MZV's in block degree 2 & the period — polynomial relations

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§1 MZV's & block filtration

Def: For $k_1, \dots, k_{d-1} \geq 1$, $k_d \geq 2$ integers

$$\zeta(k_1, \dots, k_d) = \sum_{n_1 < \dots < n_d} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}}$$

weight: $k_1 + \dots + k_d$, depth d

- Appears in knot theory, periods of mixed Tate motives, high energy physics
- Deep open questions: $\zeta(3)$ transcendental?

Write $\zeta(k_1, \dots, k_d) = (-1)^d I(0; \underbrace{10\cdots0}_{k_1}\underbrace{10\cdots0}_{k_2}\cdots\underbrace{10\cdots0}_{k_d}; 1)$

where

$$I_{(\gamma)}(a; x_1, \dots, x_n; b) = \int_{0 < t_1 < \dots < t_n < 1} \frac{d\gamma(t_1)}{\gamma(t_1) - x_1} \wedge \dots \wedge \frac{d\gamma(t_n)}{\gamma(t_n) - x_n}$$

γ = path: $a \to b$, for MZV's $\gamma(t) = t$ on $[0,1]$.

Lift to motivic MZV's (Goncharov, Brown)

Idea

$$\zeta^m(k_1, \dots, k_d) = [\text{variety}, \text{chain}, \text{de Rham form}]$$

[Maybe $(H^n_B(X), H^n_{dR}(X), \text{comp})$]

variety: $X \simeq \mathcal{M}_{0,n}$; chain $\in H^k_B(X)^\vee$; de Rham form $\in H^k_{dR}(X)$ de Rham

Then:

[Cf Brown's more refined definitions of motivic periods ...]

- all relations are geometric
- weight grading
- Hopf algebra w/ Δ

$\mathcal{H}$ = motivic MZV's have rigid structure!

Goal: understand MZV's mod products (& $\zeta(2)$)

$\rightsquigarrow$ $\mathcal{L}$ = Lie coalgebra of MZV's mod products.

Graded dual of $\mathcal{L}$ is "motivic Lie algebra"

Motivic theory $\mathfrak{g}^m \subseteq \mathbb{Q}\langle e_0, e_1\rangle$ w/ $\{\cdot,\cdot\}$ Ihara bracket

Fact: $\mathfrak{g}^m \cong \mathrm{Lie}[\sigma_3, \sigma_5, \sigma_7, \dots]$ is free on single odd wt generators (non-canonical)

cf Deligne.

$$\sigma_5 = e_1 e_0^4 + \frac{9}{2} e_1 e_0^2 e_1 e_0 + \dots$$

$\rightsquigarrow$ relation $\zeta^m(3,2) - \frac{9}{2}\zeta^m(5) = 0$ (mod prod)

via pairing $\langle I^m(0; i_1 \cdots i_m; 1), e_{j_1} \cdots e_{j_n}\rangle = \delta_{m=n} \prod \delta_{i_k = j_k}$

Define depth filtration & block filtration

$$D^n(\mathbb{Q}\langle e_0, e_1\rangle) = \{w \mid \#e_1\text{'s} \geq n\}$$
$$B^n(\mathbb{Q}\langle e_0, e_1\rangle) = \{w \mid \#e_1e_1 + \#e_0e_0 \geq n\}$$

$\hookrightarrow$ Keilthy Thesis (following Brown following idea in my thesis, relations)

↳ Blocks : $\underbrace{e_1 e_0 e_1 e_0}_{4} \mid \underbrace{e_0 e_1 e_0 e_1 e_0 e_1}_{5} \mid \underbrace{e_1 e_0 e_1}_{3}$

$\rightsquigarrow$ $(4,5,3)$ block decomposition
w/ 3 blocks , block degree 2.

Depth graded : $\mathfrak{dg} = \bigoplus \mathfrak{D}^n \mathfrak{g} / \mathfrak{D}^{n+1} \mathfrak{g}$

= "rels on MZV's mod products and lower depth"

$\boxed{\text{Problem}}$: Relations in $\mathfrak{dg}$, so not free.

