The usefulness of two-one formulas

Is this where most MZV evaluations come from?

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Geometries and Special Functions

Multiple zeta values, and their relatives

Definition

For $k_1, \ldots, k_d \in \mathbb{Z}_{\geq 1}$, $k_d \geq 2$, define multiple zeta value (MZV)

$$\zeta(k_1, \ldots, k_d) = \sum_{0<n_1<\cdots<n_d} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}},$$

and multiple t value (MtV)

$$t(k_1, \ldots, k_d) = \sum_{0<n_1<\cdots<n_d} \frac{1}{(2n_1-1)^{k_1} \cdots (2n_d-1)^{k_d}}$$

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Definition (Interpolated versions, Yamamoto)

For $f = \zeta, t$, define

$$f^r(k_1, \ldots, k_d) = \sum_{\circ = `+' \text{ or } `,'} r^{\#+'\text{s}} f(k_1 \circ k_2 \circ \cdots \circ k_d)$$

So $\zeta^0 = \zeta$, $\zeta^1 = \zeta^\star$ (defined as sum with $\leq$), also interesting is $\zeta^{1/2}$.
All variants satisfy stuffle product formulas.

Multiple zeta values, and their relatives

Definition

For $k_1, \ldots, k_d \in \mathbb{Z}_{\geq 1}$, $k_d \geq 2$, define interpolated multiple zeta value (MZV)

$$\zeta^r(k_1, \ldots, k_d) = \sum_{0 < n_1 \leq \cdots \leq n_d} \frac{r^{\delta_{n_1 = n_2} + \delta_{n_2 = n_3} + \cdots + \delta_{n_{d-1} = n_d}}}{n_1^{k_1} \cdots n_d^{k_d}} \,,$$

and multiple t value (MtV)

$$t(k_1, \ldots, k_d) = \sum_{0 < n_1 < \cdots < n_d} \frac{1}{(2n_1 - 1)^{k_1} \cdots (2n_d - 1)^{k_d}}$$

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So $\zeta^0 = \zeta$, $\zeta^1 = \zeta^\star$ (defined as sum with $\leq$), also interesting is $\zeta^{1/2}$.
All variants satisfy stuffle product formulas.

Many special evaluations ...

Theorem (Zagier, 2012)

$$\zeta(\overbrace{2,\ldots,2}^{a},3,\overbrace{2,\ldots,2}^{b}) = 2\sum_{r=1}^{a+b+1}(-1)^r\left[\binom{2r}{2a+2} - (1-2^{-2r})\binom{2r}{2b+1}\right]\zeta(2r+1)\frac{\pi^{2a+2b+2-2r}}{(2a+2b+2-2r+1)!}$$

Proof Idea: Generating series of LHS is hypergeometric ${}_3F_2'$, generating series of RHS is digammas $\psi(z) \coloneqq \frac{\mathrm{d}}{\mathrm{d}z}\log\Gamma(z)$. Equality by complex analysis wizardry.

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Guided the way for similar results for MtV's

Theorems (Murakami, 2020/21 & C, late 2021)

Explicit formulas

Murakami: $t(\overbrace{2,\dots,2}^{a},3,\overbrace{2,\dots,2}^{b}) \in \mathbb{Q}[\pi^2,\zeta(3),\zeta(5),\dots]$

C: $t(\underbrace{2,\dots,2}_{a},1,\underbrace{2,\dots,2}_{b}) \in \mathbb{Q}[\log(2),\pi^2,\zeta(3),\zeta(5),\dots]$

. . . with important motivic applications

Theorem (Brown, 2012)

A basis for motivic MZV's is given by $\zeta^{\mathfrak{m}}(k_1, \ldots, k_d)$, $k_i \in \{2, 3\}$.

Theorem (Murakami, 2020/21)

A basis for motivic MZV's (yes, MZV's) is given by $t^{\mathfrak{m}}(k_1, \ldots, k_d)$, $k_i \in \{2, 3\}$.

