Symmetries of weight 6 multiple polylogarithms, and Goncharov's Depth Conjecture

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Mathematical aspects of $\mathcal{N}=4$ SYM

Outline

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Definitions

Definition

Multiple polylogarithm (MPL) is

$$\mathrm{Li}_{k_1,k_2,\ldots,k_d}(x_1,x_2,\ldots,x_d) := \sum_{0<m_1<m_2<\cdots<m_d} \frac{x_1^{m_1}\cdots x_d^{m_d}}{m_1^{k_1}\cdots m_d^{k_d}}, \quad |x_i|<1\,.$$

- Weight is $n = k_1 + \cdots + k_d$,
- Depth is d.

Goal: One of the most fundamental challenges around multiple polylogarithms is to understand the nature, structure and properties of the depth (filtration).

Relating different depths

Functions of different depths can be related:

$$\mathrm{Li}_1(x)\,\mathrm{Li}_1(y) = \mathrm{Li}_{1,1}(x,y) + \mathrm{Li}_{1,1}(y,x) + \mathrm{Li}_2(xy)\,.$$

- stuffle product of MPL's $\rightsquigarrow$ generalises easily

More surprising is when the depth reduces

Proposition (Goncahrov, Zagier, Lewin$^{(?)}$)

For $|xy| < 1, |y| < 1$,

$$\mathrm{Li}_{1,1}(x,y) = \mathrm{Li}_2\Big(\frac{y(x-1)}{1-y}\Big) - \mathrm{Li}_2\Big(\frac{-y}{1-y}\Big) - \mathrm{Li}_2(xy)\,.$$

as a power-series identity.

Further depth reductions

Proposition ($\mathrm{Li}_{1,1,1}$ is depth 1, Goncharov, Zhao, Lewin$^{(?)}$, ...)

$$\begin{aligned}\mathrm{Li}_{1,1,1}(x,y,z) = & -\mathrm{Li}_3\left(\tfrac{1-xyz}{1-x}\right) - \mathrm{Li}_3\left(\tfrac{1-xyz}{xy(1-z)}\right) + \mathrm{Li}_3\left(\tfrac{(y-1)(1-xyz)}{(1-x)y(1-z)}\right) + \mathrm{Li}_3(xy) \\ & - \mathrm{Li}_3\left(\tfrac{y(1-x)}{y-1}\right) + \mathrm{Li}_3(1-x) - \mathrm{Li}_3\left(\tfrac{y(1-z)}{y-1}\right) + \mathrm{Li}_3\left(\tfrac{-y}{1-y}\right) + \textit{products}\end{aligned}$$

Can weight 4 be reduced to depth 1? Apparently not, but:

Proposition ($\mathrm{Li}_{3,1}$ satisfies relations, modulo depth 1, Zagier, Gangl)

$$\begin{aligned}\mathrm{Li}_{3,1}\left(\tfrac{1-x}{y}, y\right) + \mathrm{Li}_{3,1}\left(\tfrac{x}{y}, y\right) = & -\tfrac{1}{2}\mathrm{Li}_4\left(\tfrac{(1-x)y}{x(1-y)}\right) - \tfrac{1}{2}\mathrm{Li}_4\left(\tfrac{xy}{(1-x)(1-y)}\right) + \tfrac{1}{2}\mathrm{Li}_4\left(\tfrac{(1-y)y}{(1-x)x}\right) \\ & - \mathrm{Li}_4\left(\tfrac{1-y}{1-x}\right) - \mathrm{Li}_4\left(\tfrac{1-y}{x}\right) + \mathrm{Li}_4\left(-\tfrac{y}{1-y}\right) \\ & - \mathrm{Li}_4(1-x) - \mathrm{Li}_4(x) + 2\,\mathrm{Li}_4(y) + \textit{products}\end{aligned}$$

Question: How to predict, understand, find and explain such reductions and obstructions?

Aim for today: What we currently know in weight 6.

Outline

Motivic multiple polylogarithms

Recall: Iterated integral

$$I_{(\gamma)}(a; x_1, \ldots, x_n; b) := \int_{a<t_1<\cdots<t_n<b} \frac{dt_1}{t_1 - x_1} \wedge \frac{dt_2}{t_2 - x_2} \wedge \cdots \wedge \frac{dt_n}{t_n - x_n}$$

and the expression for MPL's

$$\mathrm{Li}_{k_1,k_2,\ldots,k_d}(x_1, \ldots, x_d) = (-1)^d I(0; 1, \{0\}^{k_1-1}, x_1, \{0\}^{k_2-1}, x_1x_2, \{0\}^{k_3-1}, \ldots, x_1 \cdots x_{d-1}, \{0\}^{k_d-1}; x_1x_2 \cdots x_d) .$$

- Goncharov upgraded these to framed mixed Tate motives $I^{\mathfrak{u}}(a; x_1, \ldots, x_n; b)$ in connected graded Hopf algebra, with coproduct
- Define $\mathrm{Li}^{\mathfrak{u}}_{k_1,k_2,\ldots,k_d}(x_1, \ldots, x_d)$ via $I^{\mathfrak{u}}$
- Extend (via regularisation $I^{\mathfrak{u}}(0; 0; x) = \log^{\mathfrak{u}}(x)$) to

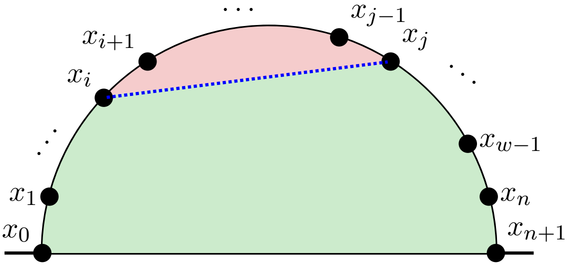
$$\mathrm{Li}^{\mathfrak{u}}_{k_0\,;\,k_1,\ldots,k_d}(x_1, \ldots, x_d) = (-1)^d I^{\mathfrak{u}}(0; \{0\}^{k_0}, 1, \{0\}^{k_1-1}, x_1, \{0\}^{k_2-1}, x_1x_2, \{0\}^{k_3-1}, \ldots, x_1 \cdots x_{d-1}, \{0\}^{k_d-1}; x_1x_2 \cdots x_d)$$

Lie coalgebra and cobracket

- $\mathcal{L}_\bullet(F)$ = complex of motivic multiple polylogs, modulo products (with values in F)

$$\Delta\colon \mathcal{L}_\bullet(F) \to \bigwedge^2 \mathcal{L}_\bullet(F)$$

$$\mathrm{I}^{\mathcal{L}}(x_0; x_1, \ldots, x_n; x_{n+1}) \mapsto \sum_{i<j} \mathrm{I}^{\mathcal{L}}(x_i; x_{i+1}, \ldots, x_{j-1}; x_j) \wedge \mathrm{I}^{\mathcal{L}}(x_0; x_1, \ldots, x_i, x_j, \ldots, x_n; x_{n+1})$$



Depth is motivic

- Filtration $\mathcal{D}_d = \{\text{depth} \leq d \text{ multiple polylogarithms}\}$

Then

$$\Delta(\mathcal{D}_d) \quad \subset \quad \mathcal{D}_d \wedge \mathcal{L}_1 \oplus \bigoplus_{\substack{i+j=d \\ i,j\geq 1}} \mathcal{D}_i \wedge \mathcal{D}_j \,.$$

Why?

