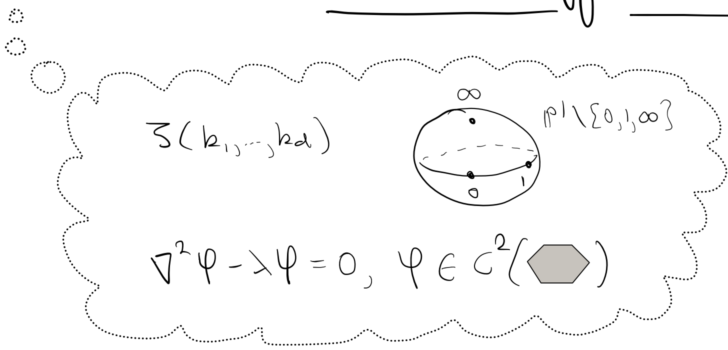
Multiple zeta values in number theory & (differential) geometry



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1st goal Some overview of multiple zeta values (and some generalisations), their theory, properties and structures.

2nd goal some appearances in differential geometry / analysis

- Dirichlet eigenvalues of regular polygons [Berghaus, Georgiev, Monien, Radchenko]
- Area expansion of families of CMC (constant mean curvature) surfaces [Heller², Traizet + Appendix by C]

§ Definitions / introduction

Starting point, the Riemann zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \operatorname{Re} s > 1.$$

Well-known importance in (analytic) number theory. Has analytic continuation to $\mathbb{C}$, meromorphic with simple pole at $s = 1$. Properties of $\zeta(s)$ control behaviour of primes.

Property $\zeta(1 + it) \neq 0 \implies \pi(N) \sim \dfrac{N}{\log N}, \quad N \to \infty$

[# {primes ≤ N}

"trivial zeros"]

Riemann Hypothesis : $\zeta(t) = 0 \overset{??}{\Rightarrow} \begin{cases} t = -2, -4, -6, \dots & \text{OR} \\ \text{Re}\, t = \frac{1}{2} \end{cases}$

"non-trivial zeros"

Result RH $\Rightarrow$ more precise formula for $\pi(N)$.

Other viewpoint on $\zeta(s)$, special values.

Basel problem Evaluate $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1.644934\dots$

Solution (Euler 1734) $\zeta(2) = \frac{\pi^2}{6}$

Euler gave formulas for $\zeta(4) = \frac{\pi^4}{90}$, $\zeta(6) = \frac{\pi^6}{945}$, and generally

$$\zeta(2k) = (-1)^{k+1} \frac{(2\pi)^{2k} B_{2k}}{2 \cdot (2k)!}$$

← Bernoulli number $\sum B_m \frac{t^m}{m!} := \frac{t}{e^t - 1}$

$\in \pi^{2k} \mathbb{Q}$ ← ie irrational, transcendental, ...

Further questions

- Evaluation of $\zeta(3) = 1.202056\dots$ Unknown
- $\zeta(3) \overset{?}{\in} \pi^3 \mathbb{Q}$?? Unknown
- $\zeta(2k+1)$ transcendental? Irrational? Unknown.

So far :

- $\zeta(3) \notin \mathbb{Q}$ Apéry 1978
- One of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ irrational Zudilin 2001

- Infinitely many $\zeta(2k+1)$ irrational — Ball–Rivoal 2000/2001.

Little else known...

Try to understand $\zeta(k)$ as part of a larger structure. More variables, more freedom, better properties?

Definition) For $k_1, \ldots, k_d \in \mathbb{Z}_{\geq 1}$ positive integers, multiple zeta value (MZV) is

$$\zeta(k_1, \ldots, k_d) := \sum_{0 < n_1 < \cdots < n_d} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}}$$

- For convergence $k_d \geq 2$.
- Weight is $w = k_1 + \cdots + k_d$
- Depth is d

} Useful measures of "complexity"

Computation & numerics)

- Some 'easy' algorithms to evaluate to high-precision via asymptotic expansion (eg, Zagier)

↳ implemented in gp/pari
↳ tangible values, and ways to experiment.

§ Properties of MZV's

Product structure

$$\zeta(a)\,\zeta(b) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{n^a m^b}$$
$$= \left(\sum_{n<m} + \sum_{n=m} + \sum_{n>m}\right) \frac{1}{n^a m^b}$$
$$= \zeta(a,b) + \zeta(a+b) + \zeta(b,a)$$

Leads already to some identities

$$\zeta(2,2) = \frac{1}{2}\left(\zeta(2)\cdot\zeta(2) - \zeta(4)\right)$$
$$= \frac{1}{2}\left(\frac{\pi^2}{6}\times\frac{\pi^2}{6} - \frac{\pi^4}{90}\right)$$
$$= \frac{\pi^4}{120}.$$

Product extends to all MZV's.

