Depth reduction of MPL's: Expectations, techniques, approaches

§ Definitions

Just to recall the notation / conventions:

$$\mathrm{Li}_{k_1 \cdots k_d}(x_1, \ldots, x_d) = \sum_{n_1 < \cdots < n_d} \frac{x_1^{n_1} \cdots x_d^{n_d}}{n_1^{k_1} \cdots n_k^{d_k}}$$

So $\mathrm{Li}_1(x) = -\log(1-x)$ via the Taylor series, and $\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is the dilog.

We can write MPL's as iterated integrals:

$$I(a; x_1, \ldots, x_N; b) = \int_{a < x_1 < \cdots < x_N < b} \frac{dt_1}{t_1 - x_1} \wedge \cdots \wedge \frac{dt_N}{t_N - x_N}$$

Then:

$$\mathrm{Li}_{k_1 \cdots k_d}(x_1, \ldots, x_d) = (-1)^d I_{k_1 \cdots k_d}\left(\frac{1}{x_1 \cdots x_d}, \frac{1}{x_2 \cdots x_d}, \ldots, \frac{1}{x_d}\right)$$

with

$$I_{k_1 \cdots k_d}(z_1, \ldots, z_d) = I(0; z_1 \{0\}^{k_1 - 1} \cdots z_d \{0\}^{k_d - 1}; 1)$$

Integrals multiply with shuffle product:

$$I(a;w;b)\,I(a;v;b) = I(a;\underbrace{w \sqcup\!\sqcup v}_{};b)$$

Permutations where w-letters and v-letters are correctly ordered "riffle shuffle".

Series multiply with stuffle product.

$$\underset{n}{Li_a}(x)\,\underset{m}{Li_b}(y) = \overset{n<m}{Li_{a,b}(x,y)} + \overset{n>m}{Li_{b,a}(y,x)} + \underset{n=m}{Li_{a+b}(xy)}$$

Can regularise I to allow integrals with leading zeros:

$0 \sqcup\!\sqcup 0^{k-1} w = k \cdot 0^k w +$ terms with $k-1$ starting 0's.

Write $I_{k_0;k_1\cdots k_d}(z_1 \cdots z_d) = I(0;0^{k_0}z_1 0^{\cdots} \cdots z_d 0^{k_d-1};1)$

Likewise $Li_{k_0;k_1\cdots k_d}(x_1 \cdots x_d)$

$$= I_{k_0;k_1\cdots k_d}\left(\frac{1}{x_1\cdots x_d}, \frac{1}{x_2\cdots x_d}, \cdots, \frac{1}{x_d}\right)$$

[These functions are somehow the crucial ones for higher weight identities.]

Remark: Writing $Li_{k_0; k_1, \dots k_d}(x_1 \dots x_d)$ via "convergent MPL's".

- Obviously we can shuffle regularise.
- But we obtain better identities via a dihedral symmetry

$$I(\infty; x_1, \dots x_N; x_{N+1})$$ "iterated integrals"

is dihedrally symmetric in $x_1, \dots x_{N+1}$ of order $2(N+1)$.

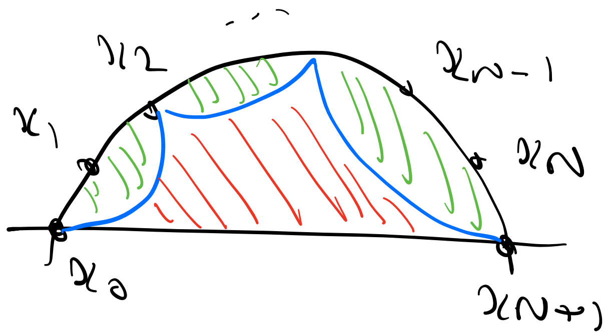
Then $I(\infty; x_1, \dots x_N; x_{N+1})$
$$= I(\infty; x_1, \dots x_N; 0) + I(0; x_1, \dots x_N; x_{N+1})$$

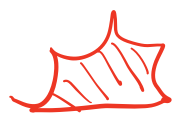
$$I(\infty; x_1 \dots x_N; 0) = I(\infty; \underbrace{x_2 \dots x_N\, 0}; x_1)$$

more 0's : lower depth.

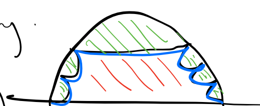
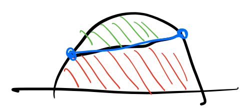
§ Lie coalgebra & Goncharov's depth conj

Iterated integrals form a Hopf algebra. w/ coproduct given by



$$\Delta I = \sum_{\mu} \prod I($$  $$) \otimes I($$  $$)$$

Quotient by products to obtain Lie coalgebra of irreducibles, with cobracket given by

Only  survives $\rightsquigarrow$ 

$$\delta = \Delta - \Delta^{op} = \Delta' \ (\text{mod } Ш)$$

This gives a map $\mathcal{L}_0(F) \longrightarrow \Lambda^2 \mathcal{L}_0(F)$

The reduced cobracket $\bar{\delta}$ (neglecting wt 1 $\wedge$ wt $(n-1)$), vanishes on classical polylogs:

$$\bar{\delta} \, Li_n(x) = 0 .$$

Conjecture: If $\bar{\delta}(X) = 0$, X linear combination of MPL's, then $X = \sum \mathrm{Li}_n$'s.

