Generators of MZV's & alternating MZV's

- Some results from Dec 2021 [2112.14613]
  ⇝ came together partly via many discussions w/ Mike Hoffman which started during his visit to MPI in 2020 (and the extremely warm stew which attended it!)

§ Definitions & Background

We define multiple zeta values (MZV) (alternating if $\varepsilon_i = \pm 1$, generally cyclotomic if ε_i root of unity) by

$$\zeta(k_1 \ldots k_d) = \sum_{0 < n_1 < \cdots < n_d} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}}$$

$k_d > 1$ for convergence

$$\zeta\binom{\varepsilon_1 \ldots \varepsilon_d}{k_1 \ldots k_d} = \sum_{0 < n_1 < \cdots < n_d} \frac{\varepsilon_1^{n_1} \cdots \varepsilon_d^{n_d}}{n_1^{k_1} \cdots n_d^{k_d}}$$

$(\varepsilon_d, k_d) \neq (1, 1)$ for convergence

If $\varepsilon_i \in \{\pm 1\}$, use shorthand
$\overline{k_i} \longleftrightarrow \varepsilon_i = -1$
$k_i \longleftrightarrow \varepsilon_i = +1$

[So $\zeta(1, \overline{3}) = \sum \frac{(-1)^{n_2}}{n_1 n_2^3}$]

Define multiple t value (MtV) by

$$t(k_1,\dots,k_d) = \sum_{\substack{0<n_1<\dots<n_d \\ n_j \text{ odd}}} \frac{1}{n_1^{k_1}\cdots n_d^{k_d}}$$

$k_d > 1$ for convergence.

[Can introduce alternating MtV's, not here]

For MtV's & MZV's define weight $= k_1 + \dots + k_d$, depth $= d$. (Useful measures of complexity.)

MtV's reintroduced ~ 2019 by Hoffman, after Nielsen ~ 1960 studied $t(n) =: t_n$ depth 1.

Note: $$t(k_1 \dots k_d) = \sum_{0<n_1<\dots<n_d} \frac{(1-(-1)^{n_1})\cdots(1-(-1)^{n_d})}{2^d\, n_1^{k_1}\cdots n_d^{k_d}}$$

$$= \sum_{\varepsilon_1 \dots \varepsilon_d \in \{\pm 1\}} \frac{\varepsilon_1 \cdots \varepsilon_d}{2^d}\, \zeta\begin{pmatrix} \varepsilon_1 & \cdots & \varepsilon_d \\ k_1 & \cdots & k_d \end{pmatrix}$$

So MtV's are a subspace/set of alternating MZV's. Proper? Sort of ...

$\zeta(\bar{1}) = -\log 2$, $t(1)$ undefined, so (modulo weight grading negative) MtV's $\subsetneq$ MZV's.

But one can regularise w/ $t(1) = \lambda \log 2 \sim$ = .

Why study?

— MZV's fit $\zeta(n)$ into shuffle algebra structure:

$$\underbrace{\zeta(a)}_{n_1}\underbrace{\zeta(b)}_{n_2} = \underbrace{\zeta(a+b)}_{n_1=n_2} + \underbrace{\zeta(a,b)}_{n_1<n_2} + \underbrace{\zeta(b,a)}_{n_1>n_2}.$$

+ Route to investigate $\zeta(5) \overset{?}{\in} \mathbb{Q}$.

+ Surprising amount of structure. shuffle structure via integral rep.

- $2^{12} = 4096$ MZV's in wt 14
  $\rightsquigarrow$ only 21 generators needed
- duality $\zeta(1,1,1,2) = \zeta(5)$
- relations in depth 2 from cusp forms.

— Applications to High Energy Physics (closed string) scattering amplitudes involve MZV's, and generalisations.)

— MtV's are a 'twisted variant', similar but different.

$\rightsquigarrow$ Compare & contrast structures

— Connections (via alternating polylogs, D. Broadhurst) to volumes of orthoschemes!

§ Structures, dimensions, bases

Computer experimentation (LLL then later proven identities) suggests (Conj, Zagier)

$$\dim(\text{weight } k \text{ MZV's}) = d_k$$

where $d_1 = 0$, $d_2 = d_3 = 1$, $d_k = d_{k-2} + d_{k-3}$

(so $d_{14} = 21$).

