New depth reductions in weight 6

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15:50 / 5th talk
[expanded notes]

[Polylogarithms, Cluster Algebras & Scattering Amplitudes
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[Notation, the same as in Rudenko's minicourse. Recalled partly, for convenience]

§ 1 Introduction / notation

Multiple polylogarithm (MPL) is

$$\mathrm{Li}_{n_1,\dots,n_k}(a_1,\dots,a_k) = \sum_{0<m_1<\dots<m_k} \frac{a_1^{m_1}\cdots a_k^{m_k}}{m_1^{n_1}\cdots m_k^{n_k}}$$

with $n_1,\dots,n_k \in \mathbb{N}$, $|a_i| < 1$
(can weaken to be $|a_1 \cdots a_i| < 1$)

Depth: k (= # args), weight: $a_1 + \dots + a_k$.

MPL's are iterated integrals via

$$\mathrm{Li}_{n_1,\dots,n_k}(a_1,\dots,a_k)$$
$$= (-1)^k I(0; \underbrace{\tfrac{1}{a_1\cdots a_k}, 0, \dots, 0}_{n_1}, \underbrace{\tfrac{1}{a_2\cdots a_k}, 0, \dots, 0}_{n_2}, \dots, \underbrace{\tfrac{1}{a_k}, 0, \dots, 0}_{n_k}; 1)$$

where

$$I_{(\gamma)}(a;x_1,\ldots,x_N;b) = \int_{\substack{0<t_1<t_2<\\ \cdots<t_N<1}} \gamma^*\left(\frac{dt_1}{t_1-x_1}\right)\wedge\cdots\wedge\gamma^*\left(\frac{dt_N}{t_N-x_N}\right)$$

(along some path $\gamma:[0,1]\to\mathbb{C}$, from a to b.)

Zagier's Polylogarithm conjecture:

Predicts that all MPL's and classical (depth 1) polylogs compute the same cohomology (for a number field)

(motivic cohomology = K-theory)

i.e.

$$\begin{array}{ccccccccc} \mathcal{L}_n & \xrightarrow{\Delta} & \Lambda^2\mathcal{L} & \to & \Lambda^3\mathcal{L} & \to & \cdots & \to & \Lambda^n F^\times \\ \cup & & \cup & & \cup & & & & \cup \\ \mathcal{B}_n & \xrightarrow{\Delta} & \mathcal{B}_{n-1}\otimes F^\times & \to & \mathcal{B}_{n-2}\otimes\Lambda^2 F^\times & \to & \cdots & \to & \Lambda^n F^\times \end{array}$$

are quasi-isomorphic.

[$\mathcal{L}$ = complex of motivic MPL's
$\mathcal{B}$ = classical polylogs (depth 1)]

⌜ More pedestrian version is that

$$\zeta_F(s) = \sum_{\substack{I \subset \mathcal{O}_F \\ \text{non-zero ideal}}} \frac{1}{N(I)^s} \quad \curvearrowleft \#(\mathcal{O}_F / I)$$

"Dedekind zeta", can be expressed via a single-valued version of Li_m. [extending the analytic class number formula when $s = 1$.]

<u>Goncharov's depth conjecture</u>:

Framework for understanding / determining when a (combination of) MPL's is depth 1 (or more generally depth d), as a route to tackling Zagier's conjecture.

Namely: $\quad \mathcal{D}_k \mathcal{L}_n :=$ depth k, wt n MPL's.

Then $\quad \Delta(\mathcal{D}_k) \subseteq \sum_{\substack{i+j=k \\ 1 \leq i,j}} \mathcal{D}_i \wedge \mathcal{D}_j$

motivic coproduct (cobracket on $\mathcal{L}$, written δ)

Goncharov conjectures

$(\mathrm{gr}^D \mathcal{L}, \overline{\Delta})$ is cofree Lie coalgebra on $D_1 = B_2 \oplus B_3 \oplus B_4 \oplus \cdots$

truncated w/o weight 1

In weight 6, depth 3: Prediction

$$(*) \quad D_3\mathcal{L}_6 \big/ D_2\mathcal{L}_6 \cong B_2 \otimes S^2 B_2 \big/ S^3 B_2$$

Explicitly in wt 6, dp 3?