One can iterate $\bar{\delta}$ to see

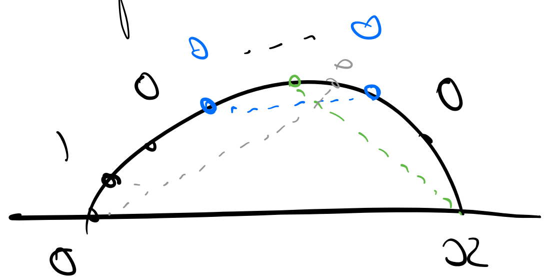
$$\bar{\delta}^2 : \mathcal{L}_\bullet(F) \xrightarrow{\delta} \Lambda^2 \mathcal{L}_\bullet(F) \longrightarrow \Lambda^3 \mathcal{L}_\bullet(F)$$

Can define a map

$$\bar{\delta}^{[d-1]} : \underbrace{\mathrm{gr}_d \mathcal{L}_\bullet(F)}_{\text{depth } d} \longrightarrow \mathrm{coLie}_d\Big(\bigoplus_{n \geq 2} \underbrace{B_n(F)}_{\mathrm{Li}_n\text{'s}}\Big)$$

Conj: If $\bar{\delta}^{[d]} X = 0$, then X = depth d.

Explicit examples: • $\mathrm{Li}_n(x) = -I(0; 10^{n-1}; x)$



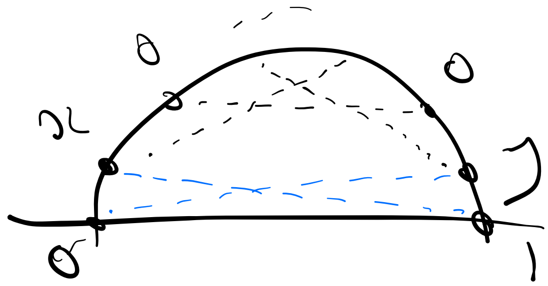
$0 \to 0 \rightsquigarrow 0$

$0(0^*)x \rightsquigarrow 0$

$0(10^*)0 \rightsquigarrow 0$

So $\bar{\delta}\, \mathrm{Li}_n(x) = 0$.

• $\bar{\delta} I_{n-1,1}(x,y) = I(0; x\, 0^{n-2}\, y; 1)$



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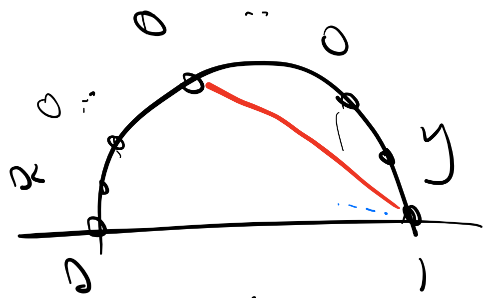


$$\sum_{i=1}^{n-3} I(0; 0^i\, y; 1) \wedge I(0; x\, 0^{n-2-i}; 1)$$

$$\rightsquigarrow \sum_{i=1}^{n-3} (-1)^i \, Li_{i+1}(y) \wedge Li_{n-1-i}(x)$$

[up to signs !!]

So

$$\delta I_{3,1}(x,y) = Li_2(x) \wedge Li_2(y) \neq 0.$$

What does this tell us?

1. $I_{3,1}(x,y)$ cannot be expressed by depth 1 functions

[as $\delta I_{3,1}(x,y) \neq 0$, but $\delta\, Li_4(x) = 0$]

2. If we take $I_{3,1}$(dilog identity, y), we should expect depth 1.

- $\bar{\delta}\, I_{4,1,1}(x,y,z)$

$$= -I_2(x) \wedge \underline{I_{1,3}(z,y)}$$

$$+ I_2\left(\frac{y}{z}\right) \wedge \underline{I_{3,1}(x,z)}$$

$$+ \text{simpler} \wedge \text{simpler}$$

$$(\text{dp } 1 \text{ or } \leq \text{wt } 3) \wedge (\text{dp } 1 \text{ or } \leq \text{wt } 3)$$

So $\bar{\delta}^{[2]} I_{4,1,1}(x,y,z)$

$$= \left(\underline{I_2(x) \wedge I_2(z)}\right) \wedge I_2\left(\frac{y}{z}\right)$$

$$+ \left(\underline{I_2\left(\frac{y}{z}\right) \wedge I_2(z)}\right) \wedge I_2(x).$$

Rmk: with $Li_{3,1,1}(x\,y\,z)$

$$= I\left(0;\, 0\,0\,0\, \frac{1}{xyz}\, \frac{1}{xy}\, \frac{1}{y};\, 1\right)$$

we got

$$\bar{\delta}^{[2]} Li_{3,1,1}(x\,y\,z) = -Li_2(x) \wedge \left(Li_2(y) \wedge Li_2(z)\right)$$

$-(\mathrm{Li}_2(x)\wedge \mathrm{Li}_2(y))\wedge \mathrm{Li}_2(z))$

What does this tell us?

