Multiple zeta values and modular forms

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Outline

($+\,\varepsilon$ of new work with Adam Keilthy?)

Introduction, definitions and motivation

Riemann zeta values

Definition (Riemann zeta function)

The Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \mathrm{Re}(s) > 1$$

Special values?

$$\zeta(2) = \frac{\pi^2}{6}, \qquad \zeta(4) = \frac{\pi^4}{90}, \qquad \zeta(6) = \frac{\pi^6}{945}, \qquad \dots$$

Euler (1730's)

The following holds:

$$\zeta(2k) = \frac{(-1)^{k+1} B_{2k} (2\pi)^{2k}}{2(2k)!},$$

where $\sum_{k=0}^{\infty} B_k \frac{x^k}{k!} := \frac{x}{e^x - 1}$ defines the Bernoulli number B_k.

Questions about $\zeta(n)$

- What is known about $\zeta(3)$?
  - No evaluation known, numerically it seems $\zeta(3)\pi^{-3} \notin \mathbb{Q}$
  - Know that $\zeta(3) \notin \mathbb{Q}$ (Apéry, 1978)
- What is known about $\zeta(5)$?
  - At least one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ irrational. (Zudilin, further work by Ball, Rivoal) (But which ones?)

Idea

To try to understand $\zeta(n)$, fit it into a (bigger) algebraic structure

Multiple zeta values (MZV's)

Definition

The multiple zeta value $\zeta(k_1, k_2, \ldots, k_d)$ is defined by

$$\zeta(k_1, k_2 \ldots, k_d) = \sum_{1 \leq n_1 < n_2 < \cdots < n_d} \frac{1}{n_1^{k_1} n_2^{k_2} \cdots n_k^{k_d}}$$

- Convergence: $k_d > 1$
- The weight is $k_1 + \cdots + k_d$, the depth is d

Motivation

Product

$$\zeta(a)\zeta(b) = \zeta(a, b) + \zeta(b, a) + \zeta(a + b)$$

Appearances

Appears in many areas

- Vassiliev knot invariants
- Dirichlet eigenvalues of regular polygons
- Scattering amplitudes in high-energy physics
- Number theory
- ...

Algebraic properties

Relations

MZV's satisfy many relations

$$\zeta(1,2) = \zeta(3) \qquad \zeta(\underbrace{1,3,\ldots,1,3}_{n \text{ repetitions}}) = \frac{2\pi^{4n}}{(4n+2)!}$$

$$\sum_{\substack{i+j+k=n \\ k>1}} \zeta(i,j,k) = \zeta(n) \qquad 28\zeta(3,9) + 150\zeta(5,7) + 168\zeta(7,5) = \frac{5197}{691}\zeta(12)$$

Dimensions

By eliminating linearly dependent elements (numerically, via LLL), we find candidate dimensions

weight w	1	2	3	4	5	6	7	8	9	10	11	12
$d_k = \dim_{\mathbb{Q}}$	0	1	1	1	2	2	3	4	5	7	9	12

Conjecture: $d_k = d_{k-2} + d_{k-3}$.

Stuffle product

Multiply series

$$\zeta(a,b)\zeta(c) = \sum_{0<i<j} \frac{1}{i^a j^b} \sum_{0<k} \frac{1}{k^c}$$
$$= \left(\sum_{0<i<j<k} + \sum_{0<i<k<j} + \sum_{0<k<i<j} + \sum_{0<i<j=k} + \sum_{0<i=k<j} \right) \frac{1}{i^a j^b k^c}$$
$$= \zeta(a,b,c) + \zeta(a,c,b) + \zeta(c,a,b) + \zeta(a,b+c) + \zeta(a+c,b)$$

More formally

Definition (Stuffle product)

For words w, v in letters z_i, recursively define

$$z_a w * z_b v = z_a(w * z_b v) + z_b(z_a w * v) + z_{a+b}(w * v)$$

This encodes $\zeta(n_1, \ldots, n_d)\zeta(m_1, \ldots, m_e)$ via $\zeta(z_{n_1} \cdots z_{n_d}) = \zeta(n_1, \ldots, n_d)$.