The coproduct on integrals is easier to describe than on MPL's [cf. Goncharov semicircular coproduct]

The (iterated) coproduct on

$$\mathrm{Li}_{3;1,1}(x,y,z) := -I(0; \underbrace{000}_{3} \tfrac{1}{xyz}\, \tfrac{1}{yz}\, \tfrac{1}{z}; 1)$$

is given by

$$\overline{\Delta}^{[2]} \mathrm{Li}_{3;1,1}(x,y,z) = \mathrm{Li}_2(x) \otimes \mathrm{Li}_2(y) \wedge \mathrm{Li}_2(z)$$

(mod symmetric terms)

This shows surjectivity of (*). For well-definedness we need the 6-fold symmetries (generated by $x \mapsto 1-x$, $x \mapsto \frac{1}{x}$), i.e.:

$$\left\{ \begin{array}{l} \mathrm{Li}_{3;1,1,1}(a,b,c) + \mathrm{Li}_{3;1,1,1}(a,b,1-c) \in D_2 \\ \mathrm{Li}_{3;1,1,1}(a,b,c) + \mathrm{Li}_{3;1,1,1}(a,b,c^{-1}) \in \end{array} \right.$$

[and in slots 1, and 2 also]

Actually we even need more:

$$\mathrm{Li}_{3;1,1,1}(a,b,\underbrace{\text{5-term in } c,d}) \in D_2$$

[Some version of 5-term dilogarithm relation, say

$$\mathrm{Li}_2(c) + \mathrm{Li}_2(d) + \mathrm{Li}_2\left(\frac{1-c}{1-cd}\right) + \mathrm{Li}_2(1-cd) + \mathrm{Li}_2\left(\frac{1-d}{1-cd}\right) = 0 \text{ (products)}]$$

Results:

Thm (Matveiakin–Rudenko, 2022)

$$\mathrm{Li}_{3;1,1,1}(a,b,\text{5-term in } c,d)$$
$$= \sum \text{depth } 2$$

equivalent up to depth 2 Private Communication CGaRa → MRn

$+ \sum$ functs of the form
$$\mathrm{Li}_{3;1,1}(x,y,z) + \mathrm{Li}_{3;1,1}(x,y,1-z)$$

$+ \sum$ functs of the form
$$\mathrm{Li}_{3;1,1}(x,y,z) + \mathrm{Li}_{3;1,1}(x,y,\tfrac{1}{z})$$

i.e. By assuming the 6-fold syms of dilogarithm, Matveiakin-Rudenko reduced the S-term to depth 2. However, the symmetries were not proven.

Thm (C, 2023)

$$\mathrm{Li}_{3;1,1}(a,b,c) + \mathrm{Li}_{3;1,1}(a,b,\tfrac{1}{c}),$$
$$\mathrm{Li}_{3;1,1}(a,b,c) + \mathrm{Li}_{3;1,1}(a,b,1-c)$$
$$\in D_2.$$

So the 6-fold symmetries hold.

Cor: Goncharov's depth conjecture holds in wt 6, dp 3.

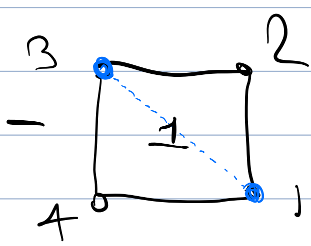
§2 Sketch of proof

We need the depth 3 QLi function.

Rudenko described depth 1 & depth 2 in the minicourse

$$QLi_n(x_1,\dots,x_4) = -Li_{n-1;1}([1234])$$

$\longleftrightarrow$ pictorially $-$

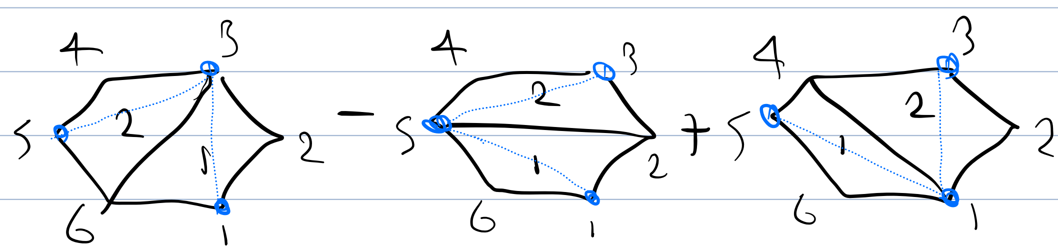


where $(1234) = [x_1 x_2 x_3 x_4] = \dfrac{(x_1-x_2)(x_3-x_4)}{(x_2-x_3)(x_4-x_1)}$