1. $I_{4,1,1}(x\,y\,z) \hookrightarrow \mathrm{Li}_{3;1,1}(x\,y\,z)$ are depth 3, and can't be reduced.

[Generally $\bar{\delta}^{(2)}\mathrm{Li}_{a,b}(x,y) = 0$.]

2. $\mathrm{Li}_{3;1,1}(\text{dilog identity}, y, z) \overset{?}{=}$ depth 2.

§ Rudenko Quadrangular polylogs

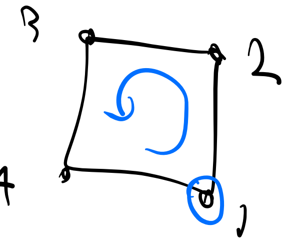
I won't recall the construction in detail, only depth 1 & 2.

Define:

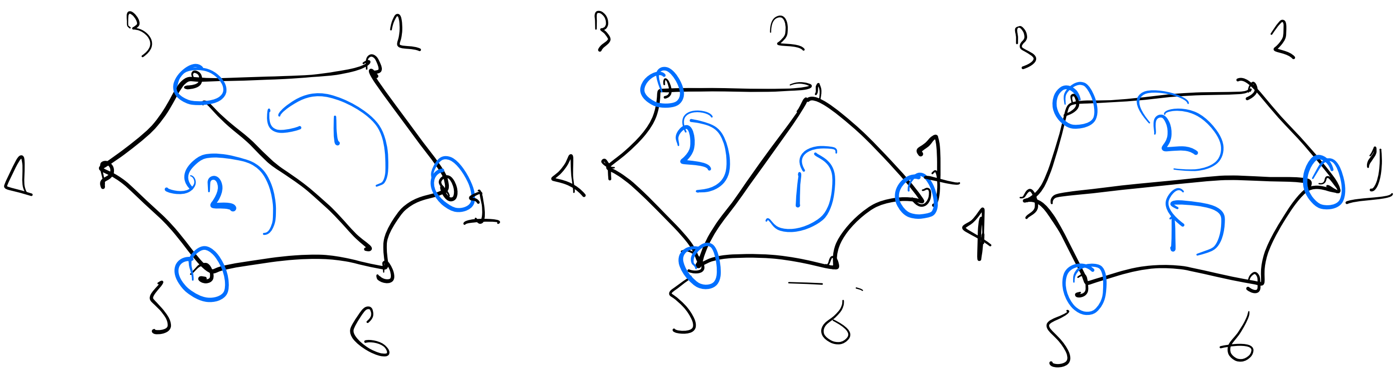
"cyclic version" $cr(x_1 x_2 x_3 x_4)$ $\downarrow$

$$\underbrace{f_1^{w}(x_1 \ldots x_4)}_{\text{QLi}} = -\mathrm{Li}_{w-1;1}\left(\frac{12\cdot 34}{23\cdot 41}\right)$$

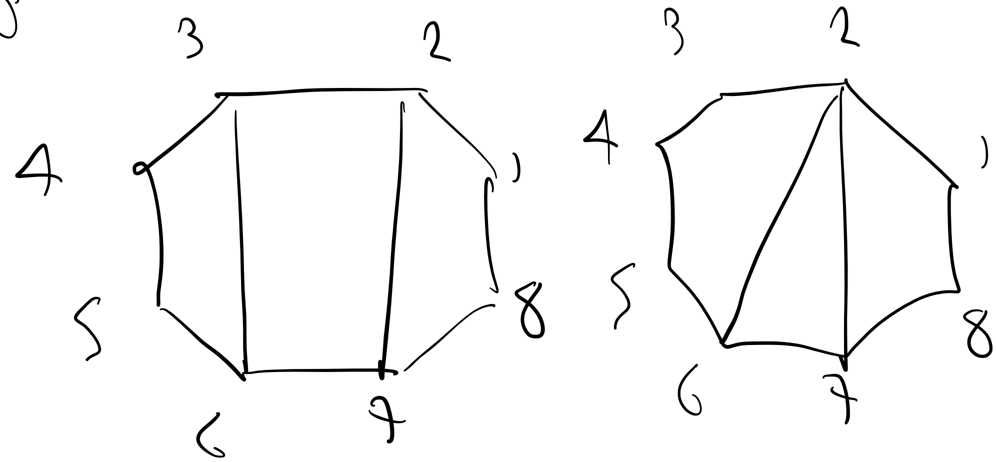
weight $w \leftrightarrow n$??
depth $1 \leftrightarrow k$