Integral representation

Proposition

$$\zeta(n_1,\ldots,n_d) = (-1)^d I(0;1,\{0\}^{n_1-1},1,\{0\}^{n_2-1},\ldots,1,\{0\}^{n_d-1};1)$$

where

$$I(x_0;x_1,\ldots,x_N;x_{N+1}) = \int_{0<t_1<\cdots<t_N<1} \frac{\mathrm{d}t}{t_1-x_1}\cdots\frac{\mathrm{d}t}{t_n-x_n}$$

Proof idea: expand geometric series

$$\int_{0<t_1<t_2<1} \frac{\mathrm{d}t_1}{1-t_1}\frac{\mathrm{d}t_2}{t_2} = \sum_{n_1=1}^{\infty}\int_{0<t_1<t_2<1} t_1^{n_1}\mathrm{d}t_1\frac{\mathrm{d}t_2}{t_2}$$
$$= \sum_{n_1=1}^{\infty}\int_{0<t_2<1} \frac{t_2^{n_1-1}}{n_1}\mathrm{d}t_2 = \sum_{n_1=1}^{\infty}\frac{1}{n_1^2} = \zeta(2)$$

Corollary (Duality)

$$\zeta(n_1,\ldots,n_d) \approx 10^{n_1-1}\cdots 10^{n_d-1} \xleftrightarrow[\substack{0\leftrightarrow 1\\ \textit{reverse}}]{t\mapsto 1-t} 10^{m_1-1}\cdots 10^{m_e-1} \approx \zeta(m_1,\ldots,m_e)$$

Shuffle product

Multiply integrals for $\zeta(2)\zeta(2)$

$$\int_{0<t_1<t_2<1} \frac{\mathrm{d}t_1}{1-t_1} \frac{\mathrm{d}t_2}{t_2} \cdot \int_{0<s_1<s_2<1} \frac{\mathrm{d}s_1}{1-s_1} \frac{\mathrm{d}s_2}{s_2} = \int_{\text{shuffles}} \frac{\mathrm{d}t_1}{1-t_1} \frac{\mathrm{d}t_2}{t_2} \frac{\mathrm{d}s_1}{1-s_1} \frac{\mathrm{d}s_2}{s_2}$$

$$= 4 \int_{0<t_1<t_2<t_3<t_4} \frac{\mathrm{d}t_1}{1-t_1} \frac{\mathrm{d}t_2}{1-t_2} \frac{\mathrm{d}t_3}{t_3} \frac{\mathrm{d}t_4}{t_4} + 2 \int_{0<t_1<t_2<t_3<t_4} \frac{\mathrm{d}t_1}{1-t_1} \frac{\mathrm{d}t_2}{t_2} \frac{\mathrm{d}t_3}{1-t_3} \frac{\mathrm{d}t_4}{t_4}$$

$$= 4\zeta(1,3) + 2\zeta(2,2)$$

More formally

Definition (Shuffle product)

For words w, v in letters e_0, e_1, recursively define

$$e_i w ⧢ e_j v = e_j(w ⧢ e_j v) + e_j(e_i w ⧢ v)$$

This encodes $\zeta(n_1, \ldots, n_d)\zeta(m_1, \ldots, m_e)$ via $\zeta(e_{i_1} \cdots e_{i_n}) = \pm I(0; e_{i_1}, \ldots, e_{i_n}; 1)$.

Double shuffle relations

Compare product structres

The shuffle- and stuffle-product give two *different* algebra structures on MZV's!