Theorem (C, 2021)

A basis for motivic MtV's is given by $t^{\mathfrak{m}}(k_1, \ldots, k_d)$, $k_i \in \{1, 2\}$, *moreover the convergent MtV's* $t^{\mathfrak{m}}(k_1, \ldots, k_{d-1}, k_d + 1)$, $k_i \in \{1, 2\}$ *are linearly independent.*

Proof ideas: Very combinatorial, using motivic coaction and 2-adic properties of the coefficients of $\zeta(2, \ldots, 2, 3, 2, \ldots, 2)$, $t(2, \ldots, 2, 3, 2, \ldots, 2)$ or $t(2, \ldots, 2, 1, 2, \ldots, 2)$ evaluations.

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Theorem (Brown, 2012)

A basis for motivic MZV's is given by $\zeta^{\mathfrak{m}}(k_1, \ldots, k_d)$*,* $k_i \in \{2, 3\}$*.*

Theorem (Murakami, 2020/21)

A basis for motivic MZV's (yes, MZV's) is given by $t^{\mathfrak{m}}(k_1, \ldots, k_d)$*,* $k_i \in \{2, 3\}$*.*

Theorem (C, 2021)

A basis for motivic MtV's is given by $t^{\mathfrak{m}}(k_1, \ldots, k_d)$*,* $k_i \in \{1, 2\}$*, moreover the convergent MtV's* $t^{\mathfrak{m}}(k_1, \ldots, k_{d-1}, k_d + 1)$*,* $k_i \in \{1, 2\}$ *are linearly independent.*

Proof ideas: Very combinatorial, using motivic coaction and 2-adic properties of the coefficients of $\zeta(2, \ldots, 2, 3, 2, \ldots, 2)$, $t(2, \ldots, 2, 3, 2, \ldots, 2)$ or $t(2, \ldots, 2, 1, 2, \ldots, 2)$ evaluations.

Surprisingly non-trivial part: Proof of the initial ζ or t evaluations.

Zhao's generalised two-one theorem

Theorem (Zhao, 2016 (reformulated))

For any multiple zeta star value

$$\zeta^\star(k_1,\ldots,k_d) = (-1)^{\delta_{k_1\neq 1}} 2^b \cdot \zeta^{1/2}(\widetilde{\ell_1 - 2}, \widetilde{\ell_2}, \ldots, \widetilde{\ell_b}) \,,$$

where we define $(\ell_1,\ldots,\ell_b)$ *by decomposing the following into alternating words*

$$0; 10^{k_1-1} 10^{k_2-1} \cdots 10^{k_d-1}; 1 \quad \rightsquigarrow \quad \underbrace{0101\cdots}_{\ell_1} \mid \underbrace{1010\cdots}_{\ell_2} \mid \cdots \mid \underbrace{\cdots 0101}_{\ell_b}$$

Moreover $\widetilde{\ell_i} = \ell_i$ *if* ℓ_i *odd, and* $\widetilde{\ell_i} = \overline{\ell_i}$ *if* ℓ_i *even (*$\overline{\ell_i}$ *denotes factor* $\frac{(-1)^{n_i}}{n_i^{\ell_i}}$ *in the series).*

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$$(1,4,6) \rightsquigarrow 0;11000100000;1 \rightsquigarrow \overbrace{0;1}^{2} \mid \overbrace{10}^{2} \mid \overbrace{0}^{1} \mid \overbrace{010}^{3} \mid \overbrace{0}^{1} \mid \overbrace{0}^{1} \mid \overbrace{0}^{1} \mid \overbrace{0;1}^{2}$$

Hence

$$\zeta^\star(1,4,6) = 2^7 \zeta^{1/2}(\overline{2},1,3,1,1,1,\overline{2})$$

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Upshot:

$$\zeta(\overbrace{\{2\}^a}^{2,\ldots,2,\text{ with } a \text{ repeats}}, 3, \{2\}^b) = (-1)^{a+b} \zeta^\star(\{2\}^b, 3, \{2\}^a) + \text{products} \qquad \text{(stuffle antipode)}$$

$$\zeta^\star(\{2\}^b, 3, \{2\}^a) = -4\zeta^{1/2}(2b+1, \overline{2a+2}) = \text{depth } 1 \qquad \text{(two-one \& parity theorem)}$$

Directly obtain Zagier's evaluation.

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Directly obtain Zagier's evaluation.