$$\mathcal{G}_2^w(x_1 \cdots x_6) = +\mathrm{Li}_{w-2;1,1}\left(\frac{12\cdot 36}{23\cdot 61}, \frac{34\cdot 56}{45\cdot 36}\right) - \mathrm{Li}_{w-2;1,1}\left(\frac{12\cdot 56}{25\cdot 61}, \frac{34\cdot 52}{23\cdot 45}\right) + \mathrm{Li}_{w-2;1,1}\left(\frac{14\cdot 56}{45\cdot 61}, \frac{12\cdot 34}{23\cdot 41}\right)$$



$\mathcal{G}_3^w(x_1 \cdots x_8)$ more complicated but comes from:



... etc

§ Quadrangular relations

Rudenko gave relations for the QLi.

wt 2, dp 1 : For 5 - pts $x_1 \dots x_5$

$$\ell_1^2(1234) - \ell_1^2(1235) + \ell_1^2(1245) - \ell_1^2(1345) + \ell_1^2(2345) = 0 \quad (Ш)$$

$\longleftrightarrow$ 5 - term relation for Li_2.

wt 2, dp 2 :

$$\ell_2^2(1 \dots 6) + \ell_1^2(1235) - \ell_1^2(1236) - \ell_1^2(1245) + \ell_1^2(1246) = 0 \quad (Ш)$$

OR: $I_{1,1}$'s notn

This gives $3 \times \mathrm{Li}_{1,1}(*,*) = \sum \mathrm{Li}_2$'s.

Can we extract anything interesting here?

NB: Since $\delta\, \mathrm{Li}_{1,1}(x,y) = 0$ (too low weight) we expect $\mathrm{Li}_{1,1} = \sum \mathrm{Li}_2$'s!

Get :

$$-\mathrm{Li}_{1,1}\left(\overset{1}{\frac{\cancel{12}\cdot\cancel{36}}{\cancel{23}\cdot\cancel{61}}} , \frac{34\cdot\cancel{56}^{\,52}}{45\cdot\cancel{63}_{\,23}} \right) + \mathrm{Li}_{1,1}\left(\overset{1}{\frac{\cancel{12}\cdot\cancel{56}}{\cancel{25}\cdot\cancel{61}}} , \frac{34\cdot 52}{23\cdot 45} \right)$$

$$-\mathrm{Li}_{1,1}\left(\frac{14\cdot\cancel{56}^{\,52}}{45\cdot\cancel{61}_{\,12}} , \frac{12\cdot 34}{23\cdot 41} \right)$$

$$-\mathrm{Li}_2\left(\frac{23\cdot 51}{12\cdot 35}\right) + \mathrm{Li}_2\left(\frac{24\cdot 51}{12\cdot 45}\right)$$

$$\cancel{+\mathrm{Li}_2\left(\frac{23\cdot \overset{21}{\cancel{61}}}{\underset{32}{\cancel{36}}\cdot 12}\right)} \cancel{-\mathrm{Li}_2\left(\frac{24\cdot \overset{21}{\cancel{61}}}{12\cdot \underset{42}{\cancel{46}}}\right)}$$

Set $x_6 = x_2$ (carefully)

$$\mathrm{Li}_{11}\left(\overbrace{\frac{14\cdot 52}{45\cdot 21}}^{A}, \overbrace{\frac{12\cdot 34}{23\cdot 41}}^{B}\right)$$

$$= \mathrm{Li}_2\left(\frac{24\cdot 51}{12\cdot 45}\right) - \mathrm{Li}_2\left(\frac{23\cdot 51}{12\cdot 35}\right) \qquad (Ш)$$

$$\rightsquigarrow \mathrm{Li}_{1,1}(A, B) = \mathrm{Li}_2(1-A) - \mathrm{Li}_2\left(\frac{1-A}{1-AB}\right)$$

wt 3, dp 2:

$$b_2^3(x_1 \ldots x_6)$$

$$+ b_1^3(1235 - 1236 - 1234 + 1246 + 1345 - 1346 + 1356 - 1456 - 2345 + 2346 - 2356 + 2456)$$

$$= 0 \quad (Ш)$$

$$\rightsquigarrow \quad 3 \times \mathrm{Li}_{21} = \sum \mathrm{Li}_3\text{'s}.$$