Example

$$2\zeta(2,2)+\zeta(4) \overset{*}{=} \zeta(2)\zeta(2) \overset{ш}{=} 4\zeta(1,3)+2\zeta(2,2) \quad\Longrightarrow\quad \zeta(1,3)=\frac{1}{4}\zeta(4)$$

Allow regularisation $\zeta(1) =: T$ (a formal object); can extend $*, ш$

Conjecture

Comparing $\zeta(\underline{\mathbf{k}})\zeta(\underline{\mathbf{l}})$ via shuffle product and stuffle gives all MZV relations.

Warning

- Need regularisation to get $\zeta(1,2)=\zeta(3)$
- Difficult to deduce given relations (need nice structure, or big linear algebra)

Double shuffle relations for double zetas

In weight k, the shuffle and stuffle product give double zeta relations

$$\zeta(a)\zeta(b) = \zeta(a,b) + \zeta(b,a) - \zeta(k)$$

$$\zeta(a)\zeta(b) = \sum_{r+s=k} \left[\binom{r-1}{a-1} + \binom{r-1}{b-1} \right] \zeta(r,s)$$

Introduce generating series

$$P(X,Y) = \sum_{r+s=k} \zeta(r)\zeta(s) X^{r-1} Y^{s-1}$$

$$Z(X,Y) = \sum_{r+s=k} \zeta(r,s) X^{r-1} Y^{s-1}$$

Then we get

$$Z(X,Y) + Z(Y,X) = P(X,Y) - \zeta(k) \frac{X^{k-1} - Y^{k-1}}{X - Y}$$

$$Z(X+Y,Y) + Z(X+Y,X) = P(X,Y)$$

Modular forms relations

Modular forms connections

Recall Eisenstein series (in some normalisation):

$$G_k(z) = \frac{1}{2} \sum_{p,q \in \mathbb{Z}}{}' \frac{1}{(p\tau + q)^k} = \zeta(k) + \sum_{n=1}^{\infty} \sigma_k(n) q^n$$

Constant term relations

Relations

$$\sum c_a G_a(z) G_{k-a}(z) = c_0 G_k(z)$$

give relations for zetas

$$\sum c_a \zeta(a) \zeta(k-a) = c_0 \zeta(a)$$

by taking constant term.

(More interesting relations from cusp forms!)

Period polynomials

Given a cusp form $f \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$, consider the *period* polynomial

$$\begin{aligned} r(f)(X,Y) &= \int_0^{\mathrm{i}\infty} f(t) Y^{2k-2}(X/Y - t)^{2k-2} \mathrm{d}t \\ &= \sum_{n=0}^{w} \mathrm{i}^{1-n} \binom{w}{n} X^{2k-2-n} Y^n r_n(f) \end{aligned}$$

where the r-th period is

$$r_n(f) = \int_0^{\infty} f(\mathrm{i}t) t^n \, \mathrm{d}t, \quad 0 \le n \le 2k-2$$

Fact

Up to multiplying by a single (transcendental) number, the $X^{\text{even}} Y^{\text{even}}$ part is in $\mathbb{Q}[X,Y]$. Similarly the $X^{\text{odd}} Y^{\text{odd}}$ part.

Eichler-Shimura

Matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in (\mathrm{P})\mathrm{SL}_2(\mathbb{Z})$ acts on homogeneous polynomials $f(X, Y)$ of degree k via

$$(P|_\gamma)(X, Y) = (cX + dY)^k P(aX + bY, cX + dY)$$

Period polynomial subspace

Since $\int_0^{i\infty} + \int_{i\infty}^0 = 0$ and $\int_0^{i\infty} + \int_{i\infty}^1 + \int_1^0 = 0$, period polynomials $P = r(f)(X, Y)$ have special properties

$$P + P|_S = 0 \qquad S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$$

$$P + P|_U + P|_{U^2} = 0 \qquad U = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$$

Theorem (Eichler-Shimura, very rough)

Period polynomials of degree k are isomorphic to cusp forms of weight k

Double shuffle, revisited

Double-zeta double shuffle relations are

$$Z(X+Y,Y) + Z(X+Y,X) - Z(X,Y) - Z(Y,X) - \zeta(k)\frac{X^{k-1}-Y^{k-1}}{X-Y} = 0$$

Can construct general solutions (realisations, with $\zeta(k) \to 0$) to these relations via

$$Z(X,Y) = A(X,Y) - A(X,X-Y) + A(Y,Y-X)$$

for $A(X,Y)$ any polynomial even with respect to Y. Can use ideas like this to connect to period polynomials.

