Introduction to MZV's

§ Definitions / motivations

Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \operatorname{Re} s > 1$$

Meromorphic continuation to $\mathbb{C}$, w/ pole at $s=1$.

Number theoretic interest:

- connects to distribution of primes via non-trivial zeros in $0 < \operatorname{Re} s < 1$.
- Generalises to Dedekind zeta $\zeta_K(s)$, encodes information about a number field $[K:\mathbb{Q}] < \infty$. (in residue at $s=1$!)

Special values: "at integers":

- $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$, ...

  Euler

$$\zeta(2n) = \frac{(-1)^{1+n} B_{2n} (2\pi)^{2n}}{2(2n)!}$$

Qn: What about $\zeta(3)$, $\underbrace{\zeta(5), \ldots}_{\infty\text{-many irrational} \quad \text{Ball-Rivoal}}$?

A: Apéry 1978: $\notin \mathbb{Q}$

(At least) One of $\zeta(5), \zeta(7), \zeta(9), \zeta(11) \notin \mathbb{Q}$

Zudilin.

Expect: π, $\zeta(3), \zeta(5), \ldots$ algebraically independent
[currently hopeless]

Introduce multiple zeta values (i.e. more arguments)

Consider

$$\zeta(a)\cdot\zeta(b) = \sum_{n,m=1}^{\infty} \frac{1}{n^a m^b}$$

$$= \Big(\underbrace{\sum_{n>m}}_{\substack{??\\ \rightsquigarrow\ \zeta(a,b)}} + \underbrace{\sum_{n=m}}_{\zeta(a+b)} + \underbrace{\sum_{n<m}}_{\substack{??\\ \zeta(b,a)}} \Big) \frac{1}{n^a m^b}$$

Generally: for integers $k_1, \ldots k_d \geq 1$, define

$$\underbrace{\zeta(k_1, \ldots, k_d)}_{\text{Numerics w/ a}} = \sum_{n_1 < \cdots < n_d} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}} \quad \text{w/ } k_d \geq 2$$

apl pkg (w) $\sum_{n_1 > \cdots > n_d}$ $\Rightarrow$ reverse arguments] [Re $k_d > 1$]

— Now have an algebra (product) structure.
$\rightsquigarrow$ can use this to understand single zetas?

Define weight: $k_1 + \cdots + k_d$, depth d.

Important properties of MZV's

"measure of complexity".

Surprising amount of structure:

weight	max # generators
0	1: $\zeta(\emptyset) = 1$
1	0: $\emptyset$
2	1: $\zeta(2)$
3	1: $\zeta(3)$ as $\zeta(1,2) = \zeta(3)$
4	1: $\zeta(4)$ (similar)
5	2: $\zeta(5)$, $\zeta(2)\zeta(3)$
⋮ ...	6: 2, 7: 3, 8: 4, 9: 5, 10: 7, 9, 12, 16, ...
12 (expect 2^{10} elts)	12;

$\lceil d_k = d_{k-2} + d_{k-3} \rfloor$

Notable observations:

up to weight 7 only products of $R\,\zeta_2\,\zeta_3$

$$\zeta(3,2) = \boxed{-\frac{9}{2}}\zeta(5) + \boxed{\frac{11}{2}}\zeta(3)\zeta(2)$$

(why so complicated.

Also wt 9. But in weight 8, see $\zeta(3,5)$ apparently irreducible.
In weight 10, $\zeta(3,7)$, in wt 11 even $\zeta(3,3,5)$

Partial explanation: for all relations (Kontsevich)

$$\zeta(k_1,\dots,k_d) = (-1)^d\, I(0; 1\{0\}^{k_1-1} 1\{0\}^{k_2-1} \cdots 1\{0\}^{k_d-1}; 1)$$

w/ $$\underset{(\gamma)}{I}(a; x_1,\dots,x_n; b) = \int_{a\le t_1<\dots<t_n\le b} \frac{dt_1}{t_1-x_1}\wedge\cdots\wedge\frac{dt_n}{t_n-x_n}$$

by geometric expansion.

Iterated integral (along path γ, for MZV's straight line $0\to 1$)

Multiplication of integrals is different to series product.

Eg:

$$\begin{array}{ccccc} a & b & \Sigma & ab+ba & a+b \end{array}$$

$$\zeta(2)\zeta(2) = 2\zeta(2,2) + \zeta(4)$$

$10 ш 10$

$$= 1010 + 1100 + 1100 + 1100 + 1100 + 1010$$

$$= 2\zeta(2,2) + 4\zeta(1,3)$$

$$\Rightarrow \zeta(1,3) = \frac{1}{4}\zeta(4) \quad !!$$

Conj: Shuffle comparison of Σ and $\int$ product gives all MZV relations

[Caveat: we must allow $\zeta(1)$ as a 'formal' symbol.

$$\zeta(2)\zeta(1) \overset{\Sigma}{=} \cancel{\zeta(2,1)} + \zeta(1,2) + \zeta(3)$$

$10^{\vee}\boxed{1}$
$= 10(1)$
$+ 1\boxed{1}0 + (1)10$

$$\overset{\int}{=} \cancel{\zeta(2,1)} + 2\zeta(1,2)$$

$$\Rightarrow \zeta(1,2) = \zeta(3).$$]

§ Period polynomial relations.

