Zagier's polylogarithm conjecture on $\zeta_F(4)$ and an explicit 4-ratio

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Outline

Joint work with H. Gangl & D. Radchenko, arXiv:1909.13869
Building on A. Goncharov & D. Rudenko, arXiv:1803.08585

Dedekind zeta function and polylogarithms

Dedekind zeta function

Throughout: F is a number field, $\mathcal{O}_F$ the ring of integers

Definition

The Dedekind zeta function is

$$\zeta_F(s) := \sum_{\substack{(0)\neq\mathfrak{a}\subset\mathcal{O}_F \\ \mathfrak{a}\text{ non-zero ideal}}} \big|\overbrace{\mathcal{O}_F/\mathfrak{a}}^{\text{norm of }\mathfrak{a}}\big|^{-s}, \quad \operatorname{Re}(s) > 1$$

- Meromorphic on $\mathbb{C}$, simple pole at $s = 1$
- For $F = \mathbb{Q}$, get the Riemann zeta

Theorem (Analytic class-number(-less) formula)

$$\operatorname{Res}_{s=1} \zeta_F(s) \sim_{\mathbb{Q}^\times} \sqrt{|\Delta_F|}\pi^{r_2} \operatorname{Reg}_F ,$$

- *Δ_F is the discriminant,*
- *r_2 is number of pairs of complex embeddings,*
- *Reg_F is a determinant of logarithms of units of F . (Mysterious!)*

Have '$\zeta_F(1)$' via logarithms. So higher values of $\zeta_F(m)$ should need *higher* logarithms.

Polylogarithms

Definition (Polylogarithm)

The weight m polylogarithm is

$$\mathrm{Li}_m(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^m}, \quad |z| < 1$$

- $\mathrm{Li}_1(z) = -\log(1-z)$
- Analytic continuation via $\mathrm{Li}_{m+1}(z) = \int_0^z \mathrm{Li}_m(t) \frac{\mathrm{d}t}{t}$

Definition (Bloch-Wigner-Ramakrishnan-Zagier polylogarithm)

A single-valued version of the polylogarithm is

$$\mathscr{L}_m(z) = \begin{cases} \mathrm{Re} \\ \mathrm{Im} \end{cases} \left(\sum_{k=0}^{m-1} \frac{2^k B_k}{k!} \mathrm{Li}_{m-k}(z) \log^k(z) \right) \quad \begin{matrix} m \text{ odd}, \\ m \text{ even} \end{matrix}$$

- B_k the k-th Bernoulli number

- $\mathscr{L}_1(z) = -\log|1-z|$
- $\mathscr{L}_2(z) = \mathrm{Im}(\mathrm{Li}_2(z) + \log(1-z)\log|z|)$
- $\mathscr{L}_3(z) = \mathrm{Re}(\mathrm{Li}_3(z) - \mathrm{Li}_2(z)\log|z| - \frac{1}{3}\log(1-z)\log^2|z|)$

Zagier's polylogarithm conjecture

- Write d_m = order of vanishing of $\zeta_F(1-m) = \begin{cases} r_1 + r_2 & m \text{ odd} \\ r_2 & m \text{ even} \end{cases}$
- Order embeddings $\sigma_i \colon F \to \mathbb{C}$, so $\sigma_i = \overline{\sigma_{i+r_1+r_2}}$, ($r_1$ real, r_2 pairs cx embeddings)
- Extend $\mathscr{L}_m, \sigma_i$ to 'formal linear combinations' in $\mathbb{Z}[F^\times]$ by linearity

Conjecture (Zagier)

Let $m \geq 2$. There exists $y_1, \ldots, y_{d_m} \in \mathbb{Z}[F^\times]$ so that

$$\zeta_F(m) \sim_{\mathbb{Q}^\times} \sqrt{|\Delta_F|}\pi^{md_{m+1}} \det\Big(\mathscr{L}_m(\sigma_i(y_j))\Big)_{i,j=1}^{d_m} .$$

Recipe to find (candidate) y_i inductively, using *numerical* algorithm.