Since

$$\mathrm{I}^{\mathcal{L}}(a; \overbrace{0, \ldots, 0}^{\geq 2}; b) = 0 \,,$$

any non-trivial weight ≥ 2 contribution to wedge must have ≥ 1 non-zero entries. This reduces the depth in the other factor. $\implies$

$$\begin{aligned}
\Delta \operatorname{Li}^{\mathcal{L}}_{3,1}(x,y) = {} & \log^{\mathcal{L}}(x) \wedge \operatorname{Li}^{\mathcal{L}}_{2,1}(x,y) - \operatorname{Li}^{\mathcal{L}}_{1}(y) \wedge \operatorname{Li}^{\mathcal{L}}_{3}(x) + \operatorname{Li}^{\mathcal{L}}_{1}(y) \wedge \operatorname{Li}^{\mathcal{L}}_{3}(xy) \\
& + \operatorname{Li}^{\mathcal{L}}_{1}(xy) \wedge \operatorname{Li}^{\mathcal{L}}_{3}(x) - \operatorname{Li}^{\mathcal{L}}_{1}(xy) \wedge \operatorname{Li}^{\mathcal{L}}_{3}(y) - \operatorname{Li}^{\mathcal{L}}_{2}(y) \wedge \operatorname{Li}^{\mathcal{L}}_{2}(xy)
\end{aligned}$$

$$\Delta \operatorname{Li}^{\mathcal{L}}_{4}(x) = \log^{\mathcal{L}}(x) \wedge \operatorname{Li}_{3}(x) \,.$$

Truncating and iterating the cobracket

- Write $\overline{\Delta}$ to mean ignore $\mathcal{L}_1$. Expressions become simpler

$$\overline{\Delta}\,\mathrm{Li}^{\mathcal{L}}_{3,1}(x,y) = -\,\mathrm{Li}^{\mathcal{L}}_2(y) \wedge \mathrm{Li}^{\mathcal{L}}_2(xy)\,, \qquad \overline{\Delta}\,\mathrm{Li}^{\mathcal{L}}_4(x) = 0\,.$$

(So $\mathrm{Li}^{\mathcal{L}}_{3,1}(x,y) \neq \sum \mathrm{Li}^{\mathcal{L}}_4$'s.)

- However:

$$\overline{\Delta}\,\mathrm{Li}^{\mathcal{L}}_{4,1,1}(x,y,z) = -\,\mathrm{Li}^{\mathcal{L}}_2(y) \wedge \mathrm{Li}^{\mathcal{L}}_{3,1}(xy,z) + \mathrm{Li}^{\mathcal{L}}_{3,1}\Big(yz, \tfrac{1}{y}\Big) \wedge \mathrm{Li}^{\mathcal{L}}_2(xyz) - \mathrm{Li}^{\mathcal{L}}_3(y) \wedge \mathrm{Li}^{\mathcal{L}}_{2,1}(xy,z)$$
$$-\,\mathrm{Li}^{\mathcal{L}}_{2,1}\Big(yz, \tfrac{1}{y}\Big) \wedge \mathrm{Li}^{\mathcal{L}}_3(xyz) + \mathrm{Li}^{\mathcal{L}}_4\text{'s} \wedge \text{weight } 2 + \mathrm{Li}^{\mathcal{L}}_n\text{'s} \wedge \mathrm{Li}^{\mathcal{L}}_m\text{'s}\,.$$

- Iterating $\overline{\Delta}^{[2]} := \overline{\Delta} \circ (\overline{\Delta} \otimes \mathrm{id})$ picks out genuine "depth 1 $\wedge$ depth 2" part

$$\overline{\Delta}^{[2]}\,\mathrm{Li}^{\mathcal{L}}_{4,1,1}(x,y,z) = -\,\mathrm{Li}^{\mathcal{L}}_2(y) \otimes \mathrm{Li}^{\mathcal{L}}_2(z) \otimes \mathrm{Li}^{\mathcal{L}}_2(xyz) \pmod{\text{shuffles}}\,.$$

And generally $\Delta^{[2]}\,\mathrm{Li}^{\mathcal{L}}_{a,b}(x,y) = 0$. (So $\mathrm{Li}^{\mathcal{L}}_{4,1,1}(x,y,z) \neq \sum$ depth 2.)

The Depth Conjecture

- $\mathrm{CoLie}_n(\underbrace{\mathcal{A}}_{\mathbb{Q}\text{-vector space}}) = \mathcal{A}^{\otimes n}/ ⧢$, so $\mathrm{CoLie}_2(\mathcal{A}) = \bigwedge^2\mathcal{A}$, $\mathrm{CoLie}_3(\mathcal{A}) = \mathrm{Sym}^2\mathcal{A}\otimes\mathcal{A}/\mathrm{Sym}^3\mathcal{A}$, ...
- Define $\overline{\Delta}^{[k]} := \overline{\Delta} \circ (\overline{\Delta}^{[k-1]} \otimes \mathrm{id}))$, and projected to CoLie_{k+1}
- $\mathcal{B}_\bullet(F)$ = classical polylogarithms (depth 1), modulo products (with values in F)

Goncharov Depth Conjecture (precise version)

For $k \geq 2$, the following map is an isomorphism:

$$\overline{\Delta}^{[k-1]}\colon \mathrm{gr}_k^{\mathcal{D}}\,\mathcal{L}(F) \xrightarrow{\cong ?} \mathrm{CoLie}_k\Big(\bigoplus_{n\geq 2}\mathcal{B}_n(F)\Big)\,.$$

Goncharov Depth Conjecture (simplified version)

A linear combination of MPL's has depth $< k$ if and only if the $(k-1)$-st iterated truncated cobracket $\overline{\Delta}^{[k-1]}$ vanishes.

Expectations and consequences (I) – Unobstructed case

Observation

If $k > \frac{n}{2}$, then

$$\overline{\Delta}^{[k-1]}\mathcal{L}_n = 0$$

Why? Because k many factors of weight 2 gives total weight $2k > n$.

Theorem (Rudenko, 2022)

Every weight n multiple polylogarithm can be expressed via depth $\leq \lfloor \frac{n}{2} \rfloor$ functions. Hence $\mathrm{gr}_k \mathcal{L}_n = 0$, if $k > \frac{n}{2}$.

Idea. Rudenko gave explicit representation via quadrangular polylogarithms $\mathrm{QLi}_n(x_1, \ldots, x_{2n+2})$, with depth $\frac{n}{2}$

Expectations and consequences (II) – First obstructed case

Proposition

$$\overline{\Delta}^{[k-1]} \underbrace{\mathrm{Li}^{\mathcal{L}}_{k\,;\,1,\ldots,1}(x_1,\ldots,x_k)}_{:=(-1)^k\,\mathrm{I}^{\mathcal{L}}(0;\{0\}^k,x_1,x_1x_2,\ldots;x_1x_2\cdots x_k))} = -\,\mathrm{Li}^{\mathcal{L}}_2(x_1)\otimes\cdots\otimes\mathrm{Li}^{\mathcal{L}}_2(x_k) \pmod{\textit{shuffles}}\,.$$

So $\mathrm{Li}^{\mathcal{L}}_{k\,;\,1,\ldots,1}$ behaves like $\mathrm{Li}^{\mathcal{L}}_2$ in each argument, and should reduce in depth under $\mathrm{Li}^{\mathcal{L}}_2$ functional equations.