More precisely:

$$\mathrm{Li}_{1,1,1}(x, y) = I\left(0; 0\, \tfrac{1}{xy}\, \tfrac{1}{y}\, 1\right)$$

$$\overset{\text{dih}}{=} I\left(0; 1\, 0\, \tfrac{1}{xy}\, \tfrac{1}{y}\right) - 2 I\left(0; \tfrac{1}{xy}\, 0\, 0; 1\right) - I\left(0; \tfrac{1}{x}\, 0\, 0; 1\right)$$

$$\mathrm{Li}_{1,1,1}(x, y) = \mathrm{Li}_{21}\left(\tfrac{1}{xy}, x\right) + 2\mathrm{Li}_3(xy) + \mathrm{Li}_3(x)$$

$$- \mathrm{Li}_{21}\left(\frac{23 \cdot 45 \cdot 16}{12 \cdot 34 \cdot 56}, \frac{12 \cdot 36}{23 \cdot 61}\right) + \mathrm{Li}_{21}\left(\frac{23 \cdot 45 \cdot 16}{12 \cdot 34 \cdot 56}, \frac{12 \cdot 56}{25 \cdot 61}\right) - \mathrm{Li}_{21}\left(\frac{23 \cdot 45 \cdot 16}{12 \cdot 34 \cdot 56}, \frac{14 \cdot 56}{45 \cdot 61}\right)$$

$$- 2\mathrm{Li}_3\left(\frac{12 \cdot 34 \cdot 56}{23 \cdot 45 \cdot 16}\right)$$

$$+ \mathrm{Li}_3\Big(-\frac{2351}{3521} + \frac{2451}{4512} - \frac{3451}{1345}$$

$(\ast)$

$$+ \frac{3452}{2345} - \frac{2461}{4612} + \frac{3461}{1346}$$

$$- \frac{3462}{2346} + \frac{2561}{1256} - \frac{3561}{1356}\Big)$$

NB: $\mathrm{Li}_{21}($

Set $x_0 = x_2$, get

$$\mathrm{Li}_{21}\Big(\overbrace{\frac{2345}{3452}}^{A},\ \overbrace{\frac{4125}{1254}}^{B}\Big)$$

$$= -2\,\mathrm{Li}_3\Big(\frac{23\cdot 45}{34\,52}\Big)$$

$$+\mathrm{Li}_3\Big(\frac{1243}{2431} - \frac{2351}{3512} - \frac{1253}{2531}$$

$$+ \frac{2451}{1245} - \frac{3451}{1345} + \frac{3452}{4523}\Big)$$

$$\leadsto \quad \mathrm{Li}_{21}(A, B) = \underbrace{-}_{(-2+1)}\mathrm{Li}_3(A) + \mathrm{Li}_3(1-B)$$

$$-\mathrm{Li}_3\Big(-\frac{A(1-B)}{1-A}\Big) + \mathrm{Li}_3\Big(\frac{1}{1-AB}\Big)$$

$$-\mathrm{Li}_3\Big(\frac{1-A}{1-AB}\Big) + \mathrm{Li}_3\Big(\frac{1-B}{1-AB}\Big)$$

Notice: $\mathrm{Li}_3\Big(\frac{1-A}{1-AB}\Big) + \mathrm{Li}_3\Big(\frac{-A(1-B)}{1-A}\Big) + \mathrm{Li}_3\Big(\frac{A(1-B)}{1-AB}\Big)$

$= 0 \quad (\mathrm{Li})$

[Really $\zeta(3)$!]

three term

$[x] + [1-x] + [1-\frac{1}{x}]$

So better:

$$\mathrm{Li}_{21}(AB) = -\mathrm{Li}_3(A) + \mathrm{Li}_3(1-B) + \mathrm{Li}_3\left(\frac{1}{1-AB}\right) - \mathrm{Li}_3\left(\frac{1-B}{1-AB}\right) + \mathrm{Li}_3\left(\frac{A(1-B)}{1-AB}\right)$$

Now, plug back into (*)

Obtain 24 terms, but replace

$$\mathrm{Li}_3\left(\frac{24\;35}{34\;25}\right) + \mathrm{Li}_3\left(\frac{23\;45}{34\;52}\right) = -\mathrm{Li}_3\left(\frac{23\;45}{24\;35}\right)$$

and $\mathrm{Li}_3\left(\frac{15\;26}{25\;16}\right) + \mathrm{Li}_3\left(\frac{12\;56}{25\;61}\right) = -\mathrm{Li}_3\left(\frac{12\;56}{15\;26}\right)$

to obtain 22-term relation of Goncharov.

Take $\left(\frac{1234}{1324}, \frac{1235}{2351}, \frac{1326}{3261} \right)$

$= \left(C, \frac{1}{B}, A \right)$

gives "signed" 22-term (up to images & sign)

