Polylogs, Zagier's conjecture & depth reduction

§1 Introduction

Recall the logarithm

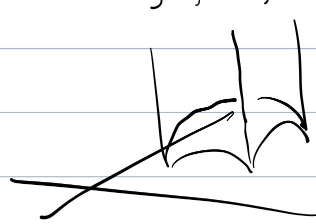
$$\underbrace{-\log(1-x)}_{\mathrm{Li}_1(x)} = \sum_{n=1}^{\infty} \frac{x^n}{n^1}, \quad |x| < 1.$$

Define "higher" variants by

$$\mathrm{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}, \quad |x| < 1$$

Simplest of which is the dilogarithm $\mathrm{Li}_2(x)$. What properties does this have / what is this good for?

$$\underbrace{\mathcal{L}i_2(z)}_{\text{single valued variant}} \longleftrightarrow \text{volume of ideal hyperbolic tetrahedron } (0, 1, \infty, z)$$



log satisfies $\log(xy) = \log x + \log y$,
Li_2 satisfies a 5-term functional equation.

$$Li_2(x) + Li_2(y) - Li_2\left(\frac{x}{1-y}\right) - Li_2\left(\frac{y}{1-x}\right) + Li_2\left(\frac{xy}{(1-x)(1-y)}\right) = -\log(1-x)\log(1-y)$$

$|x| + |y| < 1$
(power series identity)

Important application:

Zagier's conjecture on $\zeta_F(m)$:
If F = # field ($[F:\mathbb{Q}] < \infty$),
and

$$\zeta_F(s) := \sum_{\substack{I \subset \mathcal{O}_K \\ I \neq (0) \\ \text{ideal}}} \frac{1}{\underbrace{N(I)}_{\#(\mathcal{O}_K/I)}{}^{s}}$$

Then

$$\zeta_F(m) \overset{?}{=} \mathbb{Q}^{\times} \times \text{known factor}$$

involving $\pi^{r_1+r_2}$, Δ_F

$\left\{\begin{array}{l} r_2 \times r_2 \\ \text{or } (r_1+r_2)\times(r_1+r_2) \\ \text{if } m \text{ even or odd} \end{array}\right\} \times \det\left(\mathcal{L}i_m(\sigma_i(y_j))\right)$

where $\mathcal{L}i_m$ is a single-valued variant of Li_m,

$\sigma_i : F \to \mathbb{R}$ or $\mathbb{C}$ runs over all real / pairs of complex embeddings

y_j is some formal linear combination $\in \mathbb{Z}[F^\times]$.

Eg: $F = \mathbb{Q}(\sqrt{-7})$, then

$$\zeta_F(2) = \frac{4\pi^2}{21\sqrt{7}}\left(\mathcal{L}i_2\left(2\left[\frac{1+\sqrt{-7}}{2}\right] + \left[\frac{-1+\sqrt{-7}}{4}\right]\right)\right)$$

[$s=1$ $\longrightarrow$ Analytic class number formula involving $\log|\cdot|$]

§ Goncharov's programme

(Re) introduction of multiple polylogarithms

$$Li_{k_1\cdots k_d}(x_1, \ldots, x_d) = \sum_{n_1 < \cdots < n_d} \frac{x_1^{n_1} \cdots x_d^{n_d}}{n_1^{k_1} \cdots n_d^{k_d}}.$$

and their relation to iterated integrals

[depth d, wt $k_1 + \cdots + k_d$]

$$\mathrm{Li}_{k_1 \cdots k_d}(x_1 \cdots x_d) = (-1)^d I\left(0\,;\ \frac{1}{x_1 \cdots x_d}, \{0\}^{k_1-1}, \frac{1}{x_2 \cdots x_d}, \{0\}^{k_2-1}, \ldots, \frac{1}{x_d}, \{0\}^{k_d-1}\,;\ 1\right)$$

where $$I_\gamma(a\,;\ x_1, \ldots, x_N\,;\ b) = \int_{a < t_1 < \cdots < t_N < b} \gamma^*\left(\frac{dt_1}{t_1 - x_1}\right) \wedge \cdots \wedge \gamma^*\left(\frac{dt_N}{t_N - x_N}\right)$$

for γ a path $a \to b$.

[For Li, take $\mathrm{dch} : [0,1] \to [0,1]$, $t \mapsto t$

straight line path]

Important theorem/result by Goncharov.

