $\mathbf{Q}_4$, but this would require long calculations.

[10] Formulas (35) coincide with the map $\kappa(x,y)$ given by formulas (5) and (6) in [G91a].

10

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2. *Specialization relations:*[9]

$$\begin{aligned}\{x,0\}_{3,1} &:= \mathrm{Sp}_{t\to 0}\{x,t\}_{3,1} = -\{x\}_4,\\ \{x,1\}_{3,1} &= -\{1-x^{-1}\}_4 - \{1-x\}_4 + \{x\}_4.\end{aligned} \tag{32}$$

$$\rightsquigarrow$$

$$\{x,y\}_{3,1} \approx \overbrace{I_{3,1}(x,y)}^{I(0;x,0,0,y;1)} - 3\,\mathrm{Li}_4\left(\tfrac{x}{y}\right)$$

$$\begin{aligned}\{x,1\}_{3,1} &\approx I_{3,1}(x,1) - 3\,\mathrm{Li}_4(x)\\ &\approx S_{2,2}(x) \quad (\text{mod products})\end{aligned}$$

Therefore, understanding $S_{n,p}$ is necessary

Symmetries of $S_{n,p}$

Inversion (more precise in §5.3 Kölbig)

$$S_{n,p}\left(\frac{1}{z}\right) - (-1)^n S_{n,p}(z) = 0 \quad (\text{mod depth} < p, \text{products})$$

Compare with Erik Panzer's Parity Theorem for MPL's (and Goncharov's earlier version)

Reflection (more precise in §5.1 Kölbig)

$$S_{p,n}(z) = -S_{n,p}(1-z) + S_{n,p}(1) \quad (\text{mod products})\,.$$

Follows from path deconcatenation and shuffle product of iterated integrals

Simple consequences

Proposition ($S_{2,2}$ reduction – Proposition 5, CGR)

$$S_{2,2}(z) = -\operatorname{Li}_4(1-z) + \operatorname{Li}_4(z) + \operatorname{Li}_4\left(\frac{z}{z-1}\right) \quad \text{(mod products, constants)}$$

Proof.

Apply reflection to the inversion of $S_{1,3}$:

$$S_{1,3}(z^{-1}) = -S_{1,3}(z) + S_{2,2}(z) - S_{3,1}(z) \quad \text{(mod products, constants)}$$ □

Compare: Goncharov and Rudenko's specialisation $\{x,1\}_{3,1} = -\{1-x^{-1}\}_4 - \{1-x\}_4 + \{x\}_4$

Proposition ($S_{3,2}$ two-term – Proposition 12, CGR)

$$S_{3,2}(1-z) + S_{3,2}(z) = \operatorname{Li}_5(1-z) + \operatorname{Li}_5(1-z^{-1}) + \operatorname{Li}_5(z) \quad \text{(mod products, constants)}$$

Proof.

Apply reflection to the inversion of $S_{1,4}$:

$$S_{1,4}(z^{-1}) = -S_{1,4}(z) + S_{2,3}(z) - S_{3,2}(z) + S_{4,1}(z) \quad \text{(mod products, constants)}$$ □

Hopf algebra of MPL's

Motivic iterated integrals (Goncharov, Brown, . . .)

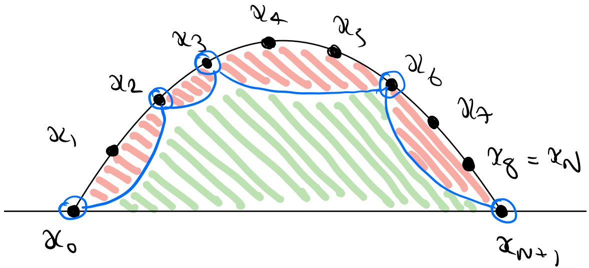
Iterated integrals $I(x_0; x_1, \ldots, x_N; x_{N+1})$ can be upgraded to framed mixed Tate motives, to define

$$I^{\mathfrak{u}}(x_0; x_1, \ldots, x_n; x_{N+1}) \,,$$

elements of a graded Hopf algebra $\mathcal{H}$ (grading is by weight N)

$$\Delta I^{\mathfrak{u}}(x_0; x_1, \ldots, x_N; x_{N+1}) =$$

$$\sum_{\substack{0=i_0<i_1<\cdots \\ <i_k<i_{k+1}=N+1}} I^{\mathfrak{u}}(x_0; x_{i_1}, \ldots, x_{i_k}, x_{N+1}) \otimes \prod_{p=0}^{k} I^{\mathfrak{u}}(x_{i_p}; x_{i_p+1}, \ldots, x_{i_{p+1}-1}; x_{i_{p+1}})$$



Lie coalgebra of MPL's

A graded Hopf algebra induces a Lie coalgebra $\mathcal{L} = \mathcal{H}/\mathcal{H}_{>0} \cdot \mathcal{H}_{>0}$, with $\delta = \Delta - \Delta^{\mathrm{op}}$