Out[2]=
$$
\begin{aligned}
&-\mathrm{Li}_3\left[\frac{(x_1-x_2)(x_3-x_4)}{(x_1-x_3)(x_2-x_4)}\right]-\mathrm{Li}_3\left[-\frac{(x_1-x_2)(x_3-x_5)}{(x_2-x_3)(x_1-x_5)}\right]+\mathrm{Li}_3\left[-\frac{(x_1-x_2)(x_4-x_5)}{(x_2-x_4)(x_1-x_5)}\right]-\mathrm{Li}_3\left[-\frac{(x_1-x_3)(x_4-x_5)}{(x_3-x_4)(x_1-x_5)}\right]-\\
&\mathrm{Li}_3\left[\frac{(x_2-x_3)(x_4-x_5)}{(x_2-x_4)(x_3-x_5)}\right]-\mathrm{Li}_3\left[\frac{(x_1-x_3)(x_2-x_6)}{(x_2-x_3)(x_1-x_6)}\right]-\mathrm{Li}_3\left[\frac{(x_3-x_4)(x_1-x_5)(x_2-x_6)}{(x_2-x_4)(x_3-x_5)(x_1-x_6)}\right]-\mathrm{Li}_3\left[-\frac{(x_1-x_2)(x_4-x_6)}{(x_2-x_4)(x_1-x_6)}\right]+\\
&\mathrm{Li}_3\left[-\frac{(x_1-x_3)(x_4-x_6)}{(x_3-x_4)(x_1-x_6)}\right]-\mathrm{Li}_3\left[\frac{(x_1-x_5)(x_4-x_6)}{(x_4-x_5)(x_1-x_6)}\right]+\mathrm{Li}_3\left[\frac{(x_1-x_2)(x_3-x_4)(x_1-x_5)(x_4-x_6)}{(x_1-x_3)(x_2-x_4)(x_4-x_5)(x_1-x_6)}\right]-\mathrm{Li}_3\left[-\frac{(x_2-x_3)(x_4-x_6)}{(x_3-x_4)(x_2-x_6)}\right]-\\
&\mathrm{Li}_3\left[\frac{(x_1-x_2)(x_3-x_5)(x_4-x_6)}{(x_1-x_3)(x_4-x_5)(x_2-x_6)}\right]-\mathrm{Li}_3\left[-\frac{(x_1-x_3)(x_5-x_6)}{(x_3-x_5)(x_1-x_6)}\right]-\mathrm{Li}_3\left[\frac{(x_1-x_2)(x_3-x_4)(x_5-x_6)}{(x_2-x_3)(x_4-x_5)(x_1-x_6)}\right]-\mathrm{Li}_3\left[\frac{(x_1-x_2)(x_5-x_6)}{(x_1-x_5)(x_2-x_6)}\right]+\\
&\mathrm{Li}_3\left[-\frac{(x_2-x_3)(x_5-x_6)}{(x_3-x_5)(x_2-x_6)}\right]-\mathrm{Li}_3\left[-\frac{(x_2-x_4)(x_5-x_6)}{(x_4-x_5)(x_2-x_6)}\right]+\mathrm{Li}_3\left[\frac{(x_1-x_2)(x_2-x_4)(x_3-x_5)(x_5-x_6)}{(x_2-x_3)(x_1-x_5)(x_4-x_5)(x_2-x_6)}\right]-\\
&\mathrm{Li}_3\left[\frac{(x_1-x_3)(x_2-x_4)(x_5-x_6)}{(x_2-x_3)(x_1-x_5)(x_4-x_6)}\right]-\mathrm{Li}_3\left[\frac{(x_3-x_4)(x_5-x_6)}{(x_3-x_5)(x_4-x_6)}\right]+\mathrm{Li}_3\left[\frac{(x_1-x_3)(x_3-x_4)(x_2-x_6)(x_5-x_6)}{(x_2-x_3)(x_3-x_5)(x_1-x_6)(x_4-x_6)}\right]
\end{aligned}
$$

Out[3]=
$$
\begin{aligned}
&\mathrm{Li}_3[A]+\mathrm{Li}_3[B]-\mathrm{Li}_3\left[-\frac{-1+B-AB}{A}\right]+\mathrm{Li}_3\left[-\frac{-1+B-AB}{AB}\right]+\mathrm{Li}_3[1-B+AB]+\mathrm{Li}_3[C]+\mathrm{Li}_3[-ABC]+\\
&\mathrm{Li}_3\left[-\frac{B(-1+A-AC)}{-1+B-AB}\right]-\mathrm{Li}_3\left[-\frac{-1+A-AC}{C}\right]+\mathrm{Li}_3\left[-\frac{-1+A-AC}{AC}\right]+\mathrm{Li}_3\left[\frac{-1+A-AC}{(-1+B-AB)C}\right]-\mathrm{Li}_3\left[\frac{-1+A-AC}{A(-1+B-AB)C}\right]+\\
&\mathrm{Li}_3[1-A+AC]+\mathrm{Li}_3\left[\frac{-1+B-AB}{A(-1+C-BC)}\right]-\mathrm{Li}_3\left[\frac{-1+B-AB}{AB(-1+C-BC)}\right]+\mathrm{Li}_3\left[-\frac{(-1+B-AB)C}{-1+C-BC}\right]-\mathrm{Li}_3\left[-\frac{-1+C-BC}{B}\right]+\\
&\mathrm{Li}_3\left[-\frac{-1+C-BC}{BC}\right]+\mathrm{Li}_3\left[-\frac{A(-1+C-BC)}{-1+A-AC}\right]+\mathrm{Li}_3\left[\frac{-1+C-BC}{B(-1+A-AC)}\right]-\mathrm{Li}_3\left[\frac{-1+C-BC}{BC(-1+A-AC)}\right]+\mathrm{Li}_3[1-C+BC]
\end{aligned}
$$