Expectations/conjectures:

- $\mathrm{Li}^{\mathcal{L}}_{k\,;\,1,\ldots,1}(1,x_2,\ldots,x_k) \overset{?}{\equiv} 0 \pmod{\text{depth} < k}$ (Nielsen-type)
- $\mathrm{Li}^{\mathcal{L}}_{k\,;\,1,\ldots,1}(x_1,x_2,\ldots,x_k) + \mathrm{Li}^{\mathcal{L}}_{k\,;\,1,\ldots,1}(\underbrace{1-x_1}_{\text{or } x_1^{-1}},x_2,\ldots,x_k) \overset{?}{\equiv} 0 \pmod{\text{depth} < k}$ (Zagier-type)
- $\mathrm{Li}^{\mathcal{L}}_{k\,;\,1,\ldots,1}(\underbrace{V(y,z)}_{\text{dilogarithm 5-term relation}},x_2,\ldots,x_k) \overset{?}{\equiv} 0 \pmod{\text{depth} < k}$ (Gangl-type)

Weight 4

These results are known in weight 4

Proposition (Nielsen-type reduction, Kölbig, Lewin, Wojtkowiak)

$$\mathrm{Li}^{\mathcal{L}}_{2;1,1}(1,x) = \mathrm{I}^{\mathcal{L}}(0;0,0,x^{-1},x^{-1};1) = -S^{\mathcal{L}}_{2,2}(x) \longleftarrow$$

Nielsen polylogarithm $S_{2,2}(x) =: I(0;1,1,0,0;x)$

$$= \mathrm{Li}^{\mathcal{L}}_4(1-x) - \mathrm{Li}^{\mathcal{L}}_4(x) - \mathrm{Li}^{\mathcal{L}}_4\Big(\frac{x}{x-1}\Big)$$

Proposition (Zagier-type reduction, Zagier ≤2000, Gangl)

Related to the $\mathrm{Li}_{3,1}(\frac{1-x}{y}, y) + \mathrm{Li}_{3,1}(\frac{x}{y}, y) = \sum \mathrm{Li}_4$*'s identity from introduction*

Theorem (Gangl-type reduction, Gangl 2012, Goncharov-Rudenko, Matveiakin-Rudenko)

$$\mathrm{Li}^{\mathcal{L}}_{2;1,1}(\underbrace{V(y,z)}_{-[y]+[z]-[\frac{z}{y}]+[\frac{1-z}{1-y}]-[\frac{y(1-z)}{(1-y)z}]}, x) = \sum_{i=1}^{122+\varepsilon} \mathrm{Li}^{\mathcal{L}}_4(f_i(x,y,z)), \quad f_i(x,y,z) \in \mathbb{Q}(x,y,z)$$

Weight 6

These results are now also known in weight 6

Theorem (Gangl-type reduction, Matveiakin-Rudenko 2022)

$$\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(V(y,z),x_2,x_3) \equiv 0 \pmod{\textit{depth} \leq 2 \ \&\ \textit{the Zagier reductions}}$$

Theorem (Nielsen-type & Zagier-type reductions, C 2023/4+)

$$\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(1,y,z) \equiv 0 \pmod{\text{depth} \leq 2}$$

$$\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(x,y,z) + \mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(1-x,y,z) \equiv 0 \pmod{\text{depth} \leq 2}$$
$$\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(x,y,z) + \mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(x^{-1},y,z) \equiv 0 \pmod{\text{depth} \leq 2}$$

Consequence: Goncharov's depth conjecture holds for weight 6 depth 3 $(k=3)$.

(Zagier-reductions seem more difficult than the Gangl-reduction in general!)

Outline

Construction of $\mathrm{QLi}_{n+k}(x_0, \ldots, x_{2n+1})$

- Family of nice multiple polylogarithm functions on $\mathfrak{M}_{0,2n+2}$ with weight $w = n + k$

Primary definition via correlators

Definition (Rudenko)

$$\mathrm{QLi}_{n+k}(x_0, \ldots, x_{2n+1}) = (-1)^{n+1} \sum_{\substack{\underline{\mathbf{s}}=(i_0,\ldots,i_{n+k}) \\ \in \mathcal{C}_{n+k}}} \mathrm{sgn}(\underline{\mathbf{s}}) \, \mathrm{Cor}^{\mathcal{L}}(x_{i_0}, \ldots, x_{i_{n+k}}) \,,$$

where

$$\mathcal{C}_{n+k} = \left\{ \begin{matrix} \text{non-decreasing sequences } 0 \leq i_0 \leq i_1 \leq \cdots \leq i_{n+k} \leq 2n+1 \\ \text{so that at least one of } \{2i, 2i+1\} \text{ appears, } 0 \leq i \leq n \end{matrix} \right\}$$

$$\mathrm{sgn}(\underline{\mathbf{s}}) = (-1)^{\#\{\text{even elements}\}}$$

Note: Think $\mathrm{Cor}^{\mathcal{L}}(x_0, \ldots, x_{n-1}, x_n)$ "=" $\mathrm{I}^{\mathcal{L}}(\infty; x_0, \ldots, x_{n-1}; x_n)$.

Quadrangulation formula

Can express $\mathrm{QLi}_{n+k}(x_0, \ldots, x_{2n+1})$ as a sum over certain quadrangles inside a $2n+2$-gon

- Introduce $abcd = [x_a, x_b, x_c, x_d] := \frac{(x_a - x_b)(x_c - x_d)}{(x_b - x_c)(x_d - x_a)}$ (cyclic) cross-ratio
- Label vertices of a $(2n+2)$-gon P with $x_0, \ldots, x_{2n+1}$
- A quadrilateral $x_{i_0}, x_{i_1}, x_{i_2}, x_{i_3}$ gives rise to a cross-ratio

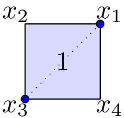
Consider any Q quadrangulation of P

- Q determines a dual tree t_Q
- Then Rudenko's arborification map attaches weight $n + k$, depth n MPL to t_Q
- There is a recursive construction, via 'products' over families of subpolygons

Pictures of QLi's - depth 1 & 2

Note: Here $\mathrm{Li}_{k_0\,;\,k_1,\dots,k_d}$ is extended by linearity to formal combinations $\sum_i \varepsilon_i [x_{i,1},\dots,x_{i,d}]$

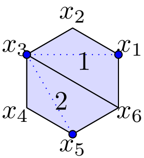
$$\mathrm{QLi}_{k+1}(x_1,\dots,x_4) = \mathrm{Li}^{\mathcal{L}}_{k\,;\,1}\Bigg(-$$



$$\Bigg)$$

$$= \mathrm{Li}^{\mathcal{L}}_{k\,;\,1}\big(-[1234]\big) = -\mathrm{Li}^{\mathcal{L}}_{k\,;\,1}\big(1234\big) = -\mathrm{Li}^{\mathcal{L}}_{k\,;\,1}\big([x_1,x_2,x_3,x_4]\big)$$

$$\mathrm{QLi}_{k+2}(x_1,\dots,x_6) = \mathrm{Li}^{\mathcal{L}}_{k\,;\,1,1}\Bigg(+$$



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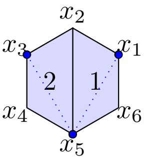

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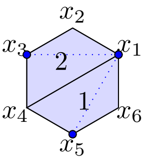
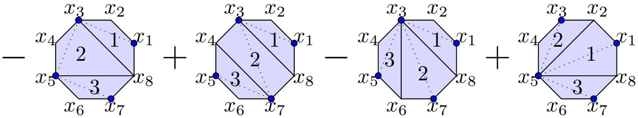


$$\Bigg)$$

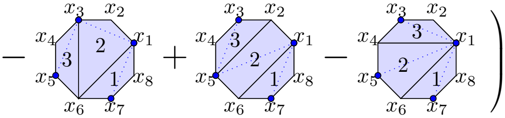
$$= \mathrm{Li}^{\mathcal{L}}_{k\,;\,1,1}\big([x_1,x_2,x_3,x_6],[x_3,x_4,x_5,x_6]\big) - \mathrm{Li}^{\mathcal{L}}_{k\,;\,1,1}\big([x_1,x_2,x_5,x_6],[x_3,x_4,x_5,x_2]\big)$$
$$+ \mathrm{Li}^{\mathcal{L}}_{k\,;\,1,1}\big([x_1,x_4,x_5,x_6],[x_1,x_2,x_3,x_4]\big)$$