Iterated integrals I can be upgraded to formal mixed Tate motives I^a

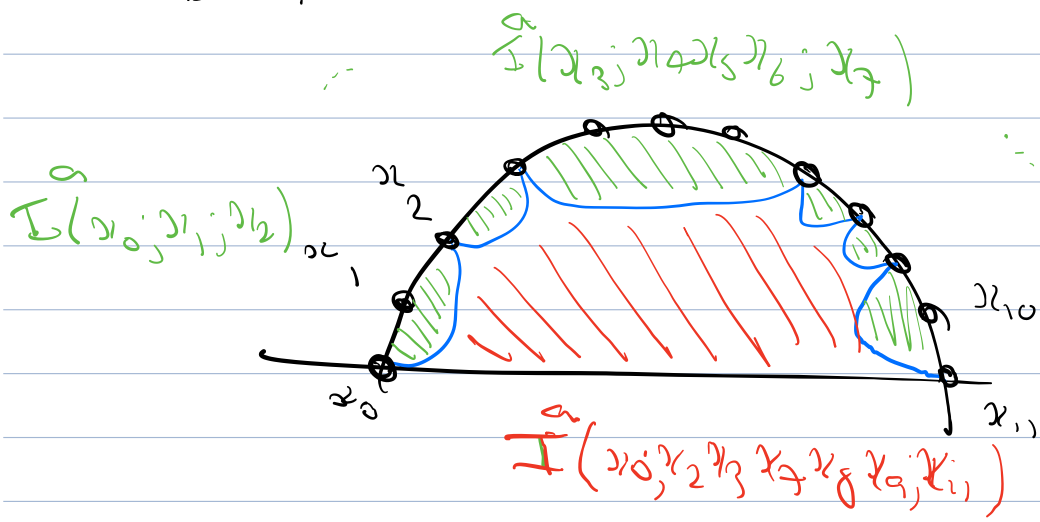
[Think: I^a keeps track of the forms and path (up to homotopy).

So identities only have a geometric origin.]

Theorem: Motivic integrals I^a form a Hopf algebra, with coproduct

$$\Delta I^a(x_0; x_1 \dots x_N; x_{N+1})$$

$$= \sum_{\substack{i_0=0<i_1<\dots \\ <i_k<i_{k+1}=N+1}} \prod_{p=0}^{k} I^a(x_{i_p} \to x_{i_{p+1}}) \otimes I^a(x_{i_0}; x_{i_1} \dots x_{i_k}; x_{i_{k+1}})$$



With this new structure, Goncharov conjectures some deep results about polylogs.

With $\delta = \Delta - \Delta^{op}$ (mod products)

$\bar{\delta} = \delta$ (mod weight 1 terms)

We have $\bar{\delta} Li_k(x) = 0$

(i.e. $\Delta Li_k(x) \simeq \sum Li_i(x) \otimes \log^{k-i} x$)

$$\bar{\delta}\, Li_{a,b}(x,y) \approx \sum Li_i(x) \wedge Li_j(y)$$
$$+ \sum Li_i(x) \wedge Li_j\left(\frac{x}{y}\right)$$
$$+ \sum Li_i(y) \wedge Li_j\left(\frac{x}{y}\right)$$
$$= \text{"depth } 1 \wedge \text{depth } 1\text{"}$$

Goncharov expects:

$$\ker \bar{\delta} = \text{depth 1 polylogs.}$$

$$\bar{\delta} X = \text{dp } 1 \wedge \text{dp } 1$$
$$\iff X = \text{dp } 2.$$

and generally $\bar{\delta}$ should detect the depth of an MPL.

Application to Zagier conjecture:

Goncharov shows how to relate $\zeta_F(m)$ with weight m Grassmannian polylog (some geometrically defined, high depth object.)

If $\overline{\delta} = 0$, it can be conjecturally reduced to classical polylogs, hence $\zeta_F(m)$ is expressed by Li_m's.

wt 2 : already by Zagier (weak version), essentially Bloch & Suslin.

Here $\overline{\delta} Li_{1,1}(x,y) = 0$ means

$$Li_{1,1}(x,y) = Li_2\left(\frac{-x}{1-x}\right) - Li_2\left(\frac{x(y-1)}{1-x}\right) + Li_1(x) Li_1(y)$$

is "expected".

wt 3 : Goncharov found reductions of $Li_{1,1,1}(x,y,z) = \sum Li_3$'s.

$\Longrightarrow$ 22-term 3 vars ("nasty") functional equation for Li_3.

Zagier's conjecture holds for $\zeta_F(3)$ (~1995) wt 4: We can't reduce Li_{1111} to Li_4's. Not even

$$I_{31}(x,y) = I(0; x\,0\,0\,y; 1)$$

as $\delta I_{31}(x,y) = Li_2(x) \wedge Li_2(y) \neq 0$.