Ihara-Takao (Schneps) : $\{\sigma_5, \sigma_7\} - 3\{\sigma_3, \sigma_9\} = 0$ (mod dp 3 4)

$\rightsquigarrow$ $14\,\zeta(3,9) + 75\,\zeta(5,7) + 84\,\zeta(7,5) = 0$ (mod dp 1)

$\Leftrightarrow$ period polynomial relations from $SL_2(\mathbb{Z})$
cusp form Δ of wt 12.
(Gangl-Kaneko-Zagier)

So need dp 4 generator in $\mathfrak{dg}$.

[Fix:] Introduce $\mathfrak{bg} = \bigoplus \mathfrak{B}^n \mathfrak{g} / \mathfrak{B}^{n+1} \mathfrak{g}$ via $\mathfrak{g} \hookrightarrow e_0 \mathbb{Q}\langle e_0, e_1 \rangle e_1$

= "rels on MZV's mod products and lower block degree".

Thm (Keilthy) $\mathfrak{bg} \cong \mathfrak{g}^m$ so is free.

Pf : Via dimension counting ; block filtration

= level filtration (#3's in Hoffman basis). Algebra basis for $\mathcal{H}$ = Hoffman–Lyndon words.]

Then $\boxed{}$ has terms of depth ≥ 3, but block degree 2, so $\neq 0$ RHS.

§ Block relations

Encode block degree $n-1$ words via monomials in $\mathbb{Q}[x_1,\dots,x_n]$

$$\underbrace{e_0e_1e_0e_1}_{b_1=4} \,|\, \underbrace{e_1e_0e_1}_{b_2=3} \,|\, \underbrace{e_1e_0e_1}_{b_3=3} \,|\, \underbrace{e_1}_{b_4=1} \to x_1^{b_1}\cdots x_n^{b_n}$$

Image is in $x_1\cdots x_n\,\mathbb{Q}[x_1,\dots,x_n]$ as $b_i \geq 1$

One has: $$\mathrm{bg}_n \hookrightarrow \underbrace{x_1\cdots x_n}_{b_i\geq 1}\,\underbrace{(x_1-x_n)}_{\substack{\text{recursive}\\\text{computation}\\\text{via } \{\cdot,\cdot\}}}\,\underbrace{\mathbb{Q}[x_1,\dots,x_n]}_{\substack{\text{Same parity}\\\text{condition.}}}$$

The part in $\mathbb{Q}[x_1,\dots,x_n]$ is (reduced) $\mathrm{bg} = \mathrm{rbg}$

Thm (Keilthy — Block relations)

For every $r \in \mathrm{rbg}_n$, $\sigma \in \mathrm{Dih}_n$

i) $r(x_{\sigma(1)}\cdots x_{\sigma(n)}) = (\mathrm{sgn}\,\sigma)^n\, r(x_1\cdots x_n)$

ii) $\prod\left(\frac{\partial}{\partial x_1} \pm \cdots \pm \frac{\partial}{\partial x_n}\right) r = 0$

iii) Regularisation condition.

For $n=1$ $\rightsquigarrow$ complete description.
For $n=2$ $\rightsquigarrow$ need something easier (explicit description)

(Relation 0) $\qquad f(\lambda x_1, \lambda x_2, \lambda x_3) = \lambda^{2n} f(x_1, x_2, x_3)$ for all $\lambda \in \mathbb{Q}$,

(Relation 1) $\qquad f(x_1, x_2, x_3) = f(x_2, x_3, x_1) = -f(x_3, x_2, x_1)$,

(Relation 2) $\qquad \frac{1}{2}(f(0,y,z) - f(0,y,-z)) = f(-y,y,z) - f(y,-z,z)$,

(Relation 3) $\qquad \frac{\partial^4 f}{\partial x_1^4} + \frac{\partial^4 f}{\partial x_2^4} + \frac{\partial^4 f}{\partial x_3^4} - 2\frac{\partial^4 f}{\partial x_1^2 \partial x_2^2} - 2\frac{\partial^4 f}{\partial x_2^2 \partial x_3^2} - 2\frac{\partial^4 f}{\partial x_3^2 \partial x_1^2} = 0$.