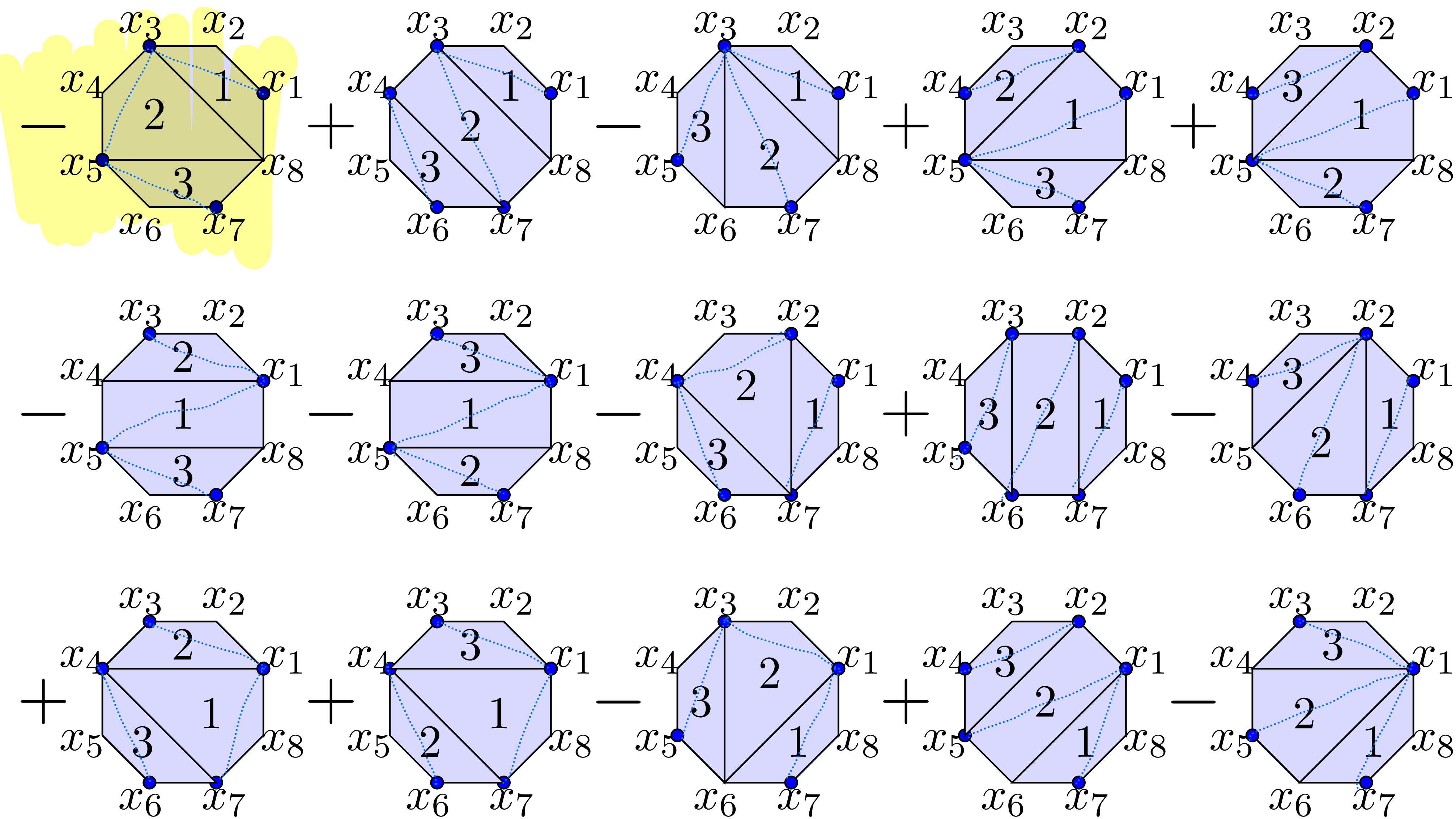
$$QLi_n(x_1,\dots,x_6)$$

$$= Li_{n-2;1,1}([1236],[3456]) - Li_{n-2;1,1}([1256],[3452]) + Li_{n-2;1,1}([1456],[1234])$$

$\longleftrightarrow$ pictorially

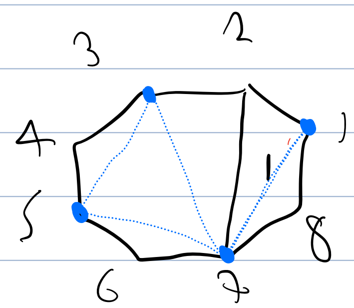


$f_3(x_1, \ldots, x_8)$ definition = depth 3 part of $QLi_6(x_1, \ldots, x_8)$



Above is the depth 3 part of $QLi_8(x_1,\dots,x_8)$. Some depth 2 terms like

$$Li_{1;12}\big([1278],\underbrace{[345672]}_{\frac{(x_3-x_4)(x_5-x_6)(x_7-x_2)}{(x_4-x_5)(x_6-x_7)(x_2-x_3)}}\big) \longleftrightarrow$$



are neglected (since we only care about depth 3 relations anyway).