- $I^{\mathfrak{u}}(x_0; x_1, \ldots, x_N; x_{N+1})$ becomes $I^{\mathfrak{L}}(x_0; x_1, \ldots, x_N; x_{N+1}) \quad (\text{mod products})$

Example

- $\delta \log^{\mathfrak{L}}(x) = 0$
- $\delta \operatorname{Li}_n^{\mathfrak{L}}(x) = \operatorname{Li}_{n-1}^{\mathfrak{L}}(x) \wedge \log^{\mathfrak{L}}(x) \in \mathcal{L}_{n-1} \wedge \mathcal{L}_1$
- $\delta^{\geq 2} S_{3,2}^{\mathfrak{L}}(x) = -\operatorname{Li}_2^{\mathfrak{L}}(x) \wedge \zeta^{\mathfrak{L}}(3) + \underbrace{\zeta^{\mathfrak{L}}(2)}_{=0} \wedge \operatorname{Li}_3^{\mathfrak{L}}(x)$

Conjecture (Freeness Conjecture, Goncharov)

The kernel of $\delta^{\geq 2}$ is generated by classical polylogarithms $\operatorname{Li}_n^{\mathfrak{L}}(x)$

Expectation

$$S_{3,2}^{\mathfrak{L}}(\text{dilogarithm relations}) = \sum \operatorname{Li}_5^{\mathfrak{L}}\text{'s}$$

Five-term for $S_{3,2}$

Theorem ($S_{3,2}$ of the five-term relation – Theorem 16, CGR)

Following identity holds between $S_{3,2}$ and Li_5:

$$\begin{aligned}\mathrm{Alt}_5 \Big\{ 11 S_{3,2}\big(\,\mathrm{cr}(x_1,x_2,x_3,x_4)\big) \quad + \quad 15\,\mathrm{Li}_5\big(\mathrm{r}_1(x_1,\dots,x_5)\big) \\ - 9\,\mathrm{Li}_5\,\big(\mathrm{r}_2(x_1,\dots,x_5)\big) + \mathrm{Li}_5\,\big(\mathrm{r}_3(x_1,\dots,x_5)\big)\Big\} = 0 \quad (\mathrm{mod\ products, constants})\,.\end{aligned}$$

Here

$$\mathrm{cr}(x_1,x_2,x_3,x_4) \coloneqq \frac{(x_1-x_3)(x_2-x_4)}{(x_1-x_4)(x_2-x_3)}$$

is the classical cross-ratio, and $\mathrm{r}_1, \mathrm{r}_2, \mathrm{r}_3$ are the following 'higher ratios'

$$\begin{aligned}\mathrm{r}_1(x_1,\dots,x_5) &\coloneqq -\frac{(x_1-x_2)(x_1-x_4)(x_3-x_5)}{(x_1-x_3)(x_1-x_5)(x_2-x_4)}\,,\\ \mathrm{r}_2(x_1,\dots,x_5) &\coloneqq -\frac{(x_1-x_2)^2(x_3-x_4)(x_3-x_5)}{(x_1-x_3)(x_1-x_4)(x_2-x_3)(x_2-x_5)}\,,\\ \mathrm{r}_3(x_1,\dots,x_5) &\coloneqq -\frac{(x_1-x_2)^3(x_1-x_5)(x_3-x_4)^2(x_3-x_5)}{(x_1-x_3)^3(x_1-x_4)(x_2-x_4)(x_2-x_5)^2}\,.\end{aligned}$$

Proof strategy

Remark

The "symbol" ($\otimes^m$-invariant) captures differential properties of MPLs. Can be computed/manipulated algorithmically via $(ab) \otimes c = a \otimes c + b \otimes c$, so can check using computer assistance.

Mod-products symbols

$$S_{3,2}(z) \rightsquigarrow -(1-z) \wedge z \otimes \big(\overbrace{(1-z) \sqcup z^{\otimes 2}}^{(1-z)\otimes z\otimes z + z\otimes(1-z)\otimes z + z\otimes z\otimes(1-z)}\big)$$

$$\mathrm{Li}_5(z) \rightsquigarrow -(1-z) \wedge z \otimes z^{\otimes 3}$$

Step 1 Show complicated irreducibles in $1 - r_i$ cancel

Step 2 Consider representation on $\bigwedge^2 V \otimes \mathrm{Sym}^3(V) \cong W_4 \oplus W_6$, $V = \mathrm{span}_{S_5}\{\mathrm{cr}(x_i, x_{i+1}, x_{i+2}, x_{i+3})\}$

Step 3 Show 6-dimensional part is trivial

Step 4 Check identity in 4-dimensional part by polynomial calculations

Corollaries and applications

Corollary (Radchenko 2016, Thesis)