Some Diff rel

§ Block degree 1

Project σ_{2k+1} to block degree 1; we need to understand

$$e_0 (e_1 e_0)^{i-1} e_1 \boxed{e_0 | e_0} (e_1 e_0)^{k-i} e_1 \qquad \rightsquigarrow \quad \zeta \to 1,2 \text{ dual}$$

$\boxed{e_1 e_0}$

$$\longleftrightarrow \quad \zeta^m(\{2\}^{i-1} 3 \{2\}^{k-i}) \equiv c_i \zeta(2k+1) \quad (\text{mod prod})$$

$$\text{w/} \quad c_i = 2(-1)^k \left[\binom{2k}{2i} - (1-2^{-2k})\binom{2k}{2k-1-2i}\right]$$

(Zagier)

Then c_i determines parity behaviour (as $\zeta(2k+1)$ appears) only 1 element

$$\rightsquigarrow \quad \sum_i c_i \; \boxed{e_0} (e_1 e_0)^{i-1} \,\vdots\, e_0 (e_1 e_0)^{k-i} \boxed{e_1} + \text{duality}$$

$$\rightsquigarrow \quad \sum_i c_i \, x_1^{2i-1} x_2^{2k+2-2i} + \text{duality}$$

$$= x_1 x_2 (x_1 - x_2) \underbrace{\left[\frac{(1-2^{2k+1})(x_1+x_2)^{2k} - (x_1-x_2)^{2k}}{2^{2k}} \right]}$$

[$r(x_1, x_2)$ satisfies i) – iii)]

§ Block degree 2

Need to understand

$$\zeta(\{2\}^a 4 \{2\}^b) \leftrightarrow e_0 (e_1 e_0)^a e_1 e_0 e_0 e_0 (e_1 e_0)^b e_1$$

$$\zeta(\{2\}^a 3 \{2\}^b 3 \{2\}^c) \leftrightarrow e_0 (e_1 e_0)^a e_1 e_0 e_0 (e_1 e_0)^b e_1 e_0 e_0 (e_1 e_0)^c e_1$$

Expectation: depth filtration $\subseteq$ block filtration
so [$D_2 \subset B_1$, then induct w/ Δ]

$$\zeta(a,b) = \sum{}' \zeta(2 \cdots 2 4 2 \cdots 2) + \sum{}' \zeta(2\cancel{3 2 3}2)$$

Thm (C-Keilthy) Mod products

$$\zeta(\{2\}^a 4 \{2\}^b) = 4(-1)^n \Big[-\zeta(2a+2, 2b+2) - \zeta(2a+3, 2b+1) + \sum_{j=1}^{2n+3} 2^{j-4-2n} \left(\binom{2n+3-j}{2b+1} - \binom{2n+3-j}{2a+1} \right) \zeta(j, 2n+4-j) \Big]$$

$$\zeta^{\mathfrak{l}}(\{2\}^a, 4, \{2\}^b) = 4(-1)^n \Big[-\zeta^{\mathfrak{l}}(2a+2, 2b+2) - \zeta^{\mathfrak{l}}(2a+3, 2b+1) + \sum_{j=1}^{2n+3} 2^{j-4-2n} \left(\binom{2n+3-j}{2b+1} - \binom{2n+3-j}{2a+1} \right) \zeta^{\mathfrak{l}}(j, 2n+4-j) \Big].$$

Proof: Motivic version ΔLHS $= \Delta$RHS, and $\zeta(\text{ev}) = 0$ mod products.

[Full identity is ugly, but was necessary step. Analytic proof ask me later.]

Cor: $\zeta(2a+1, 2n-2a+3) = \frac{(-1)^{a}}{4} \sum_{i=a}^{n-a} \zeta(\{2\}^{i} 4 \{2\}^{n-i})$

Pf: Antisymmetry of binomials & $\zeta(2a+2, 2b+2)$ in Thm.

Assemble ingredients:

i) $\mathrm{rhg}_2 \hookrightarrow \mathbb{Q}[x_1, x_2 x_3]$ satisfying dihedral sym, ∂E, $-$ neg

ii) block degree 2 relations via pairing

$gr_2^B \mathcal{L} \otimes \mathrm{rhg}_2$ $\quad [\; x_1 x_2 x_3 (x_1 - x_3)$

$I^m(0; w; 1) \otimes x_1^i x_2^j x_3^k$

$\longmapsto \delta_{\underline{L}} = (i+2, j+1, k+1) - \delta_{\underline{L}} = (i+1, j+1, k+2)$

w/ $\underline{L}$ = block decomposition (b_1, b_2, b_3) of ∂w,

iii) By Cerality, $F := \langle \zeta^m(2a+1, 2b+3), \cdot \rangle : \mathrm{rhg}_2 \to \mathbb{Q}$

is $F(x_1^i x_2^j x_3^k) = \frac{1}{4} \underbrace{\delta_{(i,j,k)=(2b+2, 0, 2a)}}_{\text{telescoping}}$