$$\zeta_{\mathbb{Q}(\zeta_5)}(2) \stackrel{?}{=} -\frac{2^3\sqrt{5}}{3\cdot 5^4}\pi^4 \det\begin{pmatrix} \mathscr{L}_2(\zeta_5) & \mathscr{L}_2(\zeta_5^2) \\ \mathscr{L}_2(\zeta_5^2) & \mathscr{L}_2(\zeta_5^4) \end{pmatrix} \approx 1.0923496617\ldots$$

Status

- $n = 2$: Zagier (weak version)
  Bloch-Suslin $\sim$1981
  Goncharov (subtle fix)
- $n = 3$: $\sim$1993 Goncharov via Li_3 breakthrough
- $n = 4$: 2018 Goncharov-Rudenko via $\mathbf{Q}_4$ new geometric identity
  (geometric understanding/reinterpretation of a result of Gangl)

Also known for special classes of field F

- Cyclotomic fields

Goncharov has a vast program which can prove specific m
(using Borel's theorem as an important starting point)

Warning

Requires heavy input of *currently unknown* $\mathscr{L}_m$-functional equations and identities

Cohomology and constructing canonical classes

Borel's Theorem

Borel defined regulator from K-theory

$$R_m^{\mathrm{bo}} \colon K_{2m-1}(\mathbb{C}) \to \mathbb{R}(m-1) =: \mathbb{R}(2\pi \mathrm{i})^{m-1}$$

Theorem (Borel, 1977)

Consider

$$\phi \colon K_{2m-1}(F) \to \bigoplus_{\sigma_i} K_{2m-1}(\mathbb{C}) \to \mathbb{Z}^{\mathrm{Hom}(F,\mathbb{C})} \otimes \mathbb{R}(m-1)$$

1. ϕ *is injective (mod torsion),*
2. *image is a lattice* Λ_m^F *(in the invariants under cx conjugation),*
3. $\zeta_F(m) \sim_{\mathbb{Q}^\times} \sqrt{|\Delta_F|}\pi^{m d_{m+1}} \operatorname{covol}(\Lambda_m^F)$.

Strategy: find formula for R_m^{bo} in terms of $\mathscr{L}_m$.

Fact

R_m^{bo} arises from certain canonical ('Borel') class $c_{2m-1} \in H^{2m-1}_{\mathrm{cts}}(\mathrm{GL}_m(\mathbb{C}), \mathbb{R}(m-1))$

Construction of c_1

Represent a class $c \in H^{m-1}_{\text{cts}}(G, \mathbb{R})$ via the cochain $\phi\colon G^m \to \mathbb{R}$.

Fact

$$\begin{aligned} \phi_1 \colon \ & \mathrm{GL}_1(\mathbb{C})^2 \to \mathbb{R} \\ & \phi_1(g_1, g_2) = \log\lvert\det(g_1^{-1} g_2)\rvert \end{aligned}$$

defines 1-cocycle, and represents c_1.

Cocycle condition:

$$\log\lvert x\rvert - \log\left\lvert\frac{x}{y}\right\rvert + \log\left\lvert\frac{1}{y}\right\rvert = 0$$

Construction of c_3

On $\mathrm{Conf}_m(n) = \{(v_1, \ldots, v_n) \mid v_i \in \mathbb{C}^m\} \big/ \mathrm{GL}_m$, write $\langle i_1 \cdots i_m \rangle = \det(v_{i_1} \cdots v_{i_m})$.

Cross-ratio on $\mathrm{Conf}_4(2)$ is

$$\mathrm{cr}(v_1, \ldots, v_4) = \frac{\langle 13 \rangle \langle 24 \rangle}{\langle 14 \rangle \langle 23 \rangle} = \frac{z_1 - z_3}{z_1 - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_3} \qquad \text{where } z_i \leftrightarrow [v_{i,1} : v_{i,2}] \in \mathbb{P}^1(\mathbb{C})$$

Theorem (Bloch)

For all $v \neq \mathbf{0} \in \mathbb{C}^2$,

$$\phi_2 : \mathrm{GL}_2(\mathbb{C})^4 \to \mathbb{R}$$
$$\phi_2(g_1, \ldots, g_4) = \mathscr{L}_2(\mathrm{cr}(g_1 v, \ldots, g_4 v))$$

defines 3-cocycle, and represents c_3.