First diagram represents

$$-Li_{3;111}([1238],[3458],[5678])$$

We have some basic results just from shuffle | stuffle & symmetries of motivic correlators $\approx I(\infty;\dots;b)$

$\approx Li_{n;1\dots1}(\dots)$ (mod lower depth)

Eg:

$$Li_{3;111}(a,b,c) = Li_{3;111}(c,b,a) \quad \text{(depth 2)}$$

"reversal"

$$Li_{3;111}(a,b,c) = Li_{3;111}\left(\tfrac{1}{a},\tfrac{1}{b},\tfrac{1}{c}\right) \quad \text{(depth 2)}$$

"inversion/parity"

To obtain more, we degenerate the wt 6 dp 3 relation

$$\sum (-1)^i QLi_6(x_1, \ldots \hat{x_i}, \ldots x_a)$$
$$= \sum QLi \text{ depth } 1 \;\&\; QLi \text{ depth } 2.$$

$\longleftrightarrow$

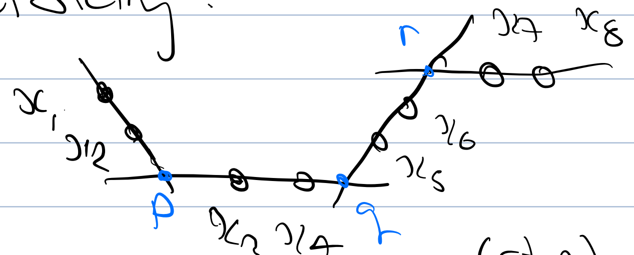
$$\sum (-1)^i f_3(x_1, \ldots, \hat{x_i}, \ldots, x_a) = \text{depth } 2.$$

to boundary components of $\overline{M}_{0,9}$, divisors and stable curves.

Idea: We let points $x_{i_1}, \ldots, x_{i_a}$ collide, which splits off a copy of $\mathbb{P}^1$ where $x_{i_1}, \ldots, x_{i_a}$ are infinitesimally close.

[Otherwise, there is always a projective transformation moving $x_{i_1}, \ldots, x_{i_a}$ to $0, 1, \infty, \ldots$ i.e. far apart!]

Pictorially:



Here x_1, x_2 collide (at p), x_5, x_6, x_7, x_8 collide (at q) and x_7, x_8 further collide (at r)

[We can compute these degenerations via limits

$$\begin{cases} x_1 = \lambda x_1' + p \\ x_2 = \lambda x_2' + p \end{cases}$$

$$\begin{cases} x_5 = \mu x_5' + q \\ x_6 = \mu x_6' + q \end{cases}$$

$$\begin{cases} x_7 = q + \mu(r + \nu x_7') \\ x_8 = q + \mu(r + \nu x_8') \end{cases}$$

as $\lambda, \mu, \nu \to 0$. For cross-ratios the results are always well-defined. For more complicated functions (appearing in dp 2 arguments), results are dependent on order/parametrisation/..., but no issue, as dp ≤ 2]

Then using $Li_{3;1,1,1}(\infty, x, y) = 0$

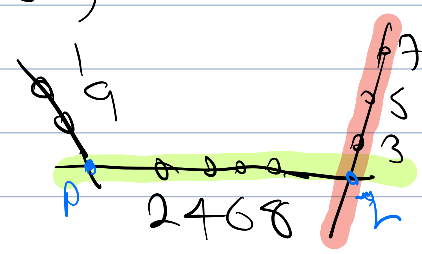
$Li_{3;1,1,1}(0, x, y) = \text{depth } 2,$

we can simplify/eliminate terms in $\beta_3|_C$, when β_3 is restricted to some stable curve.

Recall relation: $\sum (-1)^i \beta_3(x_1, \dots, \hat{x}_i, \dots, x_9) = \text{depth } 2$

Eg (~ reversal + inverse)