Numerical experiments suggest a strange "duality"

$$\zeta(3) \overset{?}{=} \zeta(1,2)$$
$$\zeta(4) = \zeta(1,1,2)$$
$$\zeta(1,1,3,5) \overset{?}{=} \zeta(1,1,1,2,1,4)$$

How to prove and understand?

Explained by an integral representation:

Idea:

$$\sum_{n=1}^{\infty} \frac{t_2^n}{n} = -\log(1-t_2) = -\int_0^{t_2} \frac{dt_1}{t_1-1}$$

Then apply

$$\int_0^{t_3} \bullet \frac{dt_2}{t_2} \rightsquigarrow \sum_{n=1}^{\infty} \frac{t_3^n}{n^2} = -\int_0^{t_3}\int_0^{t_2} \frac{dt_1}{t_1-1}\frac{dt_2}{t_2}$$

$\vdots$

$$\int_0^{t_{a+1}} \bullet \frac{dt_a}{t_a} \rightsquigarrow \sum_{n=1}^{\infty} \frac{t_{a+1}^n}{n^a} = -\int_{0<t_1<\cdots<t_a<t_{a+1}} \frac{dt_1}{t_1-1}\underbrace{\frac{dt_2}{t_2}\cdots\frac{dt_a}{t_a}}_{a-1 \text{ many}}$$

$$\rightsquigarrow \quad t_{a+1}=1 \text{ gives } \zeta(a) = -\int_{0<t_1<\cdots<t_a<1} \frac{dt_1}{t_1-1}\underbrace{\frac{dt_2}{t_2}\cdots\frac{dt_a}{t_a}}_{a-1 \text{ many}}.$$

Instead apply

$$-\int_0^{t_{a+2}} \bullet \frac{dt_{a+1}}{t_{a+1}-1} = \int_0^{t_{a+2}} \sum_{m=1}^{\infty} t_{a+1}^{m-1} \bullet$$

$$\rightsquigarrow \sum_{n=1}^{\infty} \frac{1}{n^a} \sum_{m=1}^{\infty} \frac{t_{a+1}^{n+m}}{(n+m)} = \int_{0<t_1<\cdots<t_a<t_{a+1}<t_{a+2}} \frac{dt_1}{t_1-1}\overbrace{\frac{dt_2}{t_2}\cdots\frac{dt_a}{t_a}}^{a-1}\frac{dt_{a+1}}{t_{a+1}-1}$$

$n+m>n$, so reindex via $m \mapsto m-n$.

get $\sum_{1\leq n<m} \bullet$

And so on...

Generally obtain:

$$\sum_{n_1<n_2<\dots<n_d} \frac{1}{n_1^{k_1}\cdots n_d^{k_d}} = \int_{0<t_1<\dots<t_w<1} \frac{dt_1}{t_1-a_1} \cdots \frac{dt_w}{t_w-a_w}$$

where $w = k_1+\dots+k_d$, and

$$(a_1,\dots,a_w) = (1,\underbrace{0,\dots,0}_{k_1-1},1,\underbrace{0,\dots,0}_{k_2-1},\dots,1,\underbrace{0,\dots,0}_{k_d-1})$$

Duality is now the obvious $t_i \mapsto 1-t_i$ substitution.

<u>Remark</u> Integral representation shows MZV's are so-called <u>periods</u>. A period is a complex number with real / imaginary parts of the form

$$\int_{\text{polynomial inequalities}} \text{rational function w/ } \mathbb{Q}\text{-coefficients}$$

⌜ Name <u>period</u> comes from periods of elliptic curves, which themselves are periods of (Weierstrass $\wp$) elliptic functions (which are connected with the actual <u>period</u> of a pendulum!) ⌟

Periods are numbers of an (algebraic-) geometric origin. For (smooth) variety X/k there is a canonical isomorphism

Comp : $\underbrace{H^i_{dR}(X)}_{\text{algebraic de Rham cohomology}} \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} \underbrace{H^i_B(X)}_{\text{Betti / singular cohomology.}} \otimes_{\mathbb{Q}} \mathbb{C}$

Induced by a pairing (for X analytic)

$$H^i_{dR}(X) \otimes H^B_i(X) \longrightarrow \mathbb{C}$$
$$\omega \otimes \sigma \longmapsto \int_\sigma \omega$$

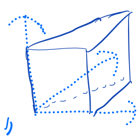
[Idea is to relate algebraic de Rham cohomology to analytic de Rham cohomology of cx manifold X^{an}.]