Cocycle condition corresponds to

Famous 5-term relation

$$\mathscr{L}_2\left([x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1-y}{1-x}\right] + \left[\frac{x(1-y)}{(1-x)y}\right]\right) = 0$$

Zagier's Conjecture for $n = 2$ follows from Borel's Theorem.

Construction of c_5

Goncharov defines a (pre-)triple-ratio

$$\mathrm{cr}_3(v_1, \ldots, v_6) = \frac{\langle 12\,4\rangle\langle 23\,5\rangle\langle 31\,6\rangle}{\langle 12\,5\rangle\langle 23\,6\rangle\langle 31\,4\rangle}$$

Theorem (Goncharov)

For all $v \neq \mathbf{0} \in \mathbb{C}^3$,

$$\phi_3 : \mathrm{GL}_3(\mathbb{C})^6 \to \mathbb{R}$$
$$\phi_3(g_1, \ldots, g_6) = \mathrm{Alt}_6\, \mathscr{L}_3(\mathrm{cr}_3(g_1 v, \ldots, g_6 v))$$

defines 5-cocycle, and represents c_5.

Zagier's Conjecture for $n = 3$ follows from Borel's Theorem.

Cocycle condition:

- $\underbrace{840}_{=7!/6}$-term Li_3 functional equation
- related 22-term functional equation

22-term and beyond

Theorem (22-term relation, Goncharov)

$$\mathscr{L}_3\left(\mathrm{Cyc}_{x,y,z}\left([z]+\left[-\frac{x(yz-z+1)}{xz-x+1}\right]+\left[\frac{yz-z+1}{y(xz-x+1)}\right]-\left[\frac{yz-z+1}{yz(xz-x+1)}\right]\right.\right.$$
$$\left.\left.+[xz-x+1]-\left[\frac{xz-x+1}{z}\right]+\left[\frac{xz-x+1}{xz}\right]\right)+[-xyz]\right)=3\,\mathscr{L}_3(1)$$

How to generalise the cross-ratio and triple-ratio?

Naïve guesses fail

Candidate

$$\mathrm{cr}_4(v_1,\ldots,v_8)=\frac{\langle 123\,5\rangle\langle 234\,6\rangle\langle 345\,7\rangle\langle 451\,8\rangle}{\langle 123\,8\rangle\langle 234\,5\rangle\langle 345\,6\rangle\langle 451\,7\rangle}$$

does not give functional equations for $\mathscr{L}_4$!

Higher ratios and Grassmannian polylogs

m-ratio and Grassmannian polylogs

Conjecture (Existence of m-ratio, Goncharov)

For $m \geq 2$, there exists
$$\sum_i \lambda_i [r_i]\,, \quad r_i \in \mathbb{Q}(\mathrm{Conf}_{2m}(m))$$
such that $\phi_m(g_1, \ldots, g_{2m}) = \mathrm{Alt}_{2m} \sum_i \lambda_i \mathscr{L}_m(r_i(g_1 v, \ldots, g_{2m} v))$, $\forall v \neq \mathbf{0} \in \mathbb{C}^m$, *is a* $(2m-1)$*-cocycle and represents the Borel class* c_{2m-1}.

Key tool to investigate: Grassmannian polylogs Gr_m.

Theorem (Goncharov)

A single-valued version of Gr_m *represents* c_{2m-1}

Cocycle condition: $\mathrm{Alt}_{2m+1} \mathrm{Gr}_m = 0$.
Manifest as terms of the symbol ($\otimes^m$-invariant) depend on $2m-1$ points.