Specialise to $C =$



$$= 19 \cup_p 2468 \cup_q 357$$

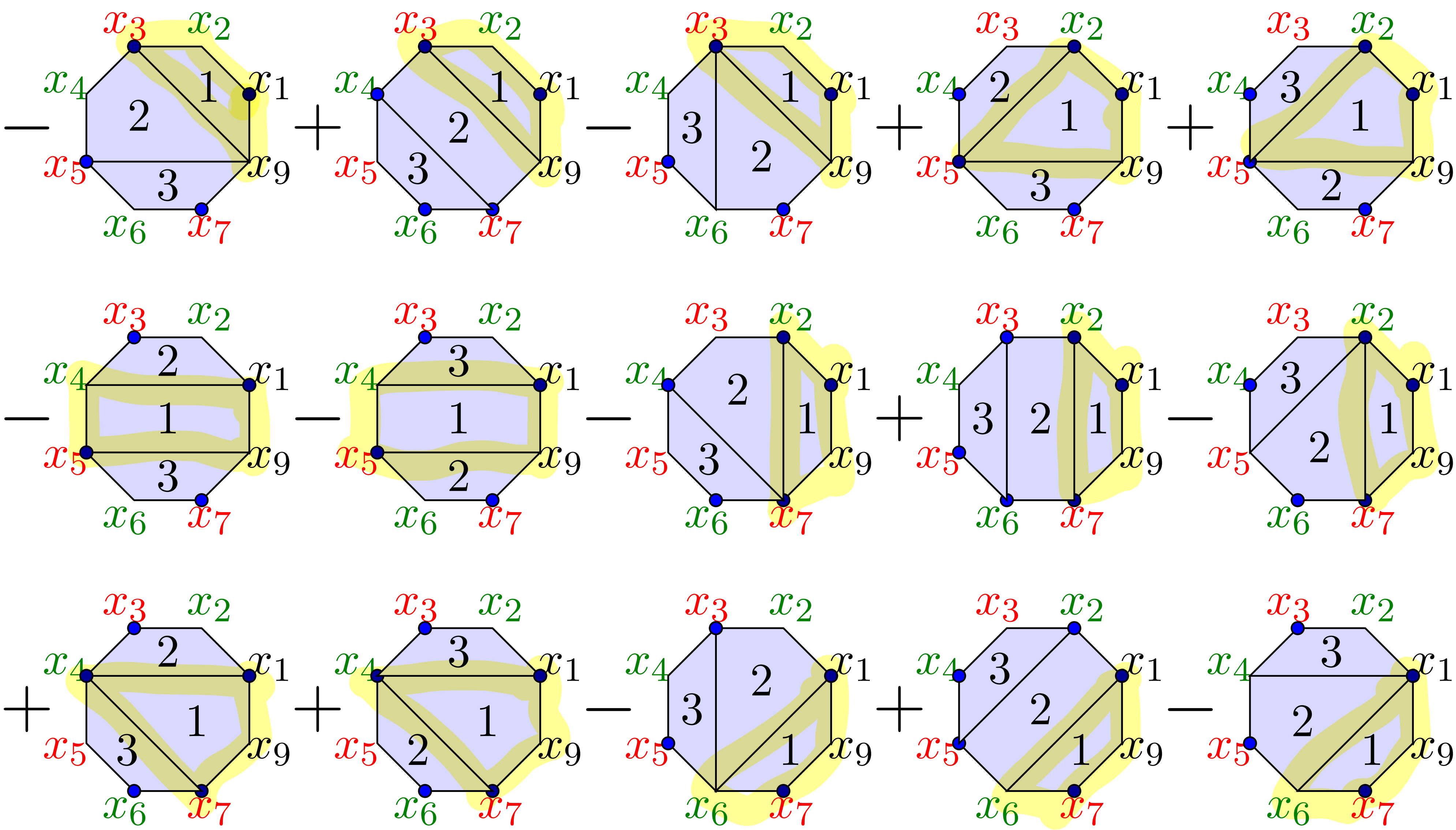
On C,

$$\beta_3(x_1, \dots, \hat{x}_i, \dots, x_9) = 0 \quad (\text{mod depth } 2)$$

for $i = 2, 3, \dots, 8$

As every term contains argument w/ consecutive entries x_1, x_9, one entry from x_3, x_5, x_7, and one from x_2, x_4, x_6, x_8

$f_3(x_1, \ldots, \widehat{x_i}, \ldots, x_9)$ on $19 \cup_p 2468 \cup_q 357$



Cross-ratios of this form ($c_1 = 3, 5, 7$, $c_2 = 2, 4, 6, 8$)

$$[1\ 9\ c_1\ c_2] = 0 \text{ or } \infty$$

on C. [As from the point of view of c_1, c_2, $x_1 = x_9 = p$, so l.o.s. $x_1 - x_9 = \lambda(x_1 - x_9)$
$\longrightarrow 0$ as $\lambda \to 0$.]