Alternating over 6 points gives a Li_5 *functional equation*

$$\mathrm{Alt}_6 \left\{ \mathrm{Li}_5 \Big(15[\mathrm{r}_1(x_1, \ldots, x_5)] - 9[\mathrm{r}_2(x_1, \ldots, x_5)] + [\mathrm{r}_3(x_1, \ldots, x_5)] \Big) \right\} \overset{\text{ш}}{=} 0$$

Dilogarithm and corresponding $S_{3,2}$ evaluations

For $\phi = \frac{1+\sqrt{5}}{2}$ the golden ratio,

$$\mathrm{Li}_2(\phi^{-2}) = \frac{2}{5}\zeta(2) - \log^2(\phi)$$

via specialisations of the 5-term relation. Hence

$$\begin{aligned} S_{3,2}(\phi^{-2}) = {} & \frac{1}{66} \mathrm{Li}_5 \left([\phi^{-6}] - 32[\phi^{-3}] + \frac{201}{2}[\phi^{-2}] - 48[\phi^{-1}] \right) \\ & + \mathrm{Li}_4\left(\phi^{-2}\right) \log(\phi) + \frac{1}{2}\zeta(5) - \frac{2}{11}\zeta(4)\log(\phi) \\ & - \zeta(3)\,\mathrm{Li}_2\left(\phi^{-2}\right) - \frac{20}{33}\zeta(2)\log(\phi)^3 + \frac{79}{330}\log(\phi)^5 \,. \end{aligned}$$

Summary

- Definitions and key properties of Nielsen polylogarithms
- Motivic yoga to determine which reductions occur
  - Connections to Zagier's polylogarithm conjecture, and
  - Goncharov's freeness conjecture
- Main result: five-term relation for $S_{3,2}$
- Evaluations for $S_{3,2}$ at dilogarithm identities

Expectations for $S_{4,2}$

Observation

$\delta^{\geq 2} S^{\mathfrak{L}}_{4,2}(z) = -\operatorname{Li}^{\mathfrak{L}}_3(z) \wedge \zeta^{\mathfrak{L}}(3)$

- Expect $S_{4,2}$ satisfies trilog functional equations modulo Li_6

Remark

So far, only know

$$S_{4,2}(z) - S_{4,2}(z^{-1}) = 4\operatorname{Li}_6(z) \quad (\text{mod products})$$

$$S_{4,2}(z) + S_{4,2}(1-z) + S_{4,2}(1-x^{-1}) = \\ 2\big(\operatorname{Li}_6(z) + \operatorname{Li}_6(1-z) + \operatorname{Li}_6(1-z^{-1})\big) \quad (\text{mod products})$$

- Can still investigate special values.

$$\operatorname{Li}^{\mathfrak{u}}_3(-1) = -\tfrac{3}{4}\zeta^{\mathfrak{u}}(3) \quad \rightsquigarrow \quad \delta^{\geq 2} S^{\mathfrak{L}}_{4,2}(-1) = 0$$

So $S_{4,2}(-1) = \zeta(1, \overline{5})$ should be depth 1?

Special values of $S_{4,2}$

Claim – §7.3 CGR

$$S_{4,2}(-1) = \frac{1}{13}\left(\frac{1}{3}\operatorname{Li}_6\left(-\frac{1}{8}\right) - 162\operatorname{Li}_6\left(-\frac{1}{2}\right) - 126\operatorname{Li}_6\left(\frac{1}{2}\right)\right) - \frac{1787}{624}\zeta(6)$$
$$+ \frac{3}{8}\zeta(3)^2 + \frac{31}{16}\zeta(5)\log(2) - \frac{15}{26}\zeta(4)\log^2(2) + \frac{3}{104}\zeta(2)\log^4(2) - \frac{1}{208}\log^6(2)\,.$$

Remark

Analytic proof via identity $\operatorname{Li}_{5,1}(-x,-1) = 9$ $S_{4,2}$ terms $+ 117$ Li_6 terms $\pmod{\text{products}}$

Motivic structure

Key points

- $\Delta' S^{\mathfrak{m}}_{4,2}(-1) = \frac{3}{4}\zeta^{\mathfrak{m}}(3) \otimes \zeta^{\mathfrak{u}}(3) + \frac{31}{16}\log^{\mathfrak{m}}(2) \otimes \zeta^{\mathfrak{u}}(5)$
- $\Delta^{(5,1)}\operatorname{Li}^{\mathfrak{m}}_6(x) = \operatorname{Li}^{\mathfrak{m}}_5(x) \otimes \log^{\mathfrak{u}}(x)$
- $\operatorname{Li}^{\mathfrak{m}}_5\left(-\frac{1}{8}\right) - 162\operatorname{Li}^{\mathfrak{m}}_5\left(-\frac{1}{2}\right) - 126\operatorname{Li}^{\mathfrak{m}}_5\left(\frac{1}{2}\right) = \frac{403}{16}\zeta^{\mathfrak{m}}(5) + \text{products}\,.$