Pictures of QLi's - depth 3

$\mathrm{QLi}_{k+3}(x_1,\ldots,x_8) = \mathrm{Li}^{\mathcal{L}}_{k\,;\,1,1,1}\Big($

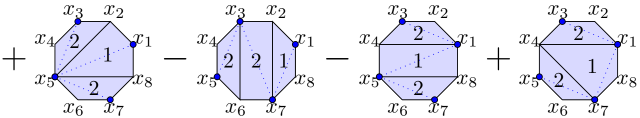


$+$ 8 more terms



$\Big)$

$+\,\mathrm{Li}^{\mathcal{L}}_{k\,;\,1,2}\Big($



$\Big)$

The key identity

Quadrangular polylogarithm functional equation

$$\mathbf{Q}_{2k}: \qquad \sum_{i=1}^{2k+3} (-1)^i \, \mathrm{QLi}_{2k}(x_1, \ldots, \widehat{x}_i, \ldots, x_{2k+2}) \equiv 0 \quad (\text{mod depth} < k)$$

- Generalisation of dilogarithm 5-term relation

$$\sum_{i=1}^{5} (-1)^i \underbrace{\mathrm{Li}^{\mathcal{L}}_{1;1}}_{=-\mathrm{Li}^{\mathcal{L}}_2}([x_1, \ldots, \widehat{x}_i, \ldots, x_5]) = 0$$

- Generalisation of Goncharov-Rudenko's weight 4 identity (for $\zeta_F(4)$ proof)
- Symmetrised version $\mathrm{QLi}^{\mathrm{sym}}$ incorporates lower depth, for exact identity

Remark: Rudenko (optimistically) expects that all MPL identities follow from this

Limits and degenerations

Given an MPL identity, specialise and consider limits, to obtain more identities.

- $\lim_{x_i \to 0} \mathrm{Li}_{k_1,\ldots,k_d}(x_1, \ldots, x_i, \ldots, x_d) = 0$
- $\lim_{x_i \to \infty} \mathrm{Li}_{k_1,\ldots,k_d}(x_1, \ldots, x_i, \ldots, x_d) =$ depth $\leq d$
- Likewise for $\mathrm{Li}^{\mathcal{L}}_{k_0\,;\,k_1,\ldots,k_d}$

Considering $\lim_{x_j \to x_i} \mathbf{Q}_k$ does give some identities

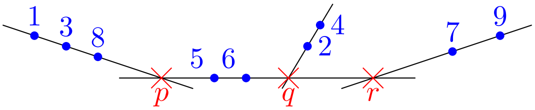
Problem: They are often complicated and certainly not enough. How else to specialise?

Stable curves

Recall

The Deligne-Mumford compactification of $\mathfrak{M}_{0,n}$ is described by (genus 0) stable curves

- Components are isomorphic to $\mathbb{P}^1$
- Only singular points are simple double points
- Number of marked & singular points per components ≥ 3



Idea: The points x_1, x_3, x_8 all degenerate to p

But: There is a always projective transformation moving (x_1, x_3, x_8, p) to $(\infty, 0, 1, z)$

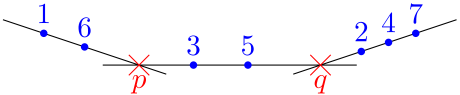
So: Points x_1, x_3, x_8 split off as a separate $\mathbb{P}^1$

Calculate: Set $x_i = \lambda_1 y_i + p$, $i = 1, 3, 8$, (similar for $i = 2, 4$ and $i = 7, 9$), with $\lambda_j \to 0$

Cross-ratios well-defined: $[x_1,x_3,x_8,x_i] = [x_1,x_3,x_8,p]$ and $[x_1,x_3,x_2,x_4] = [x_1,x_3,p,p] = 0$

Example: weight 4, Nielsen-type reduction

Specialise $\mathbf{Q}_4 = \sum_i (-1)^i \, \mathrm{QLi}_4(x_1, \ldots, \widehat{x_i}, \ldots, x_7)$ to:



$$\begin{aligned}\mathrm{QLi}_4(\widehat{x_7}) &= \mathrm{Li}^{\mathcal{L}}_{2\,;\,1,1}\big(+[1236, 3456] - [1256, 3452] + [1456, 1234]\big) \\ &\to \mathrm{Li}^{\mathcal{L}}_{2\,;\,1,1}\big(+[\underbrace{pq3p}_{=\infty}, 3q5p] - [\underbrace{pq5p}_{=\infty}, 3q5q] + [\underbrace{p45p}_{=\infty}, pq3q]\big) \equiv 0 \pmod{\text{depth } 1}\end{aligned}$$

$$\begin{aligned}\mathrm{QLi}_4(\widehat{x_6}) &= \mathrm{Li}^{\mathcal{L}}_{2\,;\,1,1}\big(+[1237, 3457] - [1257, 3452] + [1457, 1234]\big) \\ &\to \mathrm{Li}^{\mathcal{L}}_{2\,;\,1,1}\big(+[\underbrace{pq3q}_{=1}, \underbrace{3q5q}_{=1}] - [\underbrace{pq5q}_{=1}, \underbrace{3q5q}_{=1}] + [\underbrace{pq5q}_{=1}, \underbrace{pq3q}_{=1}]\big) = \mathrm{Li}^{\mathcal{L}}_{2\,;\,1,1}(1, 1)\end{aligned}$$

$\mathrm{QLi}_4(\widehat{x_5}) \to \mathrm{Li}^{\mathcal{L}}_{2\,;\,1,1}(1, 1)$ Similarly: $\mathrm{QLi}_4(\widehat{x_3}),\ \mathrm{QLi}_4(\widehat{x_2}),\ \mathrm{QLi}_4(\widehat{x_1}) \to 00$ (mod depth

$$\begin{aligned}\mathrm{QLi}_4(\widehat{x_4}) &= \mathrm{Li}^{\mathcal{L}}_{2\,;\,1,1}\big(+[1237, 3567] - [1267, 3562] + [1567, 1235]\big) \\ &\to \mathrm{Li}^{\mathcal{L}}_{2\,;\,1,1}\big(+[\underbrace{pq3q}_{=1}, 35pq] - [\underbrace{pqpq}_{=1}, 35pq] + [\underbrace{p5pq}_{=1}, pq35]\big) = \mathrm{Li}_{2\,;\,1,1}(1, pq35)\end{aligned}$$

Conclusion: Nielsen-type reduction $\mathrm{Li}_{2\,;\,1,1}(1, pq35) \equiv 0 \pmod{\text{depth } 1}$

Outline

Some general identities

We treat the following as known

$$\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(x,y,z) + \mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(x^{-1},y^{-1},z^{-1}) \equiv 0 \pmod{\text{depth } 2} \qquad \text{(inverse)}$$

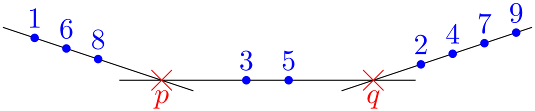
$$\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(x,y,z) - \mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(z,y,x) \equiv 0 \pmod{\text{depth } 2} \qquad \text{(reverse)}$$

$$\begin{aligned}\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(x_1,x_2,y) &+ \mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(x_1,y,x_2) \\ &+ \mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(y,x_1,x_2) \equiv 0 \pmod{\text{depth } 2}\end{aligned} \qquad (2 ⧢ 1)$$