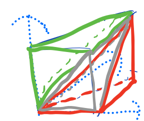
Overall: periods and relations amongst them are supposed to be <u>geometric</u> in origin.

Back to MZV's. <u>Different</u> product via integrals

$$\int_{0<t<1} \omega(t) \int_{0<s_1<s_2<1} \eta_1(s_1)\eta_2(s_2)$$

$I \times \Delta_2 = \Delta_3 \cup \Delta_3' \cup \Delta_3''$





$$= \int_{\substack{0<t<s_1<s_2<1 \\ \text{or } 0<s_1<t<s_2<1 \\ \text{or } 0<s_1<s_2<t<1}} \omega(t)\eta_1(s_1)\eta_2(s_2)$$

$$= \int_{0<u_1<u_2<u_3<1} \{\omega(u_1)\eta_1(u_2)\eta_2(u_3) + \eta_1(u_1)\omega(u_2)\eta_2(u_3) + \eta_1(u_1)\eta_2(u_2)\omega(u_3)\}$$

Denote by $\omega_1 \dots \omega_n ⧢ \omega_{n+1} \dots \omega_{n+m}$ the shuffle (riffle shuffle) product.
All permutations where $\omega_1 \dots \omega_n$ correctly ordered and $\omega_{n+1} \dots \omega_{n+m}$ correct

More MZV identities

$$\zeta(2)\zeta(2) = \int_{0<t_1<t_2<1} \frac{dt_1}{t_1-1}\frac{dt_2}{t_2} \sqcup\!\sqcup \int_{0<s_1<s_2<1} \frac{ds_1}{s_1-1}\frac{ds_2}{s_2}$$

$$= \dots$$

$$= 4\cdot\zeta(1,3) + 2\cdot\zeta(2,2)$$

1100 → 4   1010 → 2

$$\binom{2+2}{2} = 6$$

$$\Rightarrow \zeta(1,3) = \frac{1}{4}\left(\frac{\pi^2}{6}\cdot\frac{\pi^2}{6} - 2\cdot\frac{\pi^4}{120}\right)$$

$$= \frac{\pi^4}{360}.$$

Expectation: Comparing $\zeta(\underline{k})\zeta(\underline{l})$ expressions via Σ-product (stuffle) and $\int$-product (shuffle) leads to <u>all</u> MZV relations.

[Hence weight would be a grading!]

Caveat: We need to make sense of $\zeta(1)$ via "regularisation process". Afterwards we can get e.g.

$$\zeta(2)\zeta(1) \overset{\Sigma}{=} \zeta(2,1) + \zeta(1,2) + \zeta(3)$$

$$\overset{\int}{=} \zeta(2,1) + 2\cdot\zeta(1,2)$$

$10 \sqcup\!\sqcup 1 = 101 + 2\cdot 110$

$$\Rightarrow \quad \zeta(1,2) = \zeta(3).$$

MZV's in low weight. With numerical experiments, later proven via shuffle-stuffle, we find ...

		$\mathbb{Q}$-dimension $\leq$
wt 0	$\rightsquigarrow$ only $\zeta(\emptyset) = 1$	1
wt 1	$\rightsquigarrow$ $\emptyset$ (or $\zeta(1) = 0$ (regularisation))	0
wt 2	$\rightsquigarrow$ $\zeta(2)$	1
wt 3	$\rightsquigarrow$ $\zeta(3) = \zeta(1,2)$	1
wt 4	$\rightsquigarrow$ $\zeta(4)$ $\zeta(1,3) = \frac{1}{4}\zeta(4)$, $\zeta(2,2) = \frac{3}{4}\zeta(4)$ $\zeta(1,1,2) = \zeta(4)$	1
wt 5	$\rightsquigarrow$ $\zeta(5)$, $\zeta(2,3)$ [OR $\zeta(5)$, $\zeta(2)\zeta(3)$, OR $\zeta(2,3)$, $\zeta(3,2)$] $\zeta(1,1,3) = 2\zeta(5) - \zeta(2)\zeta(3)$ $= \frac{1}{6}\zeta(5) - \frac{1}{3}\zeta(2,3)$ $= -\frac{1}{5}\zeta(2,3) + \frac{1}{3}\zeta(3,2)$	2 [but can't prove ≥ 1]
wt 6	$\rightsquigarrow$ $\zeta(6)$, $\zeta(3)^2$	2
wt 7	$\rightsquigarrow$ $\zeta(7)$, $\zeta(2)\zeta(5)$, $\zeta(4)\zeta(3)$	3
wt 8	$\rightsquigarrow$ $\zeta(8)$, $\zeta(3)\zeta(5)$, $\zeta(3)^2\zeta(2)$, $\zeta(3,5)$ ($\zeta(3,5) = 0.03770767\ldots$)	4

Define $d_k = d_{k-2} + d_{k-3}$, $d_1 = 0$, $d_2 = d_3 = 1$.