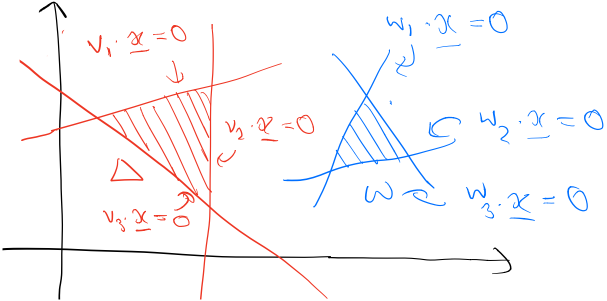
Aomoto and Grassmannian polylogs

Definition (Grassmannian polylog)

Gr_m is the multivalued analytic function defined by

$$\mathrm{d}\,\mathrm{Gr}_m(v_1,\ldots,v_{2m}) = \mathrm{Alt}_{2m}\,\mathcal{A}(v_1,\ldots,v_m \mid v_{m+1},\ldots,v_{2m}) \cdot \mathrm{d}\log\langle m+1,\ldots,2m\rangle$$

where $\mathcal{A}(v_1,\ldots,v_m \mid w_1,\ldots,w_m)$ is geometrically defined Aomoto polylogarithm.



$$\mathcal{A}(v_1,v_2,v_3 \mid w_1,w_2,w_3) = \int_{\Delta} \mathrm{d}\log\left(\frac{w_2\cdot\mathbf{x}}{w_1\cdot\mathbf{x}}\right)\wedge \mathrm{d}\log\left(\frac{w_3\cdot\mathbf{x}}{w_1\cdot\mathbf{x}}\right)$$

Reduction of Gr_m

Goal: Rewrite Gr_m in terms of $\mathscr{L}_m$.

Problem: An obstruction exists (non-zero motivic cobracket), meaning this is impossible (for $m \geq 4$).

Fix: Can modify Gr_m by trivial coboundary terms depending on $\leq 2m-1$ points. Find trivial coboundary correction which kills obstruction.

Goncharov already gave correction in weight 4:

$$\mathrm{Alt}_8\, I_{3,1}(\mathrm{cr}(34|2567), \mathrm{cr}(67|1345)) \,,$$

with projected cross-ratio $\mathrm{cr}(ab|cdef) = \dfrac{\langle ab\, ce\rangle}{\langle ab\, cf\rangle} \dfrac{\langle ab\, df\rangle}{\langle ab\, de\rangle}$ defined on $\mathrm{Conf}_8(4)$.

Explicit reduction of Gr_4 and a 4-ratio

$I_{3,1}$ and ρ-coordinates

Definition ($I_{3,1}$)

$I_{3,1}$ multiple polylog is defined by

$$I_{3,1}(x,y) = \mathrm{Li}_{3,1}\left(\frac{y}{x}, \frac{1}{y}\right) = \sum_{0<n<m} \frac{y^{n-m}x^{-m}}{n^3 m}$$

Definition (ρ-coordinates)

Coordinates on $\mathrm{Conf}_8(4)$

$$\rho_i = \underbrace{\rho^{7,8}_{i,i+1,i+2}}_{\text{mod } 6} = \frac{\langle i, i+1, i+2, 7\rangle}{\langle i, i+1, i+2, 8\rangle}$$

Shorthand $\rho_{i,j} = \rho_i - \rho_j$

Reduction of Gr_4

Theorem (CGR, 2019)

Modulo products

$$\frac{7}{144}\,\mathrm{Gr}_4 = \mathrm{Alt}_8\Big[I_{3,1}\Big(\frac{\rho_{1,2}\rho_{3,4}}{\rho_{3,2}\rho_{1,4}}, \frac{\rho_1}{\rho_{1,4}}\Big) + 2I_{3,1}\Big(\frac{\rho_{1,2}}{\rho_1}, \frac{\rho_{3,2}}{\rho_{3,4}}\Big) + 6\,\mathrm{Li}_4\Big(\frac{\rho_1\rho_{3,2}}{\rho_{1,2}\rho_{3,4}}\Big)\Big].$$

Proof.

Found with computer assistance. Explicit calculation of the symbol ($\otimes^m$-invariant) by hand. □

Makes explicit first step of Goncharov-Rudenko.

Remark

There is some structure in this reduction.
(Cyclic symmetry, cross-ratio-like structure in $\rho_i$and $0, \infty$)

Behaviour of $I_{3,1}$

Heuristic: modulo Li_4 terms $I_{3,1}(x,y) \sim \mathrm{Li}_2(x) \wedge \mathrm{Li}_2(y)$.