- Can be shown via general theory of multiple polylogarithms
- Can be derived from $\mathbf{Q}_6$ directly
- Are used implicitly in many of the following lemmas

Nielsen-type reduction (I)

Specialise $\mathbf{Q}_6$ to



Obtain

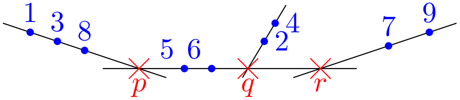
Lemma

$$\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(1,1,pq35) \equiv 0 \pmod{\text{depth } 2}$$

Remark: Naturally generalises the same result & curve as in weight 4

Nielsen-type reduction (II)

Specialise $\mathbf{Q}_6$ to



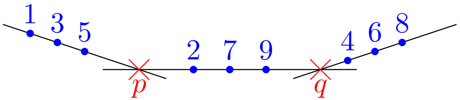
Obtain

Lemma (Symmetry 1, Degenerate)

$$\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(1, A, B) \equiv \mathrm{Li}^{\mathcal{L}}_{3;1,1,1}\Big(1, \underbrace{\frac{A(1-B)}{1-AB}}_{pq6r}, \underbrace{\frac{AB-1}{AB}}_{p56q}\Big) \pmod{\text{depth } 2}$$

with $A = pq5r, B = 56pr$.

Specialise $\mathbf{Q}_6$ to



Obtain

Lemma (Symmetry 2, Degenerate)

$$\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(1, A, B) \equiv -\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}\Big(1, A, \underbrace{\frac{1-AB}{A(1-B)}}_{q927}\Big) \pmod{\text{depth } 2}.$$

with $A = 2pq7, B = 9p27$

Nielsen-type reduction (III)

Play these symmetries against each other

- Write $g(x_1,\ldots,x_5) \coloneqq \mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(1,[x_3,x_1,x_4,x_2],[x_5,x_1,x_3,x_2])$ and notice

$$g(x_1,x_2,x_3,x_4,x_5) \equiv -g(x_2,x_1,x_3,x_4,x_5) \pmod{\text{depth } 2}\,, \qquad \text{(inverse)}$$

$$g(x_5,p,r,q,x_6) \equiv \quad g(p,x_6,q,r,x_5) \pmod{\text{depth } 2}\,, \qquad \text{(sym 1)}$$

$$g(p,x_7,x_2,q,x_9) \equiv -g(q,x_2,x_7,p,x_9) \pmod{\text{depth } 2}\,. \qquad \text{(sym 2)}$$

Nielsen-type reduction (IV) - The payoff

Play these symmetries against each other

- Write $g(x_1, \ldots, x_5) \coloneqq \mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(1, [x_3, x_1, x_4, x_2], [x_5, x_1, x_3, x_2])$ and notice

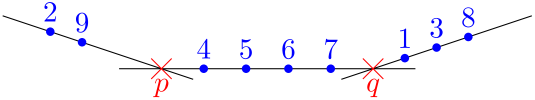
$$
\begin{aligned}
& g(x_1, x_2, x_3, x_4, x_5) \\
\equiv\ & -g(x_2, x_1, x_3, x_4, x_5) && \text{(inverse)} \\
\equiv\ & g(x_4, x_3, x_1, x_2, x_5) && \text{(sym 2)} \\
\equiv\ & g(x_4, x_3, x_2, x_1, x_5) && \text{(sym 1} \times \text{3)} \\
\equiv\ & -g(x_1, x_2, x_3, x_4, x_5) \quad (\text{mod depth } 2)\,. && \text{(sym 2)}
\end{aligned}
$$

Theorem (Nielsen-type reduction)

$$\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(1, X, Y) \equiv 0 \quad (\text{mod depth } 2)$$

Zagier-type reduction (I)

Specialise $\mathbf{Q}_6$ to



Obtain

Lemma (Symmetry 1, Full)

$$\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(A,B,C) \equiv -\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}\Big(\underbrace{1-A}_{q67p}, \underbrace{\frac{B}{B-1}}_{qp56}, \underbrace{1-C}_{q45p}\Big) \pmod{\text{depth } 2}$$

with $A = p67q, B = pq56, C = p45q$

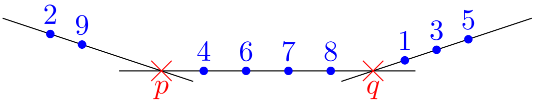
With Inverse and Reverse, this gives 12 symmetries.

Remark: Similar starting point in weight 4, namely

$$\mathrm{Li}^{\mathcal{L}}_{2;1,1}(A,B) \equiv \mathrm{Li}^{\mathcal{L}}_{2;1,1}\Big(1-A, \tfrac{B}{B-1}\Big)$$

Zagier-type reduction (II) - A four term relation

Specialise $\mathbf{Q}_6$ to



Obtain

Lemma (Four term relation)

$$\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}\left(A, \frac{C}{B}, \frac{1-C}{1-B}\right) + \mathrm{Li}^{\mathcal{L}}_{3;1,1,1}\left(A, \frac{C}{B}, \frac{1-B}{1-C}\right) - \mathrm{Li}^{\mathcal{L}}_{3;1,1,1}\left(A, \frac{1}{B}, 1-C\right) - \mathrm{Li}^{\mathcal{L}}_{3;1,1,1}\left(A, \frac{1}{B}, C\right) \equiv 0 \pmod{\text{depth } 2}$$

with $A = p67q, B = pq56, C = p45q$

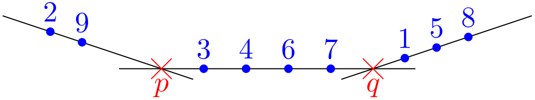
This shows the two Zagier-type reductions are equivalent:

X & $1-X$ expresses/can be expressed via X & X^{-1}.

Remark: A similar to a step in weight 4. (We already informed Matveiakin and Rudenko about weight 6; their theorem states modulo depth ≤ 2 & Zagier of X & $1-X$ type.)

Zagier-type reduction (III)

Specialise $\mathbf{Q}_6$ to



Obtain

Lemma (Symmetry 2, Full)

$$2\,\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(A,B,C) + 2\,\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}\left(A, \frac{B}{B-1}, \frac{1}{1-C}\right) \equiv 0 \pmod{\text{depth } 2}$$

with $A = 67qp, B = 4q6p, C = q34p$

Now have 216 symmetries:

$$\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(x,y,z) \mapsto \varepsilon\,\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(x^\pi, y^\sigma, z^\tau),\ \varepsilon\,\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(z^\tau, y^\sigma, x^\pi)\,,$$

for $\pi, \sigma, \tau \in \mathfrak{S}_3$, with $\mathrm{sgn}(\pi\sigma\tau) = \varepsilon$, $\mathrm{sgn}(\pi) = \mathrm{sgn}(\tau)$, and action via the anharmonic ratios

$$z^{(1\,3)} = \frac{1}{z}\,, \qquad z^{(2\,3)} = 1 - z\,.$$

Zagier-type reduction (IV) - The payoff

Play the symmetries and relations against each other.

- Consider $W(a,b,c) := \mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(a,b^{-1},c) + \mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(a,b^{-1},1-c)$
- Symmetries, and four-term relation gives

$$W(a,b,c) \equiv -W\Big(a, \frac{c}{b}, \frac{1-c}{1-b}\Big) \pmod{\text{depth } 2}\,.$$

- The transformation $(\underbrace{b}_{=[\infty,1,0,b]}, \underbrace{c}_{=[\infty,1,0,c]}) \mapsto \Big(\underbrace{\frac{c}{b}}_{=[b,\infty,c,0]}, \underbrace{\frac{1-c}{1-b}}_{=[b,\infty,c,1]}\Big)$ has order 5. Permutation $(\infty\, b\, 0\, c\, 1)$
- After 5 applications, find $W(a,b,c) \equiv -W(a,b,c) \pmod{\text{depth } 2}$

Theorem (Zagier-type reduction)

$$\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(a,b^{-1},c) + \mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(a,b^{-1},1-c) \equiv 0 \pmod{\text{depth } 2}$$

Remark: This version is simplification due to Danylo Radchenko. I originally had a more computational proof via another stable curve degeneration.