Ohno & Zudilin investigated cases: $\zeta^\star(1, \{2\}^{a_1}, \cdots, 1, \{2\}^{a_n}) \leftrightarrow \zeta^{1/2}(2a_1 + 1, \ldots, 2a_n + 1)$

A two-one theorem for $t^\star \mapsto t^{1/2}$

Theorem (Li-Yan, 2022 (reformulated))

For any multiple t star value

$$t^\star(k_1,\ldots,k_d) = (-1)^{\delta_{k_1\neq 1}} 2^{b-1} \cdot t^{1/2}\big(\overset{(\overline{})}{\ell_1 - 1}, \overset{(-)}{\ell_2}, \ldots, \overset{(-)}{\ell_b}\big)\,,$$

where we define $(\ell_1,\ldots,\ell_b)$ by decomposing the following into alternating words

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Choice of signs $\overset{(-)}{\bullet}$, more complicated but direct. Note: $t(\overline{1}) = \frac{\pi}{4}$.

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Upshot:

$$t(\overbrace{\{2\}^a}^{2,\ldots,2, \text{ with } a \text{ repeats}}, c, \{2\}^b) = (-1)^{a+b} t^\star(\{2\}^b, c, \{2\}^a) + \text{products} \qquad \text{(stuffle antipode)}$$

$$\left.\begin{aligned} t^\star(\{2\}^b, 1, \{2\}^a) &= -\frac{8}{\pi} t^{1/2}(2b+1, \overline{2a+1}) = \text{depth } 1 \\ t^\star(\{2\}^b, 3, \{2\}^a) &= -\frac{8}{\pi} t^{1/2}(\overline{2b+2}, 2a+2) = \text{depth } 1 \end{aligned}\right\} \qquad \text{(two-one \& parity theorem)}$$

Directly obtain Murakami's evaluation and mine.

A two-one result for $\zeta^{1/2} \mapsto$ "$t^\star$"

Theorem-To-Be (In progress, with M. E. Hoffman)

For $(k_1, \ldots, k_d)$ an index (with $k_i \neq 1$) and associated $(\ell_1, \ldots, \ell_d)$ obtained by decomposing the following into alternating words

$$\zeta^{1/2}(\ell_1, \ldots, \ell_b) = 2^{2d-1} \sum_{i=1}^{d} \Big(t^\star(k_1, \ldots, k_i) - \frac{\zeta^\star(k_1, \ldots, k_i)}{2^{k_1 + \cdots + k_i}} \Big) t(k_d, \ldots, k_{i+1}) \,,$$

Some explicit modifications needed if (m)any $k_i = 1$.

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Some explicit modifications needed if (m)any $k_i = 1$.

Consequences:

- Explicit form of Murakami's MtV Galois descent, e.g.
  $t(3,9) = \frac{1}{8}\zeta^{1/2}(1,3,1,1,1,1,1,1,2) + \frac{1}{4096}\zeta^\star(3,9) + \frac{1533}{2048}\zeta(3)\zeta(9) - \frac{4095}{4096}\zeta(12)$ is level 1.

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Evaluations:

- $\zeta^{1/2}(1, \{\{1\}^{a-3}, 3\}^{b-1}, \{1\}^{a-3}, 2) \in \mathbb{Q}[\zeta(a), \zeta(2a), \zeta(3a), \ldots]$, here $\underline{\ell} \leftrightsquigarrow \underline{\mathbf{k}} = (\{a\}^b)$
- $\zeta^{1/2}(3, 1, \ldots, 1, 2) \in \mathbb{Q}[$depth 2 MZV's$]$, here $\underline{\ell} = (3, \{1\}^n, 2) \leftrightsquigarrow \underline{\mathbf{k}} = (2, n+3)$
- $\zeta^{1/2}(2, 1, \ldots, 1, 2) \in \mathbb{Q}[\pi^2, \zeta(3), \zeta(5), \ldots]$, here $\underline{\ell} = (2, \{1\}^n, 2) \leftrightsquigarrow \underline{\mathbf{k}} = (1, n+3)$

Questions

- What other sorts of two-one theorems exist?
  - Partial answer: iterated beta integrals (Hirose and Sato)
- Versions for truncated MZV's/MtV's/...?
- Versions for q-analogues?
  - Zhao gave a new proof of MZV version via a q-analogue.
- Some version for elliptic MZV's, or higher?
- Can all 'nice' MZV/MtV/... evaluations be understood this way?