Theorem (Gangl-Kaneko-Zagier, 2006)

Let $f \in S_{w+2}(\mathrm{SL}_2(\mathbb{Z}))$, with (even part) of period polynomial $P_f(X,Y)$, and define a_r by

$$\sum_{r=0}^{w} a_r X^r Y^{w-r} = P_f(X+Y,X)\,.$$

Then

$$\sum_{\substack{r=0 \\ \text{even}}}^{w-2} a_r r!(w-r)!\zeta(r+1,w+1-r) \in \mathbb{Q}\zeta(w+2)$$

Examples

Weight 12, have one degree 10 period polynomial

$$P_f(X,Y) = -X^8Y^2 + 3X^6Y^4 - 3X^4Y^6 + X^2Y^8$$

$$P_f(X+Y,X) = \cancel{-8X^7Y^3} - 28X^6Y^4 - \cancel{38X^5Y^5} - 25X^4Y^6 - \cancel{8X^3Y^7} - X^2Y^8$$

Relation: $-28 \cdot 4!6!\zeta(7,5) - 25 \cdot 4!6!\zeta(5,7) - 2!8!\zeta(3,9) = -4!5! \cdot \frac{5197}{691}\zeta(12)$

Weight 16, only one degree 14 period polynomial

$$P_f(X,Y) = -2X^{12}Y^2 + 7X^{10}Y^4 - 11X^8Y^6 + 11X^6Y^8 - 7X^4Y^{10} + 2X^2Y^{12}$$

$$P_f(X+Y,X) \rightsquigarrow -132X^{10}Y^4 - 675X^8Y^6 - 686X^6Y^8 - 125X^4Y^{10} - 2X^2Y^{12} + \cancel{X^{\text{odd}}Y^{\text{odd}}}\text{'s}$$

Relation: $-2 \cdot 2!12!\zeta(3,13) - 125 \cdot 4!10!\zeta(5,11) - 686 \cdot 6!8!\zeta(7,9)$

$$-675 \cdot 6!8!\zeta(9,7) - 132 \cdot 4!10!\zeta(11,5) = -6!8! \cdot \frac{78967}{3617}\zeta(12)$$

Propagation of modular relations

Theorem (C-Keilthy, 2022?, Schematic)

Following evaluation holds modulo products

$$\zeta(\overbrace{\{2\}^a}^{2,\ldots,2 \text{ repeated } a \text{ times}}, 4, \{2\}^b) = 4(-1)^n\Big[-\zeta(2a+2, 2b+2) - \zeta(2a+3, 2b+1) + \sum_{j=1}^{2n+3} 2^{j-4-2n}\left(\binom{2n+3-j}{2b+1} - \binom{2n+3-j}{2a+1}\right)\zeta(j, 2n+4-j)\Big]$$

Corollary

Modulo products,

$$\sum_{i=a}^{n-a} \zeta(\{2\}^i, 4, \{2\}^{n-i}) = 4(-1)^n \zeta(2a+1, 2n-2a+3)\,.$$

So the modular relations propagate to $\zeta(2,\ldots,2,4,2,\ldots,2)$.

Summary

- Definition of multiple zeta values
- Appears throughout physics/maths
- Integral representation and duality
- Shuffle and stuffle product
  - double shuffle relations
  - (need for) regularisation
- Period polynomials
- Double zeta relations from cusp forms
  - Explicit recipe from period polynomials
- Modular relations for $\zeta(2,\ldots,2,4,2,\ldots,2)$