Summary

- Depth reductions of MPL's
- Goncharov's Depth Conjecture
  - Cobracket on (motivic) MPL's
  - Truncation and iteration of the cobracket
  - Cobracket conjecturally detects depth
  - Conjectured Nielsen-, Zagier-, & Gangl-type reductions of $\mathrm{Li}^{\mathcal{L}}_{k\,;\,1,\ldots,1}$
- Quadrangular polylogarithms
  - Fundamental functional equation
  - Pictorial depiction via quadrangulations (depth ≤ 3)
- Specialisations and degenerations
  - Stable curves describe $\overline{\mathfrak{M}_{0,n}}$
  - Produced via many points $x_1, \ldots, x_k$ colliding
- Proof of Nielsen-type and Zagier-type reductions in weight 6

Gangl-type reduction - a word on the proof

Theorem (Matveiakin-Rudenko 2022)

$$\mathrm{Li}^{\mathcal{L}}_{3\,;\,1,1,1}(V(y,z),x_1,x_2) \equiv 0 \quad (\text{mod depth 2 \& Zagier reductions})$$

Idea: Specialise $\mathbf{Q}_6$ to many divisors $x_i = x_j$

Find combinations of specialisations which show symmetries of

$$\mathrm{Li}^{\mathcal{L}}_{3\,;\,1,1,1}(V(y,z),x_1,x_2) \quad (\text{mod depth 2 \& Zagier})$$

Play symmetries against each other, to deduce

$$\mathrm{Li}^{\mathcal{L}}_{3\,;\,1,1,1}(V(y,z),x_1,x_2) \equiv 0 \quad (\text{mod depth 2 \& Zagier})$$

Remark: Reduces more directly to computation in $\mathbb{C}[\mathfrak{S}_9]$, so can apply linear-algebra & rep theory machinery.

Zagier reductions involve more awkward changes of variables, and non-obvious simplifications modulo the $2 \sqcup 1$ identity, and four-term identity. (Matveiakin and Rudenko tried establish these reductions, but were unsuccessful.)

Higher weight?

- Investigations in progress.
- Some patterns and structures in common in weight 4 and weight 6.
- Matveiakin and Rudenko could not yet find Gangl-type reduction in wight 8.
- More Nielsen-type reductions needed? More complications?

$$\mathrm{Li}^{\mathcal{L}}_{4;1,1,1,1}(1,x,1,y) \text{ and } \mathrm{Li}^{\mathcal{L}}_{4;1,1,1,1}(1,1,x,y)$$

Nielsen-type reduction (V) - Symmetry 1

$$\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(1,1,A) = \mathrm{Li}^{\mathcal{L}}_{5,1}\left(-\left[\tfrac{A-1}{A},\tfrac{A}{A-1}\right]-2\left[A,\tfrac{1}{A}\right]\right) + \mathrm{Li}^{\mathcal{L}}_{6}\left(-\left[\tfrac{1}{1-A}\right]+7\left[\tfrac{1}{A}\right]-4\left[\tfrac{A-1}{A}\right]\right).$$

$$\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(1,A,B) - \mathrm{Li}^{\mathcal{L}}_{3;1,1,1}\left(1,\tfrac{A(1-B)}{1-AB},-\tfrac{1-AB}{AB}\right) =$$

$$\mathrm{Li}^{\mathcal{L}}_{4,2}\left(-\left[\tfrac{B-1}{B},\tfrac{B}{B-1}\right]-\left[\tfrac{B-1}{(1-A)B},\tfrac{B}{B-1}\right]+\left[\tfrac{B}{B-1},\tfrac{AB-1}{AB}\right]-\left[\tfrac{(1-A)B}{B-1},\tfrac{1}{AB}\right]+\left[\tfrac{(1-A)B}{B-1},\tfrac{B-1}{(1-A)B}\right]\right)$$

$$+\mathrm{Li}^{\mathcal{L}}_{5,1}\left(-2\left[\tfrac{A(1-B)}{A-1},\tfrac{A-1}{A(1-B)}\right]-2\left[\tfrac{B-1}{(1-A)B},\tfrac{B}{B-1}\right]-\left[\tfrac{AB}{A-1},\tfrac{1}{B}\right]-\left[\tfrac{AB}{A-1},\tfrac{AB-1}{AB}\right]\right.$$

$$+2\left[\tfrac{B}{B-1},\tfrac{AB-1}{AB}\right]-2\left[\tfrac{(1-A)B}{B-1},\tfrac{1}{AB}\right]+4\left[\tfrac{(1-A)B}{B-1},\tfrac{B-1}{(1-A)B}\right]-\left[AB,\tfrac{1}{B}\right]$$

$$\left.-2\left[AB,\tfrac{1}{AB}\right]+2\left[\tfrac{A-1}{A},\tfrac{A}{A-1}\right]+\left[A,\tfrac{1}{A}\right]-2\left[\tfrac{B-1}{B},\tfrac{B}{B-1}\right]\right)$$

$$+\mathrm{Li}^{\mathcal{L}}_{6}\left(-5\left[\tfrac{B-1}{(1-A)B}\right]+4\left[\tfrac{A-1}{A(1-B)}\right]-\left[\tfrac{AB}{AB-1}\right]+10\left[\tfrac{A-1}{AB}\right]\right.$$

$$\left.+10\left[\tfrac{1}{AB}\right]-\left[\tfrac{1}{1-A}\right]-\left[\tfrac{1}{A}\right]+6\left[\tfrac{A-1}{A}\right]+6\left[\tfrac{B-1}{B}\right]\right).$$

Nielsen-type reduction (VI) - Symmetry 2

$$\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(1,A,B)+\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}\left(1,A,\tfrac{1-AB}{A(1-B)}\right)=$$