Tongue in cheek idea is

Conj (Hoffman).

$$\zeta(k_1, \ldots, k_d), \quad k_i \in \{2, 3\}$$

"ζ(2's and 3's)"

give a basis for MZV's.

Thm (Brown 2012).

$\zeta^{(m)}$(2's and 3's) span MZV's

are a basis for motivic MZV's

purely algebraic object $\boxed{\text{above}}$ MZV's.

Recently

Thm (Murakami 2020/21)

$t^{(m)}$(2's and 3's) span MZV's
basis for motivic MZV's,

$\hookrightarrow$ $$\zeta(5) = -\frac{286}{31} t(2,3) + \frac{192}{31} t(3,2)$$

In fact $t(\geq 1, \geq 1, \ldots, \geq 1)$ is a sum of MZV's. (Murakami)

$$t(2,3,4) = \frac{38669}{6144}\zeta(9) + \frac{21}{256}\zeta(7)\zeta(2) - \frac{135}{512}\zeta(5)\zeta(4) + \frac{7}{768}\zeta(3)^3 - \frac{9}{1120}\zeta(3)\zeta(6).$$

And MtV dimensions? We can allow $t(\ldots, 1)$ via regularisation (extend the shuffle product formally, with $t(1) := \lambda \log 2$, $\lambda \in \mathbb{Q}$)

dim (weight k convergent MtV's) $\overset{?}{=} \begin{cases} F_{k}, & k>1 \\ 0, & k=1 \end{cases}$

dim (weight k regularised MtV's) $\overset{?}{=} F_{k+1}$

where $F_1 = F_2 = 1$, $F_k = F_{k-1} + F_{k-2}$ are Fibonacci numbers

n :	1	2	3	4	5	6
F_n :	1	1	2	3	5	8

$\zeta(2) \leftarrow t(2)$

$\zeta(3) \leftarrow t(3)$

$\zeta(2)\log 2 \leftarrow t(1,2)$

$\log^2 2 \leftarrow t(1,1)$

$\log^3 2 \leftarrow t(1,1,1)$

$t(2,2) \to t(4) \to \zeta(4)$

$t(1,3) \to \zeta(\bar{3}) + \zeta(3)\log 2$

$t(1,1,2) \to \zeta(2)\log^2 2$

$t(1,1,1,1) \to \log^4 2$

$t(3,1) \to \zeta(\bar{3}) + \zeta(3)\log 2$
$\zeta(3)\log 2)$

Thm (C, 2021 arXiv — ≥ 2023 prob?)

i) t^{m}(1's and 2's, 2 or 3) are linearly independent.

[Saha conjectures a basis for convergent MtV's]

ii) $t^{(m)}$(1's and 2's) span / basis regularised MtV's, & alternating MZV's.

Application: w/ Hoffman have a symmetry theorem

$$t(k_1 \ldots k_d) \equiv (-1)^{k_1 + \cdots + k_d - 1} t(k_d \ldots k_1) \pmod{t \text{ prods}}$$

We showed (via explicit generating series)

$$t(k_1 \ldots k_d) + (-1)^{k_1 + \cdots + k_d} t(k_d \ldots k_1)$$

$$= t \times t + \zeta \times \zeta \times t$$

and wrote $\zeta = \sum t$ via Murakami.

Extension to alternating MtV's, and we write alt ζ's $= \sum t$ via my result.

§ Proof Ideas

In Brown's case, for simplicity.

$\boxed{\text{Ingredient I}}$ Identity for

$$\zeta(\underbrace{2 \ldots 2}_{a} 3 \underbrace{2 \ldots 2}_{b})$$

$$= 2 \sum_{r=1}^{a+b+1} (-1)^r \left(\binom{2r}{2a+2} + (1 - 2^{-2r}) \binom{2r}{2b+1} \right) \cdot \zeta(2r+1)\, \zeta(\underbrace{2 \ldots 2}_{a+b+1-r})$$

$$\zeta(\underbrace{2..2}_{k}) = \frac{\pi^{2k}}{(2k+1)!}$$

Proven by Zagier:

$$\sum_{a,b} \text{LHS}\, x^{2a} y^{2b} = {}_3F_2' \text{ hypergeometric derivative}$$

$$\sum_{a,b} \text{RHS}\, x^{2a} y^{2b} = \sum \text{sine/cosine} \times \text{digamma}$$

14 terms

$$\psi(x) = \frac{d}{dx} \log\Gamma(x)$$

Then i) explicity growth conditions

ii) Agreement for $(x,y) = (z,z)$,
$= (n, y)$
$\quad n \in \mathbb{Z}$
$= (x, k)$
$\quad n \in \mathbb{Z}$.