Whereas on C:

$$\ell_3(x_1, \ldots, x_8) = -Li_{3;111}([p6q8],[p4q6],[p2q4])$$

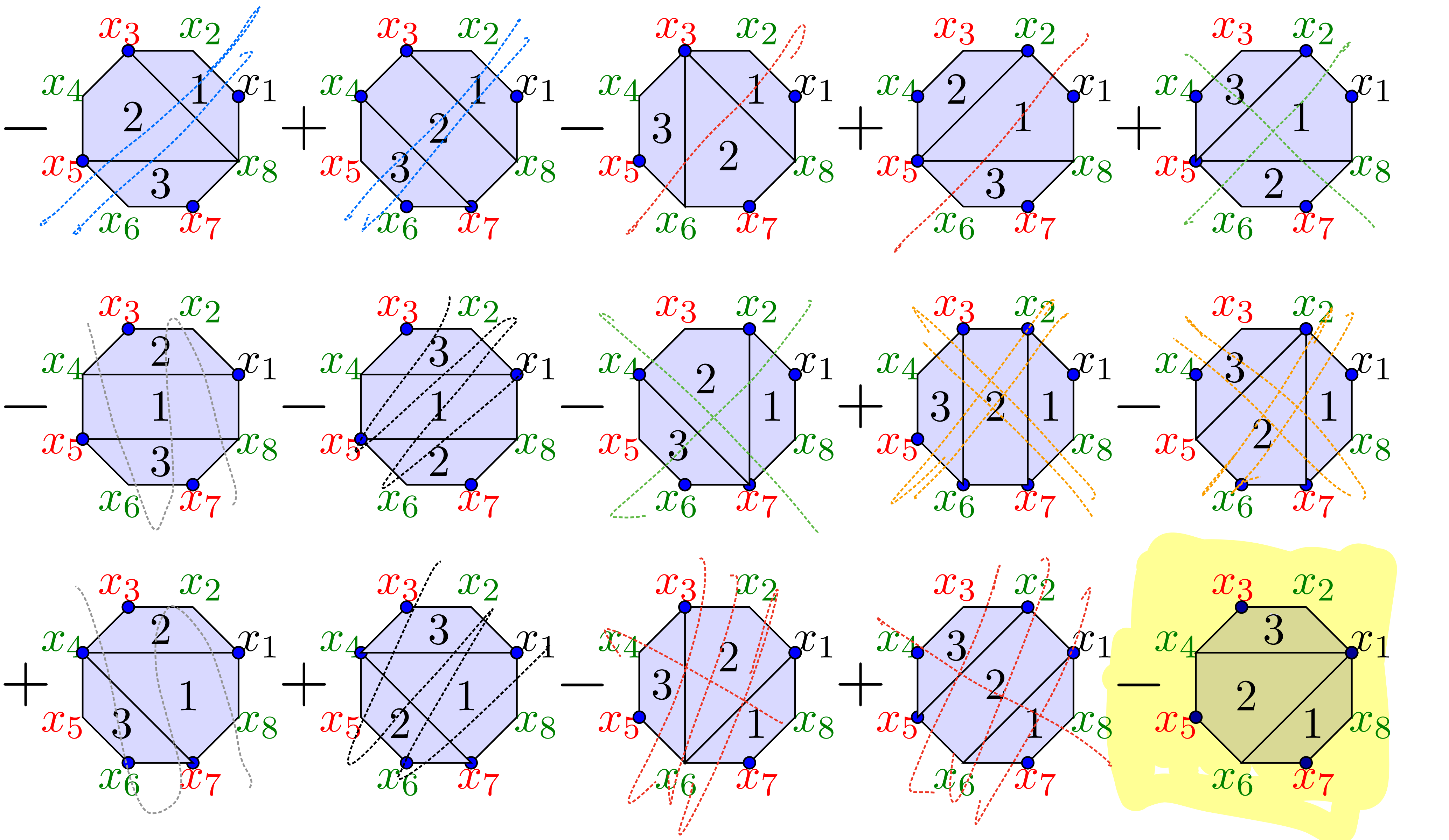
$$\ell_3(x_2, \ldots, x_9) = -Li_{3;111}([2q4p],[4q6p],[6q8p])$$

[Pairwise cancellation of terms, as illustrated below.]

Term 1 $\rightsquigarrow -Li_{3;111}([1238], 1, 1)$ } cancels.
Term 2 $\rightsquigarrow +Li_{3;111}([1238], 1, 1)$

as for example $[x_3x_4x_5x_8] = [q4q8] = 1$ on C.