Now take $B = 1 \iff x_8 = x_3$,
then get

$$\text{22-term} \rightsquigarrow \mathrm{Li}_3(1) + 2\mathrm{Li}_3(A) + 2\mathrm{Li}_3(C) + 2\mathrm{Li}_3(-AC) + 2\mathrm{Li}_3\left(\frac{A-1-AC}{A}\right) - \mathrm{Li}_3\left(\frac{1-A+AC}{C}\right) + \mathrm{Li}_3\left(\frac{1-A+AC}{A^2C}\right) + 2\mathrm{Li}_3\left(\frac{1-A+AC}{AC}\right) - \mathrm{Li}_3(C(1-A+AC)) + 2\mathrm{Li}_3(1-A+AC)$$

This is

$$\mathrm{Kummer}(1-A+AC, C),$$

so set $A = \frac{1-A'}{1-C}$, to get the usual presentation

$$\mathrm{Kummer}(A,C) \Longrightarrow$$

$$2\mathrm{Li}_3(A) + 2\mathrm{Li}_3(C) - \mathrm{Li}_3(AC) - \mathrm{Li}_3\left(\frac{A}{C}\right) + 2\mathrm{Li}_3\left(\frac{1-A^{-1}}{1-C^{-1}}\right) + 2\mathrm{Li}_3\left(\frac{1-A}{1-C^{-1}}\right) + 2\mathrm{Li}_3\left(\frac{1-A^{-1}}{1-C}\right)$$

$$+\,2\,\mathrm{Li}_3\left(\frac{1-A}{1-C}\right) \quad - \mathrm{Li}_3\left(\frac{(1-A)^2 C}{A(1-C)^2}\right).$$

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wt 3, dp 3: We also get identity in dp 3.

$$\ell_8^3(x_1 \ldots x_8) + \mathrm{dp} \leq 2 = 0 \ (\mathcal{L}\mathcal{D})$$

By setting $x_7 = x_8 = x_3$, we deduce

$$\mathrm{Li}_{111}(x, y, z) = \mathrm{dp}\ 1 \quad (\text{via } \mathrm{Li}_{2,1} \text{ reduction})$$

[Also expected $\bar{\delta}\, \mathrm{Li}_{111} = 0$, and known.]

Now lets give a overview of wt 4, 5, 6 results.

wt 4: Have $7 \times \ell_2^4(x_1 \ldots \hat{x}_i \ldots x_7)$

$$+ \ \ell_1^4(\ldots) = 0\ (\mathcal{L}\mathcal{D})$$

$$\rightsquigarrow \ \sum 2 \times I_{3,1}(-,-) = \mathrm{Li}_4\text{'s}.$$

Know $I_{3,1}$ doesn't reduce individually, but what about

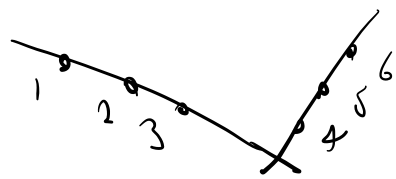
$$I_{3,1}(x,y) + I_{3,1}(1-x,y) \overset{?}{=} \{Li's\}$$

$$I_{3,1}(x,y) + I_{3,1}\left(\tfrac{1}{x},y\right) \overset{?}{=} \{Li's\}$$

Yes – Zagier / (Gangl).

Conceptually? Goncharov–Rudenko (slight reformulation here...) $\boxed{\S 6\ GR}$

Need degenerations to stable curves: (describing points in Deligne–Mumford compactification of $M_{0,n}$.)



Roughly $x_1 x_2 x_3$ collide, and $x_4 x_5 x_6$ collide which leads to $x_1 x_2 x_3$ being infinitesimally close from viewpoint of $x_4 x_5 x_6$

So $\sigma(x_1 x_4 x_5 x_6)$
$= \sigma(x_2 x_4 x_5 x_6)$
$= \sigma(x_3 x_4 x_5 x_6)$
on $123 \cup 456$.

limits: set

$x_1 \to \lambda x_1$ $\quad x_4 \to \mu x_4$
$x_2 \to \lambda x_2$ $\quad x_5 \to \mu x_5$
$x_3 \to \lambda x_3$ $\quad x_6 \to \mu x_6$.

and take limit to $\mu \to 0$.