$$\mathrm{Li}^{\mathcal{L}}_{4,2}\Big(+\big[1,\tfrac{A}{A-1}\big]+\big[\tfrac{A-1}{A},\tfrac{A}{A-1}\big]-\big[AB,\tfrac{1}{B}\big]+\big[AB,\tfrac{1}{AB}\big]+\big[\tfrac{1-B}{1-AB},\tfrac{1-AB}{1-B}\big]$$
$$-\big[\tfrac{1-B}{1-AB},\tfrac{1-AB}{A(1-B)}\big]-\big[\tfrac{(1-A)B}{1-AB},\tfrac{AB-1}{AB}\big]-\big[\tfrac{AB}{AB-1},\tfrac{1-AB}{(1-A)B}\big]\Big)$$
$$+\mathrm{Li}^{\mathcal{L}}_{5,1}\Big(+\big[\tfrac{A(1-B)}{(1-A)(1-AB)},\tfrac{1-AB}{A(1-B)}\big]-\big[\tfrac{A(1-B)}{(1-A)(1-AB)},\tfrac{AB-1}{AB}\big]+\big[\tfrac{1-B}{1-AB},\tfrac{1-AB}{1-B}\big]-\big[(1-A)B,\tfrac{1-AB}{(1-A)B}\big]$$
$$+\big[\tfrac{A-1}{A(1-B)},\tfrac{B-1}{(1-A)B}\big]-2\big[\tfrac{1-B}{1-AB},\tfrac{1-AB}{A(1-B)}\big]-2\big[\tfrac{(1-A)B}{1-AB},\tfrac{AB-1}{AB}\big]-2\big[\tfrac{AB}{AB-1},\tfrac{1-AB}{(1-A)B}\big]$$
$$-\big[\tfrac{1}{1-B},\tfrac{1-B}{1-AB}\big]+\big[(1-A)B,\tfrac{1}{B}\big]-2\big[AB,\tfrac{1}{B}\big]+\big[AB,\tfrac{1}{AB}\big]+2\big[\tfrac{A-1}{A},\tfrac{A}{A-1}\big]-2\big[1-A,\tfrac{1}{1-A}\big]-2\big[A,\tfrac{1}{A}\big]\Big)$$
$$+\mathrm{Li}^{\mathcal{L}}_{6}\Big(+5\big[\tfrac{A(1-B)}{(1-A)(1-AB)}\big]+2\big[\tfrac{(1-A)B}{1-AB}\big]+2\big[\tfrac{A-1}{A(1-B)}\big]+5\big[\tfrac{1}{(1-A)B}\big]-2\big[\tfrac{B-1}{(1-A)B}\big]+\big[\tfrac{A(1-B)}{1-AB}\big]$$
$$-\big[\tfrac{1-B}{1-AB}\big]+8\big[\tfrac{1}{AB}\big]-\big[\tfrac{1}{1-AB}\big]+5\big[\tfrac{1}{1-A}\big]+9\big[\tfrac{1}{A}\big]-4\big[\tfrac{A-1}{A}\big]-2\big[\tfrac{1}{1-B}\big]-3\big[\tfrac{1}{B}\big]\Big).$$

Nielsen-type reduction (VII) - The payoff

$$\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(1,A,B) =$$

$$\begin{aligned}
&\tfrac{1}{2}\,\mathrm{Li}^{\mathcal{L}}_{4,2}\Big(-\left[\tfrac{A(1-B)}{1-AB},\tfrac{1-AB}{A(1-B)}\right]-\left[\tfrac{(1-A)B}{1-AB},\tfrac{1-AB}{(1-A)B}\right]+\left[\tfrac{1-AB}{A(1-B)},\tfrac{1-B}{1-AB}\right]+\left[\tfrac{1-AB}{(1-A)B},\tfrac{1-A}{1-AB}\right]+\left[\tfrac{A-1}{A(1-B)},\tfrac{B-1}{(1-A)B}\right]\\
&\quad-\left[\tfrac{A(1-B)}{A-1},\tfrac{A-1}{A(1-B)}\right]-\left[\tfrac{(1-A)B}{1-AB},\tfrac{AB-1}{AB}\right]-\left[\tfrac{AB}{AB-1},\tfrac{1-AB}{A(1-B)}\right]-2\left[\tfrac{AB}{AB-1},\tfrac{1-AB}{(1-A)B}\right]+\left[\tfrac{1}{1-AB},\tfrac{1-AB}{1-A}\right]\\
&\quad+\left[\tfrac{AB}{AB-1},\tfrac{AB-1}{AB}\right]+\left[\tfrac{1-AB}{1-A},\tfrac{1}{1-AB}\right]-\left[\tfrac{A}{A-1},\tfrac{A-1}{A(1-B)}\right]-\left[\tfrac{A}{A-1},\tfrac{AB-1}{AB}\right]-\left[\tfrac{1}{1-B},\tfrac{1-B}{1-AB}\right]-\left[AB,\tfrac{1}{B}\right]\\
&\quad-\left[1-B,\tfrac{A-1}{A(1-B)}\right]+\left[AB,\tfrac{1}{AB}\right]+2\left[\tfrac{A-1}{A},\tfrac{A}{A-1}\right]-\left[1-A,\tfrac{1}{1-A}\right]+\left[1-B,\tfrac{1}{1-B}\right]-\left[1,\tfrac{1}{1-A}\right]+\left[1,\tfrac{A}{A-1}\right]\Big)\\
&+\tfrac{1}{2}\,\mathrm{Li}^{\mathcal{L}}_{5,1}\Big(-\left[\tfrac{A(1-B)}{(A-1)(1-AB)},\tfrac{1-AB}{1-B}\right]-\left[\tfrac{(1-A)B}{(B-1)(1-AB)},\tfrac{1-AB}{1-A}\right]+\left[\tfrac{(A-1)(1-AB)}{A(1-B)},\tfrac{1}{1-AB}\right]+\left[\tfrac{(B-1)(1-AB)}{(1-A)B},\tfrac{1}{1-AB}\right]\\
&\quad-3\left[\tfrac{A(1-B)}{1-AB},\tfrac{1-AB}{A(1-B)}\right]-4\left[\tfrac{(1-A)B}{1-AB},\tfrac{1-AB}{(1-A)B}\right]-\left[\tfrac{1-AB}{(1-A)(1-B)},\tfrac{1-A}{1-AB}\right]-\left[\tfrac{1-AB}{(1-A)(1-B)},\tfrac{1-B}{1-AB}\right]+2\left[\tfrac{A-1}{A(1-B)},\tfrac{B-1}{(1-A)B}\right]\\
&\quad-4\left[\tfrac{A(1-B)}{A-1},\tfrac{A-1}{A(1-B)}\right]-2\left[\tfrac{(1-A)B}{1-AB},\tfrac{AB-1}{AB}\right]-2\left[\tfrac{AB}{AB-1},\tfrac{1-AB}{A(1-B)}\right]-4\left[\tfrac{AB}{AB-1},\tfrac{1-AB}{(1-A)B}\right]+\left[\tfrac{1-AB}{A(1-B)},\tfrac{1-B}{1-AB}\right]\\
&\quad+\left[\tfrac{1-AB}{(1-A)B},\tfrac{1-A}{1-AB}\right]+4\left[\tfrac{AB}{AB-1},\tfrac{AB-1}{AB}\right]+2\left[\tfrac{1}{1-AB},\tfrac{1-AB}{1-A}\right]-\left[\tfrac{1-B}{1-AB},\tfrac{1-AB}{1-B}\right]+2\left[\tfrac{1-AB}{1-A},\tfrac{1}{1-AB}\right]-\left[\tfrac{AB}{A-1},\tfrac{AB-1}{AB}\right]\\
&\quad+\left[\tfrac{(1-A)B}{B-1},\tfrac{1}{AB}\right]-2\left[\tfrac{A}{A-1},\tfrac{A-1}{A(1-B)}\right]-2\left[\tfrac{A}{A-1},\tfrac{AB-1}{AB}\right]-3\left[\tfrac{1}{1-B},\tfrac{1-B}{1-AB}\right]-\left[\tfrac{B}{B-1},\tfrac{AB-1}{AB}\right]-2\left[1-B,\tfrac{A-1}{A(1-B)}\right]+\left[B,\tfrac{1}{B}\right]\\
&\quad-\left[\tfrac{AB}{A-1},\tfrac{1}{B}\right]+\left[(1-A)B,\tfrac{1}{B}\right]-\left[(1-A)B,\tfrac{1-AB}{(1-A)B}\right]+\left[AB,\tfrac{1}{AB}\right]+4\left[1-B,\tfrac{1}{1-B}\right]-3\left[AB,\tfrac{1}{B}\right]+6\left[\tfrac{A-1}{A},\tfrac{A}{A-1}\right]-2\left[1-A,\tfrac{1}{1-A}\right]\Big)\\
&+\tfrac{1}{2}\,\mathrm{Li}^{\mathcal{L}}_{6}\Big(-10\left[\tfrac{(1-A)B}{(B-1)(1-AB)}\right]+10\left[\tfrac{(1-A)(1-B)}{1-AB}\right]-10\left[\tfrac{A(1-B)}{(A-1)(1-AB)}\right]+3\left[\tfrac{A(1-B)}{1-AB}\right]+3\left[\tfrac{(1-A)B}{1-AB}\right]-2\left[\tfrac{1-B}{1-AB}\right]+\left[\tfrac{A-1}{A(1-B)}\right]\\
&\quad+5\left[\tfrac{1}{(1-A)B}\right]-2\left[\tfrac{B-1}{(1-A)B}\right]+10\left[\tfrac{A-1}{AB}\right]+7\left[\tfrac{AB}{AB-1}\right]+14\left[\tfrac{1}{AB}\right]-2\left[\tfrac{1}{1-A}\right]+6\left[\tfrac{1}{A}\right]-8\left[\tfrac{1}{1-B}\right]+4\left[\tfrac{B-1}{B}\right]-5\left[\tfrac{1}{B}\right]\Big).
\end{aligned}$$