$f_3(x_1,\ldots,x_8)$ on $19 \cup_p 2468 \cup_q 357$



So $\rightsquigarrow$ On C, relation becomes

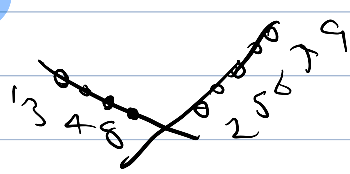
$$\mathrm{Li}_{3;111}(a,b,c) + \mathrm{Li}_{3;111}\left(\frac{1}{c},\frac{1}{b},\frac{1}{a}\right) = \text{depth } 2$$

⌜The depth 2 terms can explicitly be found by degenerating the rest of $\mathrm{QLi}_8(x_1,\dots,x_8)$ + depth 2 + depth 1 to the same curve C⌟

We use this idea to find more & more useful identities, to finally obtain our Thm.

Step 1: "three term relation"

On $1348 \cup 25679$



$\rightsquigarrow$

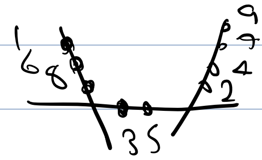
$$\mathrm{Li}_{3;111}(a,b,c) + \underbrace{\mathrm{Li}_{3;111}(b,a,c)}_{\equiv \mathrm{Li}_{3;111}(c,a,b) \text{ by reversal}} + \mathrm{Li}_{3;111}(b,c,a) = \text{depth } 2$$

$\rightsquigarrow$

$$\mathrm{Li}_{3;111}(a,b,c) + \mathrm{Li}_{3;111}(b,c,a) + \mathrm{Li}_{3;111}(c,a,b) = \text{depth } 2.$$

Step 2: Degeneration w/ two 1's.

On $168 \cup 35 \cup 2479$



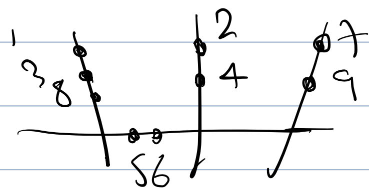
$\rightsquigarrow \quad Li_{3,1,1}(1,1,x) = 0$

[with x in other slots via reversal, then three-term.]

Step 3: Degeneration w/ one 1.

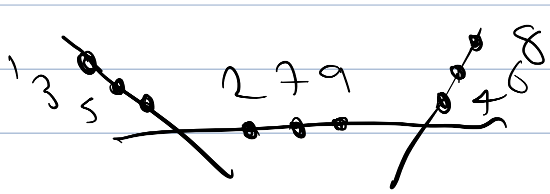
Write $g(x_1, \ldots, x_5) = Li_{3,1,1}(1, [3142], [5132])$

Step 3a: On



$\rightsquigarrow \quad g(abcde) \equiv g(bedca)$ (mod $dp2$)

Step 3b: On $135 \cup 279 \cup 468$



$$\rightsquigarrow \quad g(abcde) \equiv -g(dcbae) \pmod{dp\ 2}$$

[Inversion] parity gives $g(abcde) \equiv -g(bacde) \pmod{dp 2}$

So:

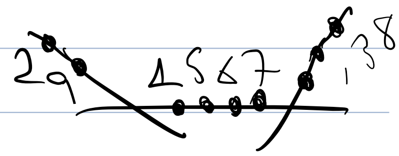
$$\begin{aligned} g(abcde) &\overset{\text{inv}}{\equiv} -g(bacde) \\ &\overset{a}{\equiv} g(dcabe) \\ &\overset{3\times b}{\equiv} g(dcbae) \\ &\overset{a}{\equiv} -g(abcde) \pmod{dp\ 2} \end{aligned}$$

So g is trivial modulo depth 2,

ie. $\mathrm{Li}_{3;1,1}(1,x,y) =$ depth 2.

Step 4: Some symmetries & relations

On $29 \cup 4567 \cup 138$



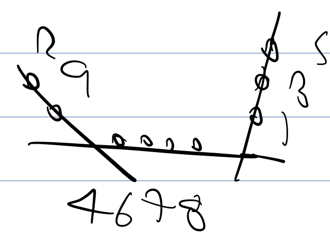
$$\rightsquigarrow \mathrm{Li}_{3;111}(a,b,c) \equiv -\mathrm{Li}_{3;111}\left(1-a, \frac{b}{b-1}, 1-c\right)$$

(mod dp 2)

(w/ inversion + reversal, get 12 syms, Similar to GoRu Lemma 6.3c)

On $29 \cup 4678 \cup 135$

"Four term relation"



$$\mathrm{Li}_{3;111}\left(a, \frac{1}{b}, c\right) + \mathrm{Li}_{3;111}\left(a, \frac{1}{b}, 1-c\right)$$

$$\equiv \mathrm{Li}_{3;111}\left(a, \frac{c}{b}, \frac{1-c}{1-b}\right) + \mathrm{Li}_{3;111}\left(a, \frac{c}{b}, \left(\frac{1-c}{1-b}\right)^{-1}\right)$$