On $135 \cup 246$ we find

$$I_{3,1}(1,x) = -2I_4(x) - I_4(1-x) + I_4(1-\tfrac{1}{x})$$

Setting $x_7 = x_2$, then $x_3 = x_1$

$$\to \quad I_{3,1}(B,C) \equiv I_{3,1}\left(\frac{1}{1-B}, \frac{1}{1-C}\right)$$

mod Li_4

$x_7 = x_2$, then $134 \cup 256$

$$\to \quad I_{3,1}\left(\frac{1}{C}\frac{1}{B}\right) = -I_{3,1}(BC)$$

mod Li_4

(GR say $x_7 = x_0$ gives
$I_{3,1}(x,y) = -I_{3,1}(y,x)$, but
for me this is implicit in convolution / $Li_{2,1,1}(\cdot,\cdot)$ definition.)

More interesting: $x_7 = x_2$, $x_4 = x_1$

$\leadsto$

$$-I_{3,2}(B, C) - I_{3,2}\left(B, \frac{C}{C-1}\right)$$

$$- I_{3,2}\left(-\frac{B}{(1-B)C}, -\frac{B(1-C)}{C}\right)$$

$$+ I_{3,2}\left(-\frac{B(1-C)}{C}, -\frac{(1-B)C}{B}\right)$$

$$= 0 \quad (\text{Li's})$$

At $B = \frac{1}{1-x}$, $C = \frac{1}{y}$ get

$$I_{3,2}(x, 1-y) + I_{3,2}(x, y)$$

$\mathrm{low}(x, y)$ " $+ I_{3,2}\left(\frac{x}{y}, \frac{1-x}{1-y}\right) + I_{3,2}\left(\frac{x}{y}, \frac{1-y}{1-x}\right)$

$= \text{Li's}.$

Li's can relate $I_{3,2}(\cdot, \text{one-minus})$
to $I_{3,2}(\cdot, \text{one over})$

It turns out one can write

$$I_{3,2}(x, y) + I_{3,2}(x, 1-y)$$

$$= \frac{1}{2}\mathrm{low}\left(\frac{1}{1-x}, \frac{1}{1-y}\right) + \frac{1}{2}\mathrm{low}\left(\frac{1}{x}, \frac{1}{y}\right)$$

[GR use projective involution, to see sym + antisym here $= 0$]

So get 2-term id's!

Finally: idea for $I_{3,1}$(3-term, 2)?

Write special term:

$$I_{3,1}(1234,\ 2345 + \text{cycle } 2\to3\to4\to5\to6) =: F(123456)$$

Then

i) this has "internal" syms via 2-term ids, namely sym in 234, antisym in 56

ii) From D72, see "exotic" sym

$$F(651234) = F(312456).$$

Now:
$$\begin{aligned}
&F(651234) \quad \text{\} above}\\
&= F(312456)\\
&\qquad\text{sym}\\
&= F(341256) \quad \text{\} above}\\
&= F(512643)\\
&= -F(561234)
\end{aligned}$$

So $F(ab\_\_\_) = -F(ba\_\_\_)$

But then
$$
\begin{aligned}
& abc\_\_\_ \\
&= -bac\_\_\_ \\
&= -bca\_\_\_ \\
&= cba\_\_\_ \\
&= cab\_\_\_ \\
&= \cdots = bac\_\_\_
\end{aligned}
$$

so sym + antisym in ab $\Rightarrow$ trivial

---

wt 5: Expect $I_{4,1}(x,y) + I_{4,1}(x,\tfrac{1}{y}) \overset{\delta}{\Longrightarrow} Li_2(x) \wedge Li_3(y)$ to reduce weight

x = dilog fe, or y = trilog.

Many partial results.

True for x = 2-term, y = 3-term (inverse known)

True for $x = \text{five}(a_i, b)$, $y = b$

True for $x = \text{five}(c, b)$, $y = 1$.

True for $x = \text{five}(a,b)$, $y = \frac{1-a}{b}$

Reduce δ y = kummer to
$x = \sum 8 \times \text{five}$

Reduce δ y = 22-term to
$x = \sum \text{mag} \times \text{five}$.

---

wt 6: $\quad Li_{3,1,1}(x\,y\,z) \xrightarrow{\bar{\delta}} \left(Li_2(x) \wedge Li_2(y)\right) \wedge Li_2(z) + Li_2(x) \wedge \left(Li_2(y) \wedge Li_2(z)\right)$

Andrei showed

$$Li_{3,1,1}(5\text{-ter}, y, z) = \sum Li_{3,1,1}\left(x + \tfrac{1}{a}, *, *\right) + \text{depth } 2.$$

1 type of 15 syms

C showed (recently!)

$$\begin{cases} Li_{3,1,1}(x\,y\,z) + Li_{3,1,1}(1-x\,y\,z) = dp\,2 \\ Li_{3,1,1}(x\,y\,z) + Li_{3,1,1}(\tfrac{1}{x}\,y\,z) = dp\,2. \end{cases}$$

[in all slots!]

Similar to $\mathcal{L}_3$, more intricate, but suggestive of some general structures...