Conjecture $\dim_{\mathbb{Q}}(\text{weight } k \text{ MZV's}) = d_k$

We know $\dim_{\mathbb{Q}}(\text{weight } k \text{ MZV's}) \leq d_k$ (motivic argument, involving algebraically defined motivic MZV's ζ^m, which keep track of chains and forms)

Expectation $\zeta(3,5)$ is irreducible, i.e. not a polynomial in single zeta values $\zeta(n)$.

Also know $\zeta^m(3,5)$ is irreducible on the motivic level, using coproduct and Hopf algebra structure.

Take-away: MZV's become interesting in weight 8, if you see $\zeta(\leq 7)$ appearing, look for $\zeta(3,5)$ next.

§ Dirichlet eigenvalues of regular polygons

[Berghaus, Georgiev, Monien, Radchenko]

Setup: Region $\Omega \subseteq \mathbb{R}^2$, bounded, piecewise smooth $\partial\Omega$.

Investigate: $\left\{ \begin{array}{l} \Delta\varphi + \lambda\varphi = 0 \text{ in } \Omega, \\ \varphi|_{\partial\Omega} = 0 \end{array} \;\middle|\; \varphi \in C^2(\Omega) \cap C(\bar{\Omega}) \right\}$

i.e. the modes of oscillation of domain Ω.

Known to have discrete spectrum

$$0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots$$
$$\lambda_n(\Omega) \to \infty$$

For Ω fixed, Weyl's law gives

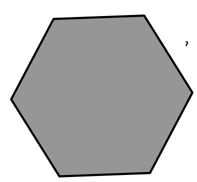
$$\#\{\lambda_n(\Omega) \leq T\} \sim \frac{\text{area}(\Omega)}{4\pi} T$$

For fixed n, changing Ω, Faber-Krahn gives

$$\min_{\text{area}(\Omega)=\pi} \lambda_1(\Omega) = \lambda_1(\text{disc } D \text{ in } \mathbb{R}^2)$$

with $\lambda_1(\Omega) = \lambda_1(\text{disc})$ iff $\Omega = \text{disc}$.

Let P_N = regular N-gon w/ area π.



Conj (Polyá-Szegö): $\lambda_1(P) \geq \lambda_1(P_N)$, for any P = N-gon of area π.

So as $N \to \infty$, we expect $\lambda_1(P_N) \to \lambda_1(\text{disc})$

More precisely, expect asymptotic expansion

$$\frac{\lambda(P_N)}{\lambda(\text{disc})} \sim 1 + \sum_{i=1}^{\infty} \frac{c_i}{N^i}$$

The numbers c_i are of interest.

i	1	2	3	4
c_i	0	0	$4\zeta(3)$	0

[Grinfeld–Strang 2004]

It is known $\lambda_1 = \lambda_1(\text{disc}) = j_{0,1}^2$, where $j_{0,1}$ is first non-trivial zero of

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-x/2)^{2n}}{(n!)^2}$$

ie $j_{0,1} \approx 2.404825557\ldots$

Then

i	5	6
c_i	$(-2\lambda_1 + 12)\ \zeta(5)$	$(4\lambda_1 + 8)\ \zeta(3)^2$

[M Boady, 2015]

Finally, numerically conjectured

i	7	8
c_i	$(-\frac{1}{2}\lambda_1^2 - 12\lambda_1 + 36)\ \zeta(7)$	$(2\lambda_1^2 + 8\lambda_1 + 48)\zeta(3)\zeta(5)$

[Jones 2017]