Similar to GoRu Lemma 6.3d)

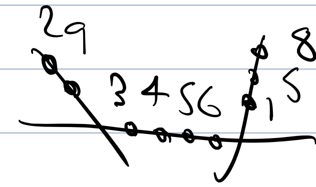
NB: This shows the claim

$\mathrm{Li}_{3;111}(a,b,c) + \mathrm{Li}_{3;111}(a,b,1-c) =$ depth 2

$\iff \mathrm{Li}_{3;111}(a,b,c) + \mathrm{Li}_{3;111}(a,b,\tfrac{1}{c}) =$ depth 2

as used in MRu 5-term Theorem.

On $29 \cup 3456 \cup 158$



$\rightsquigarrow Li_{3;111}(a,b,c) + Li_{3;111}\left(a, 1-b, \frac{c-1}{c}\right)$

$= \text{depth } 2.$

[Explicitly needs $3 \times$ three term
$+ 3 \times$ four term
plus the 12 symmetries of $Li_{3;111}$
to fully simplify.]

Finally: play these symmetries against each other and the four-term relation

At this point, we have 216 symmetries already:

$$Li_{3;111}(a,b,c) \equiv \mathrm{sgn}(\pi\sigma\tau) Li_{3;111}(a^{\pi}, b^{\sigma}, c^{\tau})$$

$$\equiv \mathrm{sgn}(\pi\sigma\tau) Li_{3;111}(c^{\tau}, b^{\sigma}, a^{\pi\tau})$$

with $\mathrm{sgn}(\sigma\tau) = \mathrm{sgn}(\tau)$, any σ.

$\ulcorner$ viewing $x \mapsto x,\ 1-x,\ \frac{1}{x},\ \frac{x}{x-1},\ \frac{1}{1-x},\ \frac{x-1}{x}$

id (23) (12) (13) (123) (132)

as S_3-permutations as is standard via permutations of arguments $1, 2, 3$ in $x = [0\,0\,1\,-x]$ $\lrcorner$

Since $1-c$, and $\frac{1}{c}$ have same signature (or $1-\frac{1}{c}$ has signature 1) we can rewrite the

$$Li_{3;j111}(\alpha, \beta, \gamma) + Li_{3;j111}(\alpha, \beta, \tfrac{1}{\gamma})$$

in far-term as

$$Li_{3;j111}(\alpha, \beta, \gamma) + Li_{3;j111}(\alpha, \beta, 1-\gamma)$$

(mod dp 2)

So far-term becomes

$$(\ast) \quad V(a, b, c) + V\left(a, \frac{c}{b}, \frac{1-c}{1-b}\right) = 0$$

(mod dp2)

w/ $V(a, b, c) := Li_{3;j111}(a, \tfrac{1}{b}, c) + Li_{3;j111}(a, \tfrac{1}{b}, 1-c)$

Since $(b,c) \longmapsto \left(\frac{c}{b}, \frac{1-c}{1-b}\right)$ is S-periodic, we get after S iterations of $(\quad)$

$$V(a,b,c) \equiv -V(a,b,c) \pmod{\mathrm{dp}\,2}$$

i.e. $V(a,b,c) \equiv 0 \pmod{\mathrm{dp}\,2}$, and so

$$\mathrm{Li}_{3;111}(a,b,c) + \mathrm{Li}_{3;111}(a,b,1-c) = \text{depth } 2.$$

And also

$$\mathrm{Li}_{3;111}(a,b,c) + \mathrm{Li}_{3;111}\left(a,b,\tfrac{1}{c}\right) = \text{depth } 2.$$

So the main result is proven.

Final note: In wt 4, the story from GoRu (strongly) used a projective involution to establish

$$\begin{cases} \mathrm{Li}_{2;11}(a,b) + \mathrm{Li}_{2;11}(a,1-b) = \text{depth } 1 \\ \mathrm{Li}_{2;11}(a,b) + \mathrm{Li}_{2;11}\left(a,\tfrac{1}{b}\right) = \text{depth } 1 \end{cases}$$

[Or equivalently with I_{31} instead GoRu Lemma 6.4]

This was necessary to split the GoRu four-term relation into its two pieces.

HOWEVER: One can obtain

$$\mathrm{Li}_{2;11}(a,b) + \mathrm{Li}_{2;11}(a,1-b)$$
$$= \text{depth } 1$$

from a (simple) sum of specialisations of the GoRu four-term relation.

No projective involution necessary!

⇝ Gives hope to try to establish some general results?