So factors through totally even part of rhg_2,

iv) Rep theory: $P_e: \mathrm{rhg}_2 \to \mathrm{rhg}_2$ "totally ev proj"

$\rightsquigarrow \dim \operatorname{Im} P_e = \lfloor \frac{n}{3} \rfloor$,

$$\dim \ker P_e = \left\lfloor \frac{n-1}{2} \right\rfloor$$

$$= \underbrace{\dim \langle \{\sigma_k, \sigma_l\} \mid k+l = 2n+2 \rangle}_{=: \mathfrak{g}^m_{2,2n+2}}$$

[35] L. Schneps. On the poisson bracket on the free lie algebra in two generators. *J. Lie Theory*, 16(1), 2006.

Corollary 4.2

It is known: wt $2n+2$ dbl MZV's mod products have

$$\mathrm{gr}^D_2\, \mathfrak{g}^m_{2,2n+2} = \left\lfloor \frac{n-1}{2} \right\rfloor - \underbrace{\dim S_{2n+2}}_{\text{cusp forms}}$$

$$= \left\lfloor \frac{n}{3} \right\rfloor$$ ⟵ case by case

Conclusion:

Every elt of $\mathrm{Im}\, P_e$ induces dbl MZV relation, here from block relations.

The period polynomial relations for S_{2n+2} define dbl MZV relations.

Since $\dim \mathrm{gr}^D_2\, \mathfrak{g}_{2,2n+2} = \dim \mathrm{Im}\, P_e$, the block relations imply the period polynomial relations.

Extra:

Explicit connection:

Write ε_{2k+1} = image of σ_{2k+1} in

rhg_1: projected to $\mathbb{Q}[x_1^2, x_2^2]$

Then

$$\ker(\{\cdot,\cdot\} : \Lambda^2 \oplus \mathbb{Q}\,\varepsilon_{2k+1} \to P_e \mathrm{rgb}_2)$$

is isomorphic to the space of period polynomials

$$\Gamma \quad \begin{aligned} &f(x,0) = f(0,y) = 0, \\ &f(x,y) + f(y,x) = 0, \\ &f(x,y) + f(x,x+y) + f(x+y,y) = 0 \end{aligned}$$

relations amongst modular periods.

Such f can construct a dbl MZV relation explicitly (Gangl–Kaneko–Zagier)

Upshot: We have a model of $\mathfrak{dg}$ in depth 1 & 2 by projecting to totally even part of rhg.

Doesn't extend to depth 4, as we still need the extra generator.

Observation:

$\zeta(2\cdots 2\,3\,2\cdots 2\,3\,2\cdots 2)$ connects only to the odd part (is depth 4)

$\zeta(2\cdots 2\,4\,2\cdots 2)$ connects to both odd & even

Our $\zeta(2\cdots 2\,4\,2\cdots 2)$ combination only to the even part

We expect $\sum * \zeta(2\cdots 2\,6\,2\cdots 2)$ to connect to the even part.

Proof (Analytic) write $\zeta(\{2\}^a 4 \{2\}^b)$

1) via $\zeta^{\star}(\{2\}^b 4 \{2\}^a)$, 2) apply 2-1 Thm (Zhao)

$<\ \rightsquigarrow\ \leq$

$\sim \zeta^{\frac{1}{2}}(\underbrace{2b+1,1}_{\text{weight w/ } \frac{1}{2} \text{ when } n_i = n_{i+1}}, \underbrace{\overline{2a+2}}_{(-1)^{n_3}})$

3) Apply depth parity $\rightsquigarrow \sim \sum \zeta(\overline{2k}, \overline{2l})$ (using Glanois dihedral symmetry)

4) Use Glanois dihedral sym + generalized doubling identity

to solve $\zeta(\overline{2k}, \overline{2l}) \pm \zeta_{2k-1}(\overline{2l}, 1) =$ depth ≤ 2 classical MZVs

$\boxed{\text{Galois descent}}$: $\zeta(\overline{2k}, \overline{2l}) =$ classical dbl MZVs