First guess for a basis (generating set!) is maybe

$$\{\zeta(\overset{>1}{\text{odd}},\dots,\overset{>1}{\text{odd}})\,\zeta(2)^n\}$$

Can check # elements in wt k is d_k. ✓

works in wt 8: $\cancel{\zeta(1,7)}$, $\zeta(3,5)$, $\zeta(5,3)$, $\zeta(2)\zeta(3,3)$, $\zeta(2)^4$.

and wt 10: $\cancel{\zeta(1,9)}$, $\zeta(3,7)$, $\zeta(5,5)$, $\zeta(7,3)$, $\zeta(2)^5$
$+\zeta(2)\zeta(3,5) + \zeta(2)\zeta(5,3)$
$+\zeta(2)^2\zeta(3,3)$.

But in weight 12, a surprise:

$$\boxed{14\zeta(3,9) + 75\zeta(5,7) + 84\zeta(7,5) = \frac{5197}{2\cdot 691}\zeta(12)}$$

Where does this come from??

$\boxed{\text{Modular forms}}$

"Big picture explanation": (Gangl-Kaneko-Zagier)

One can define a double Eisenstein series

$$G_{r,s}(\tau) = \sum_{\substack{0\prec m\prec n\\ m,n\in\mathbb{Z}+\tau\mathbb{Z}}} \frac{1}{m^r n^s}$$

w/ $0\prec n$ meaning $n = x+\tau y$ w/ $x>0$ or $x=0$ & $y>0$
and $m\prec n \iff 0\prec n-m$.

[Generalises $G_r(\tau) = \sum_{m\in\mathbb{Z}+\tau\mathbb{Z}} \frac{1}{m^r}$ signed

Eisenstein series of even weight r, modular for $SL_2(\mathbb{Z})$, i.e.

$$G_r\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^r G_r(\tau)$$

Fact: Constant Fourier coefficient of $G_r(\tau)$ is $\zeta(r)$. ($\times \mathbb{Q}$)

Fact: Constant Fourier coefficient of $G_{r,s}(\tau)$ is $\zeta(r,s)$ [r,s odd] ($\times \mathbb{Q}$)

Fact: $G_{r,s}(\tau)$ spans the space of modular forms.

So, find a cusp form (vanishing constant coefficient), get a relation between odd zeta values. Eg:

$$\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n) \cdot (2\pi)^{12}.$$

This gives

$$\boxed{14\,\zeta(3,9) + 75\,\zeta(5,7) + 84\,\zeta(7,5) = \frac{5197}{2\cdot 691}\zeta(12)}$$

above.

Remark: There is procedure to construct this from the period polynomial of the cusp form.

$$p_f(X,Y) = \int_0^{i\infty} f(t)\, Y^{2k-2}\left(\frac{X}{Y}-t\right)^{2k-2} dt$$

$$= \sum_{n=0}^{w} i^{1-n}\binom{w}{n} X^{2k-2-n}\, Y^{n}\, r_n(f)$$

w/ $r_n(f) = \int_0^{\infty} f(it)\, t^n\, dt$, $0 \le n \le 2k-2$, the nth period

↳ Upto single (transcendental) $X^{ev}Y^{ev}$ part $\in \mathbb{Q}[X,Y]$. also $X^{od}Y^{od}$ part.

Space of relations such is encoded by

$$\left\{ \begin{array}{l} p(x,0) = p(0,y) = 0, \\ p(x,y) + p(y,x) = 0, \\ p(x,y) + p(x,x+y) + p(x+y,y) = 0 \end{array} \right\}.$$

$\int_0^{i\infty} = -\int_{i\infty}^{0}$

$0 = \int_0^{i\infty} + \int_{i\infty}^{1} + \int_1^{0}$

Eg Δ has (even) period polynomial

$$p(x,y) = -x^8y^2 + 3x^6y^4 - 3x^4y^6 + x^2y^8$$

↳ construct coefficients chosen via: ...

Double shuffle, revisited

Double-zeta double shuffle relations are

$$Z(X+Y,Y)+Z(X+Y,X)-Z(X,Y)-Z(Y,X)-\zeta(k)\frac{X^{k-1}-Y^{k-1}}{X-Y}=0$$

Can construct general solutions (realisations, with $\zeta(k)\to 0$) to these relations via

$$Z(X,Y)=A(X,Y)-A(X,X-Y)+A(Y,Y-X)$$

for $A(X,Y)$ any polynomial even with respect to Y. Can use ideas like this to connect to period polynomials.

Theorem (Gangl-Kaneko-Zagier, 2006)

Let $f\in S_{w+2}(\mathrm{SL}_2(\mathbb{Z}))$, with (even part) of period polynomial $P_f(X,Y)$, and define a_r by

$$\sum_{r=0}^{w} a_r X^r Y^{w-r} = P_f(X+Y,X)\,.$$

Then

$$\sum_{\substack{r=0\\ \text{even}}}^{w-2} a_r r!(w-r)!\zeta(r+1,w+1-r)\in\mathbb{Q}\zeta(w+2)$$

Examples

Weight 12, have one degree 10 period polynomial

$$P_f(X,Y)=-X^8Y^2+3X^6Y^4-3X^4Y^6+X^2Y^8$$
$$P_f(X+Y,X)=\cancel{-8X^7Y^3}-28X^6Y^4-\cancel{38X^5Y^5}-25X^4Y^6-\cancel{8X^3Y^7}-X^2Y^8$$

Relation: $-28\cdot 4!6!\zeta(7,5)-25\cdot 4!6!\zeta(5,7)-2!8!\zeta(3,9)=-4!5!\cdot\frac{5197}{691}\zeta(12)$

↳ Modular form / period polynomial relations make it harder to understand the structure of MZV's.
Relations in depth 2 propagate / are reflected by corrections in depth 4.