Thm Berghaus, Gergov, Monien, Radchenko

There exists $c_n(\lambda) \in \mathcal{Z}_n[\lambda]$, a sequence of polynomials whose n^{th} term has coefficients in $\mathcal{Z}_n$, weight n MZVs, such that

$$\frac{\lambda_1(P_N)}{\lambda_1(\text{disc})} = 1 + \sum_{n=1}^{\infty} \frac{c_n(\lambda_1)}{N^n}$$

as $N \to \infty$

Remark

- They explicitly computed c_n, $1 \le n \le 14$.
- Genuine MZVs appear in weight 11. namely

$$\zeta^{sv}(3,5,3) := -2\zeta(3,5,3) - 2\zeta(3)\zeta(3,5) + 10\zeta(3)^2\zeta(5)$$

Thm BGMR

$c_n(0)$ and $c_n'(0)$ are products of odd single-zeta values, with explicit generating series

$$\sum c_n(0) w^n = \frac{\Gamma^2(1+w)\Gamma(1-2w)}{\Gamma^2(1-w)\Gamma(1+2w)}$$

$$\sum c_n'(0) w^n = \left(\frac{\Gamma^2(1+w)\Gamma(1-2w)}{\Gamma^2(1-w)\Gamma(1+2w)}\right)^2 \times \left(1 + w^2\psi^{(1)}(1+w) - w^2\psi^{(1)}(1-w)\right)$$

where $\psi^{(1)}(z) = \frac{d}{dz}\log\Gamma(1+z)$, $\Gamma(1+z) = \exp\left(-\gamma z + \sum_{n=2}^{\infty} \frac{\zeta(n)(-z)^n}{n}\right)$

Idea of proofs]

Use polylogarithm $\mathrm{Li}_{k_1,\dots,k_d}(z) = \sum_{n_1<\dots<n_d} \frac{z^{n_d}}{n_1^{k_1}\cdots n_d^{k_d}}$

(which gives $\zeta(k_1,\dots,k_d)$ at $z=1$), to describe the coefficients of asymptotic expansion of the conformal map f: disc $\to P_N$

$$f(z) = * \, z \cdot {}_2F_1\left(\begin{matrix} 2/N & 1/N \\ 1+1/N & \end{matrix} ; z^N\right),$$

in order to translate $\lambda_n(P_N)$ into $\lambda_n(\text{disc})$.

Remark]
- Unknown what to expect for $c''_n(0)$, as higher depth needed
- They expect $c_n(\lambda) \in \mathcal{Z}^{sv}[\lambda]$, a certain "special" subspace of so-called single valued MZV's, i.e. values of single-valued versions of $\mathrm{Li}_{n_1,\dots,n_d}(z)$, etc.

§ Area expansion of families of CMC surfaces

[Heller², Traizet, plus MZV calculations by C]

[Details sketchy; I'm not a differential geometer.]

Minimal surfaces (i.e. locally area minimising), or more generally Constant Mean Curvature (where minimal $\iff$ (cmc = 0) one of interest.

Lawson constructed compact embedded minimal surfaces of all genera in the 3-sphere. $\xi_{1,g}$.

Geometric properties of $\xi_{1,g}$ are difficult to compute. ξ_g. area is not known for $\xi_{1,g}$, $g \geq 2$ explicitly. [Numerics known?]

[???]
Heller^2, Traizet used DPW method to obtain a family of CMC surfaces f_φ^g deforming $\xi_{1,g}$, from which geometric properties can be extracted.

Def - Prop] Multiple polylogarithm

$$\mathrm{Li}_{k_1 \ldots k_d}(z_1, \ldots, z_d) = \sum_{n_1 < \cdots < n_d} \frac{z_1^{n_1} \cdots z_d^{n_d}}{n_1^{k_1} \cdots n_d^{k_d}}$$

$$= \int_0^1 \frac{dt_1}{t_1 - a_1} \cdots \frac{dt_1}{t_w - a_w}$$

with $w = k_1 + \cdots + k_d$ weight, and

$$(a_1, \ldots, a_w) = \Big(\frac{1}{z_1 \cdots z_d}, \underbrace{0, \ldots, 0}_{k_1 - 1}, \frac{1}{z_2 \cdots z_d}, \underbrace{0, \ldots, 0}_{k_2 - 1}, \ldots, \frac{1}{z_d}, \underbrace{0, \ldots, 0}_{k_d - 1}\Big)$$

Remark]
- At $z_1 = \cdots = z_d$, recover $\zeta(k_1, \ldots, k_d)$
- $d = 1$ gives $\mathrm{Li}_1(z) = -\log(1 - z)$, and $\mathrm{Li}_n(z)$ the classical polylogarithm.