Reason: Motivic cobracket of $I_{3,1}$, leads to obstruction for $\mathrm{Gr}_4 = \mathrm{Li}_4$'s.

Results: Explicit expressions (by Zagier, and by Gangl, ~ 12 terms) for

$$I_{3,1}(x,y) + I_{3,1}(x^{-1},y) = \mathrm{Li}_4\text{'s}\,, \qquad I_{3,1}(x,y) + I_{3,1}(1-x,y) = \mathrm{Li}_4\text{'s}\,.$$

Theorem (Gangl, 2016)

There exists $f_i(x,y,z)$ rational functions and $c_i \in \mathbb{Q}$, so that modulo products

$$I_{3,1}\Big(z, [x] + [y] + \Big[\frac{1-x}{1-xy}\Big] + [1-xy] + \Big[\frac{1-y}{1-xy}\Big]\Big) = \sum_{i=1}^{122} c_i \,\mathrm{Li}_4(f_i(x,y,z)) =: V(z,[x,y])\,.$$

Found with computer assistance. Goncharov-Rudenko have a geometric derivation.

Gr_4 to Li_4's

Theorem (Reduction of Gr_4, CGR, 2019)

$$\frac{7}{144}\,\mathrm{Gr}_4 + 2\,\mathrm{Alt}_8\, I^{\mathrm{sym}}_{3,1}(\mathrm{cr}(34|2567), \mathrm{cr}(67|1345)) =$$

$$\mathrm{Alt}_8 \Big\{ -V\Big(\frac{\rho_4}{\rho_1};\ \Big[\frac{\rho_{4,2}}{\rho_{4,1}}; \frac{\rho_{4,1}}{\rho_{4,3}}\Big] - [\mathrm{cr}(43|2685); \mathrm{cr}(48|7653)] + \frac{1}{4}[\mathrm{cr}(43|1256); \mathrm{cr}(43|1268)] - \frac{1}{12}[\mathrm{cr}(43|1256); \mathrm{cr}(42|1365)]\Big)$$

$$+ V\Big(\frac{\rho_2}{\rho_1};\ -[\mathrm{cr}(43|2685; \mathrm{cr}(48|7653)] + [\mathrm{cr}(48|7235; \mathrm{cr}(48|7263)] + \frac{1}{2}[\mathrm{cr}(46|5238; 43|2568)]\Big)$$

$$+ 6\,\mathrm{Li}_4\Big(\frac{\rho_1\rho_{3,2}}{\rho_{1,2}\rho_{3,4}}\Big) \quad + \quad \mathrm{Li}_4\text{'s from } I_{3,1}\text{-symmetrising}\Big\}.$$

Corollary (Explicit 4-ratio)

Obtain a new Li_4 functional equation with 1775 S_8-orbits. Computer assistance gives 368 S_8 orbits. Candidate for $K_7(F)$ via generators and relations.

Higher weight

Theorem (Grassmann reduction, CGR, 2019)

One term expression under Alt_{2m} for Gr_m via 'iterated integrals' using generalised ρ-coordinates.

Theorem (Aomoto reduction, CGR, 2021/22)

One term expression under $\mathrm{Alt}_{m,m}$ for the Aomoto $\mathcal{A}_{m-1}(v_1, \ldots, v_m \mid w_1, \ldots, w_m)$ polylog via 'iterated integrals' using generalised ρ-coordinates.

Theorem (Gr_5 reduction and coboundary, CGR, 2019)

Expression for Gr_5 in terms of four $I_{4,1}$ terms and 2 Li_5, under Alt_{10}.
Coboundary correction term expressed via two $I_{4,1}$ terms.

This is a starting point for reduction in weight ≥ 5. (In progress.)

Summary

- Zagier's polylogarithm conjecture on $\zeta_F(m)$
  - Statement and progress
- Goncharov's program for proving $\zeta_F(m)$
  - Conjecture: existence of m-ratios?
  - Borel's theorem and canonical classes c_{2m-1}
  - Construction of c_1, c_3, c_5
- Expressions for Grassmannian polylogs
- Explicit reduction of Gr_4
  - Explicit expression for 4-ratio
  - New functional equations for Li_4
- Progress in weight 5