Zagier-type reduction (V) - Symmetry 1

$$\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(A,B,C)+\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}\left(1-A,\tfrac{B}{B-1},1-C\right)=$$

$$\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}\Big(-\Big[1,\tfrac{ABC-A-C+1}{A(B-1)C},\tfrac{B-1}{B(ABC-A-C+1)}\Big]-\Big[\tfrac{(1-A)(1-C)}{ABC-A-C+1},1,\tfrac{B(ABC-A-C+1)}{B-1}\Big]+\Big[1-A,1,\tfrac{B(1-C)}{B-1}\Big]$$
$$-\Big[\tfrac{(1-A)(1-C)}{ABC-A-C+1},\tfrac{B(ABC-A-C+1)}{B-1},1\Big]+\Big[1-A,\tfrac{B(1-C)}{B-1},1\Big]+\Big[1-C,1,\tfrac{(1-A)B}{B-1}\Big]+\Big[1-C,1,\tfrac{1-AB}{A(1-B)}\Big]$$
$$+\Big[1-C,\tfrac{(1-A)B}{B-1},1\Big]-\Big[\tfrac{1-C}{1-BC},1,\tfrac{1}{A}\Big]+\Big[1,\tfrac{1}{A},\tfrac{1}{BC}\Big]+\Big[1,\tfrac{1}{C},\tfrac{1}{AB}\Big]+\Big[\tfrac{1}{B},1,\tfrac{1}{A}\Big]-\Big[1-C,1,\tfrac{B}{B-1}\Big]\Big)$$
$$+\mathrm{Li}^{\mathcal{L}}_{4,2}\Big(+\Big[\tfrac{(1-A)B(1-C)}{B-1},\tfrac{ABC-A-C+1}{(1-A)(1-C)}\Big]-\Big[\tfrac{(1-A)B(1-C)}{B-1},\tfrac{1}{1-C}\Big]+\Big[ABC,\tfrac{1}{BC}\Big]-\Big[ABC,\tfrac{1}{ABC}\Big]\Big)$$
$$+\mathrm{Li}^{\mathcal{L}}_{5,1}\Big(+7\Big[\tfrac{(1-A)B(1-C)}{B-1},\tfrac{ABC-A-C+1}{(1-A)(1-C)}\Big]-\Big[\tfrac{A(B-1)C}{ABC-A-C+1},\tfrac{ABC-A-C+1}{A(B-1)C}\Big]+2\Big[ABC,\tfrac{ABC-A-C+1}{A(B-1)C}\Big]$$
$$-\Big[\tfrac{(1-B)C}{(1-A)(1-BC)},\tfrac{1-BC}{(1-B)C}\Big]-2\Big[\tfrac{(1-AB)(1-C)}{A(1-B)},\tfrac{1}{1-C}\Big]+3\Big[\tfrac{(1-A)B(1-C)}{B-1},\tfrac{B-1}{B(1-C)}\Big]-7\Big[\tfrac{(1-A)B(1-C)}{B-1},\tfrac{1}{1-C}\Big]$$
$$+2\Big[\tfrac{1-C}{A(1-BC)},\tfrac{1-BC}{1-C}\Big]+\Big[\tfrac{(1-B)C}{1-BC},\tfrac{1-BC}{(1-B)C}\Big]+\Big[\tfrac{1-A}{1-AB},\tfrac{1-AB}{1-A}\Big]+3\Big[\tfrac{B(1-C)}{B-1},\tfrac{1}{1-C}\Big]+\Big[\tfrac{(1-AB)C}{1-A},\tfrac{1}{C}\Big]$$
$$+\Big[\tfrac{1-C}{1-BC},\tfrac{1-BC}{1-C}\Big]-3\Big[AB,\tfrac{1}{B}\Big]-2\Big[\tfrac{1}{ABC},C\Big]+2\Big[ABC,\tfrac{1}{BC}\Big]-\Big[1-A,\tfrac{1}{1-A}\Big]-\Big[B,\tfrac{1}{B}\Big]-\Big[1-C,\tfrac{1}{1-C}\Big]\Big)$$
$$+\mathrm{Li}^{\mathcal{L}}_{6}\Big(-5\Big[\tfrac{B-1}{B(ABC-A-C+1)}\Big]-\Big[\tfrac{A(B-1)C}{ABC-A-C+1}\Big]-3\Big[\tfrac{(1-B)C}{(1-A)(1-BC)}\Big]+3\Big[\tfrac{A(1-B)}{(1-AB)(1-C)}\Big]+3\Big[\tfrac{1-C}{A(1-BC)}\Big]$$
$$-15\Big[\tfrac{B-1}{(1-A)B(1-C)}\Big]+8\Big[\tfrac{B-1}{(1-A)B}\Big]+3\Big[\tfrac{1-A}{1-AB}\Big]+2\Big[\tfrac{A(1-B)}{1-AB}\Big]-6\Big[\tfrac{B-1}{B(1-C)}\Big]-3\Big[\tfrac{1-A}{(1-AB)C}\Big]$$
$$+3\Big[\tfrac{1-C}{1-BC}\Big]+3\Big[\tfrac{(1-B)C}{1-BC}\Big]+\Big[\tfrac{1}{BC}\Big]+10\Big[\tfrac{1}{AB}\Big]-15\Big[\tfrac{1}{ABC}\Big]+2\Big[\tfrac{1}{A}\Big]+5\Big[\tfrac{1}{B}\Big]-3\Big[\tfrac{B-1}{B}\Big]+3\Big[\tfrac{1}{1-C}\Big]\Big).$$

Table of all results

$\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(A,B,C)$	With degenerate $\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(\ldots,1,\ldots)$'s				Purely depth ≤ 2		
Identity	$\mathrm{Li}^{\mathcal{L}}_{3;1,1,1}(\ldots,1,\ldots)$	$\mathrm{Li}^{\mathcal{L}}_{4,2}$	$\mathrm{Li}^{\mathcal{L}}_{5,1}$	$\mathrm{Li}^{\mathcal{L}}_{6}$	$\mathrm{Li}^{\mathcal{L}}_{4,2}$	$\mathrm{Li}^{\mathcal{L}}_{5,1}$	$\mathrm{Li}^{\mathcal{L}}_{6}$
Full Symmetry 1	13	4	19	20	226	357	129
Four-term	75	23	93	73	768	1052	329
Full Symmetry 2	190	63	178	129	1453	2472	763
$1-B$ result	537	162	437	299	3814	6561	1976
B^{-1} result	555	168	442	303	3931	6793	1991
$(1-C)^{-1}$ result	384	110	314	223	2763	4780	1418
$1-C$ result	6363	1788	4529	2704	39784	70453	19749
C^{-1} result	6374	1789	4524	2711	39839	70546	19694