• At $z_i = \pm 1$, we obtain alternating MZV's

$$\zeta(\overline{k_1}, k_2, \overline{k_3})$$

$z_1 = -1 \quad z_2 = 1 \quad z_3 = -1.$

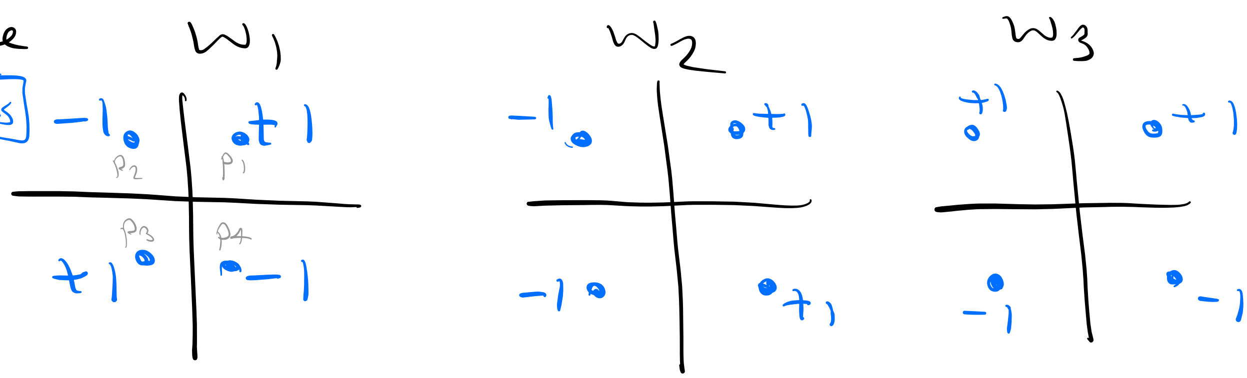
Thm (Heller², Tsanet) There is an iterative algorithm to compute the DPW potential, area (around $g = \infty$) and Willmore energy of ξ^g_φ via multiple polylogarithms.

At $\varphi = \frac{\pi}{4}$, calculations produce values in terms of

$$\Omega_{i_1 \cdots i_n} = \int_0^1 w_{i_1} \cdots w_{i_n}$$

where

residues



$$p_1 = e^{2\pi i/8}, \; p_2 = -\overline{p_1}, \; p_3 = -p_1, \; p_4 = \overline{p_1}$$

Remark: • Generally $\Omega_{i_1 \cdots i_n}$ is a sum of 4^n weight n MPL's at $z_i = (e^{2\pi i/8})^*$.

Observation Using the shuffle product of $\Omega_{i_1 \cdots i_n}$'s the α_i coefficients of

$$\text{Area}(\xi_{1,g}) = 8\pi\left(1 + \frac{\alpha_1}{(2g+1)} + \frac{\alpha_3}{(2g+1)^3} + \cdots\right)$$

can be reduced to a "strongly" structured set of indices. [Checked for $\alpha_1, \alpha_3, \alpha_5, \alpha_7$.]

Result These "strange" $\Omega_{i_1 \dots i_n}$'s are always alternating MZV's using the results of Hirose-Sato iterated beta integrals.

E.g. $\Omega_{3321} \overset{!}{=} -i\pi \cdot \zeta(\bar{2}, \bar{1}, \bar{1})$

(4^4 MPL's at $e^{2\pi i/8}$)

Thm $H^2T + C$ With $t = \frac{1}{2g+2}$,

$$\text{Area}(\xi_{1,g}) =$$

$$8\pi \Big(1 - (\log 2)\, t \quad \leftarrow \text{Known by } H^2T \text{ earlier}$$

$$- \frac{9}{4}\zeta(3)\, t^3 \quad \leftarrow \text{Numerical conjecture by } H^2T,\ \text{proven by } C$$

$2.70462\ldots \rightsquigarrow$

$$- \Big(-8\,\zeta(1,1,\bar{3}) + \frac{121}{16}\zeta(5)$$

$3.699626\ldots \rightsquigarrow$

$$+ \frac{2\pi^2}{3}\zeta(3) - 21\,\zeta(3)\log^2 2 \Big) t^5$$

$-53.168800\ldots \rightsquigarrow$

$$- (\text{explicit weight } 7)\, t^7$$

$$+ O(t^9) \Big)$$

In progress / future:

- Want to better understand the "strange" factorisation property of these Ω-values. (How) is it connected to iterated beta integrals?

- Improve algorithms and results to compute higher weight coefficients (at least α_9, α_{11})

- General / generating series expressions for α_i? Convergence of $\mathrm{Area}(\Xi_{1,g})$?