Goncharov could modify Grassmannian to still represent $\zeta_F(m)$ while having $\delta = 0$

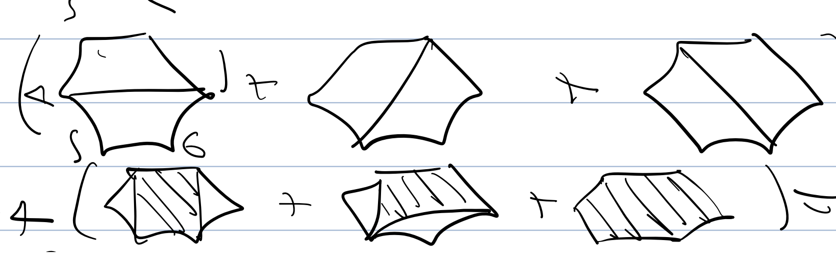
If we knew how to reduce

(*) I_{31}(dilog δ-term, 2) $= \sum Li_4$'s

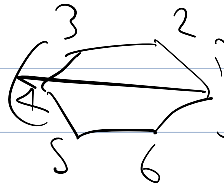
then we can write modified Grassmannian via Li_4's.

Gangl gave 122 Li_4's terms for (*)

typical term $Li_4\left(\frac{cr \times cr \times cr \times cr}{cr \times cr \times cr \times cr}\right)$

$\mathrm{Cyc}_7\Big(I_{31}\big($ 

$\big) + Li_4\big(\cdots \big)\Big) = 0$

where $I_{3,1}($$) \longmapsto I\big(cr(x_1,x_2,x_3,x_4),\, cr(x_4,x_5,x_6,x_1)\big)$

Goncharov – Rudenko derived (*) via a new geometric identity for wt 4 MPL's (+ and filled in many technical details relating cycle-rooted / multiindexed Grassmannian).

$\Rightarrow$ Zagier for $\zeta_F(4)$ (2018)

§ Recent work

CGRa (2019/2023) Non Rudy. Geometric identities for wt 5, 6, 7

$$\sum I_{3,1,1} + I_{3,2} + I_{4,1} + I_5 = 0$$

↳ implies $I_{3,1,1}, I_{3,2} = \sum I_{4,1} + I_5$'s.

Key expectations $I_{4,1}^{(sym)}(5\text{-term}, z) = \sum I_5$'s ?

$I_{4,1}^{(sym)}(x, 22\text{-term}) = \sum I_5$'s ?

Thm (C non-public) We can express $I_{4,1}^{(sym)}(x, 22\text{-term}) = \sum I_{4,1}^{(sym)}(5\text{-term}, z_i) + I_5$'s.

[We are still trying to find the ζ-term! :)]

However for degenerate cases:

Thm (CGR19) $I_{41}^{(sym)}(\zeta\text{-term}, 1) = \sum I_5$'s.

(really Nielsen S_{32}, but equivalent. Explains Li's FE from Ra Tinges.)

Thm (C) non-public: $I_{41}^{(sym)}(\zeta\text{-term in}(x,y), y) = \sum I_5$'s.

Thm (CGR19) Expression for Grassmannian ζ via $I_{41} + I_5$.

+ suitable modification so $\delta = 0$.

[Thm (CGR19) Expression for Gr wt m via $I_{21\cdots1}$]

Once we find $I_{41}^{(sym)}(\zeta\text{-term}, z) = \sum I_5$'s.
then $\zeta_F(5)$ is nearly solved.
[Still many technicalities, but this is main combinatorial step :)]

Thm (MRu 2020) Geometric (cluster?) functional equation in all weights (independently found!)

Thm (MRu 2022) Reduction of

$I_{4,1,1}^{(sym)}$ (S-term, z, w) = depth 2.

(modulo 2-term symmetries

$$I_{4,1,1}^{(sym)}(x,y,z) \mp I_{4,1,1}^{(sym)}(1-x,y,z)$$
$$I_{4,1,1}^{(sym)}(x,y,z) + I_{4,1,1}^{(sym)}(\tfrac{1}{x},y,z)\ )$$

(which are equivalent — C)

Thm (CRu non-public, ~1 month ago)

$$I_{4,1,1}^{(sym)}(x,y,z) + I_{4,1,1}^{(sym)}(1-x,y,z) = \text{dp } 2.$$
$$I_{4,1,1}^{(sym)}(x,y,z) + I_{4,1,1}^{(sym)}(\tfrac{1}{x},y,z) = \text{dp } 2$$

Thm (MRu 2020)

Every wt n MPL has depth $\leq \lfloor \frac{n}{2} \rfloor$

Thm (CGRaRn, 2022–2023)

Every depth d MPL can be reduced to $I_{n+1-d}(\underbrace{1,\dots,1}_{d})$

(i.e. a single function in every depth)

[Depth $d \geq 3$ in progress

$d = 2$ a surprisingly simple 'partial fractions' argument.]

Corollary: Goncharov's conjecture on ζ in depth 1 implies all higher depths.
(over a quadratically closed field.)

Viewpoint: Resolution of $\zeta_F(5)$ and $\zeta_F(6)$ (maybe), later to dp2 cf. magic.