Complex analysis result $\Rightarrow$ agreement everywhere.

[Similar identity for $t(2..232..2)$, by Murakami, w/ similar proof.]

[Similar identity for $t(2..212..2)$, by C. Asymptotic properties of ${}_4F_3$ needed to deal with $t(2..21)$ divergent)

regularised terms. Reduces to Zagier's
${}_3F_2' = \sum \text{trig} \times \text{digamma}$ identity :)

$\boxed{\text{Ingredient 2}}$ $\boxed{\cancel{\text{Motivic framework}}}$

Something of a black box.
Properties more important than construction.

Philosophy: $\underset{\text{analytic}}{\zeta(k_1, \dots k_d)} \underset{\text{lifts}}{\rightsquigarrow} \underset{\text{algebraic}}{\zeta^m(k_1, \dots, k_d)}$

Roughly:

$$\zeta^m(k_1, \dots, k_d) \approx [\underbrace{H^n(X \setminus A)}_{\approx (\mathbb{P}^1 \setminus 0,1,\infty)^n}, \underbrace{\text{simplex } \Delta}_{t_1 < t_2 < \dots < t_k}, \underbrace{\text{form}}_{\frac{dt}{t-1}\left(\frac{dt}{t}\right)^{k_1-1} \dots \frac{dt}{t-1}\left(\frac{dt}{t}\right)^{k_d-1}}]^m$$

symmetries)...

Idea:

$$\zeta(k_1, \dots, k_d) = \int_{0 < t_1 < \dots < t_k < 1} \frac{dt_1}{t_1 - 1} \wedge \overbrace{\frac{dt_2}{t_2} \wedge \dots \wedge \frac{dt_{k_1}}{t_{k_1}}}^{k_1 - 1} \wedge \dots \wedge \frac{dt_*}{t_* - 1} \wedge \frac{dt_*}{t_*} \wedge \dots \wedge \frac{dt_*}{t_*}$$

$$=: \operatorname{per} \zeta^m(k_1, \dots, k_d)$$

by keeping track of form and simplex separately we avoid "accidental" identities.

Conjecture (Grothendieck) per is injective (in most general setting). All relations are motivic.

Relations on ζ^m arise geometrically

— change of variables (map on varieties)
— linearity
— Stokes theorem

...

Upshot: ζ^m is more rigid, forms a graded Hopf algebra comodule, graded by weight.

(So $\zeta^m(2), \zeta^m(3), \zeta^m(5), \ldots$ are irrational as $\mathbb{Q}$ has weight 0, also algebraically independent via given calculations...)

Exists coproduct / coaction.

$$\Delta \zeta^m = \sum \underbrace{\zeta^a}_{\substack{\text{mod } \zeta^m(2) \\ \text{or mod } (i\pi)^m}} \otimes \zeta^m$$

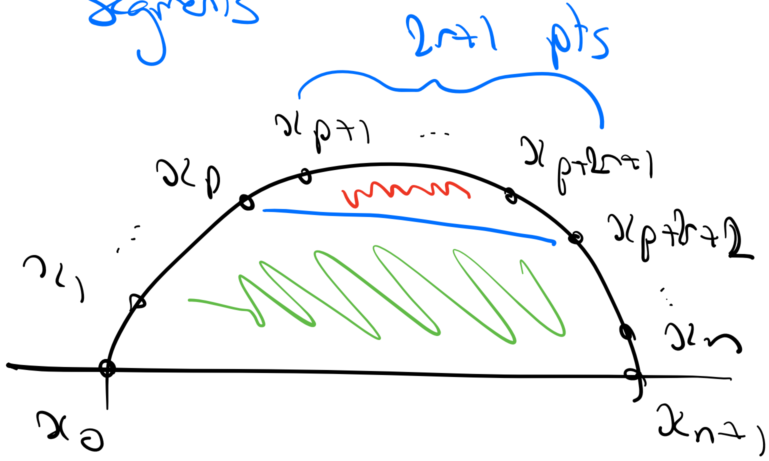
Simpler to consider linearised version

$$D_{2r+1} \zeta^m = \sum \underbrace{\zeta^L}_{\text{mod products}} \otimes \zeta^m$$

Computation mnemonic $\displaystyle\int_{x_0}^{x_{n+1}} \frac{dt_1}{t_1-x_1}\wedge\cdots\wedge\frac{dt_n}{t_n-x_n}$ motivic version.

$$D_{2r+1}\, I(x_0;\, x_1,\dots,x_n;\, x_{n+1})$$

$$= \sum_{\text{segments}} I^{L}(\quad)\otimes I^{m}(\quad)$$



Thm (Brown)

$$\ker \underbrace{D_3\oplus D_5\oplus\cdots\oplus D_N}_{D_{<N}} = \zeta^m(N)\,\mathbb{Q}.$$

So can recursively understand identities.

Brown's proof:

Show ζ^m(2's & 3's) linearly independent, as bound $\dim(\text{motivic MZVs}) \le d_k$ is known. (Deligne–Goncharov, Terasoma)

Filter ζ^m(2's & 3's) by level = #3's.

Show (after gr_L)

$gr_L D_{2r+1}\, \zeta^m(\text{level } L)$

$$= \sum \boxed{*}\, \zeta^L(2r+1) \otimes \zeta^m(\text{level } L-1)$$

↳ coeff of r = const. term in types 22322 eval

0 in $gr_L D_{2r+1}$

$$+ \cancel{\underbrace{\text{complicated}}_{\text{from } \zeta^L(2\cdots232\cdots232\cdots2)} \otimes \zeta^m(\text{level} \le L-2)}$$

Project $\zeta^L(2r+1)$ to 1, we can recover $gr_L D_{2r+1}$ by $\zeta^L(2r+1) \otimes \cdots$. Define

$$\partial_{N,L}\, \zeta^m(\text{level } L, \text{wt } N) = \sum \underbrace{\boxed{\;*\;}}_{\text{coefficient from } \zeta^L(2^\alpha 3 2^\beta)} \zeta^m(\text{level } L-1, \text{wt} < N)$$

Idea:
- Apply induction on level.
- $\partial_{N,L}$ would push level L relations down to level $L-1$. If $\partial_{N,L}$ injective, level $L-1$ relation is non-trivial.

- By induction assumption no relation in level $L-1$
- No relation in level 0, as
  $\zeta^m(2)$, $\zeta^m(2,2)$, $\zeta^m(2,2,2)$, ...
  are different weight.

Injectivity of $\partial_{N,L}$ relies on arithmetic properties of $\zeta(2..232...2)$ coefficients.

$\partial_{N=10,L=2}$:

	223	232	23	322	32	3
2233	3		$-7/2$			28
2323		3	$-11/2$			
2332			$9/2$			
3223			12	3		$-291/16$
3232					$9/2$	$75/8$
3322				-2	12	$-291/16$
			$\times 2$		$\times 2$	$\times 16$

integer

odd

even

So $\det = \dfrac{\text{odd}}{2^k} \neq 0$

Murakami filters $t^m(2\text{'s} \,\&\, 3\text{'s})$ by $\#\,3\text{'s}$
$\rightsquigarrow$ $t^m(\text{level}) = \sum \zeta^m(\text{level} \leq L)$
& vice versa.

C: i) filtration by # 1's + # 3's on t^m(1's & 2's, 2 or 3)

ii) filtration by # 1's on t^m(1's & 2's)

Maps $\partial^{*}_{N,L}$ injective, so get independence.

Rmk: For ii) we have bound dim (alt MZV's weight k) ≤ F_{k+1}
⇒ basis

For i) don't know bound yet dim (convergent MtV's wt k) ≤ F_k.

Discussions w/ M. Hirose & A. Keilthy suggest approaches, using $\mathcal{D}_j$ or $\Delta^{(1,n-1)}$ to motivically characterise convergent MtV's. (In progress.)