Depth reductions of multiple polylogarithms:
(an overview of) the state of the art around Goncharov's Depth Conjecture

Steven Charlton

Abstract. I will review the setup surrounding Goncharov's Depth Conjecture, and types of the multiple polylogarithm depth reductions it predicts. I will then discuss some recent results (of subsets of Gangl, Matveiakin, Radchenko, Rudenko, and C) proving identities and reductions in this direction.

0. Introduction

Definition 1. Multiple polylogarithm (MPL)

$$\mathrm{Li}_{a_1,\dots,a_d}(x_1,\dots,x_d) = \sum_{0<n_1<\cdots<n_d} \frac{x_1^{n_1}\cdots x_d^{n_d}}{n_1^{a_1}\cdots n_d^{a_d}}\,, \quad |x_i|<1\,.$$

- Weight: $a_1+\cdots+a_d$,
- Depth: d.

Goal/Challenge: Understand functional equations of MPL's and (properties/structure/behaviour) of depth of MPL's

For $|x|+|y|<1$ we have five-term relation (Abel, Spence, Kummer, ...)

$$\mathrm{Li}_2(x)+\mathrm{Li}_2(y)-\mathrm{Li}_2\left(\frac{x}{1-y}\right)-\mathrm{Li}_2\left(\frac{y}{1-x}\right)+\mathrm{Li}_2\left(\frac{xy}{(1-x)(1-y)}\right) \\ = -\log(1-x)\log(1-y)$$

Expect weight is grading (no identities between different weights), but depth is only a filtration.

Simple example:

$$\mathrm{Li}_{1,1}(x,y)+\mathrm{Li}_{1,1}(y,x)+\mathrm{Li}_2(xy) = \mathrm{Li}_1(x)\,\mathrm{Li}_1(y) \equiv 0 \pmod{\textit{products}}\,.$$

But this has easy generalisation via stuffle product. More interesting power-series identity [Goncharov, Zagier, Lewin$^{(?)}$]

$$\mathrm{Li}_{1,1}(x,y) = \mathrm{Li}_2\Big(\frac{y(1-x)}{y-1}\Big) - \mathrm{Li}_2\Big(\frac{y}{y-1}\Big) - \mathrm{Li}_2(xy)\,, \quad |xy|<1, |y|<1\,.$$

Similar reduction in weight 3 [Goncharov, Zhao, Lewin$^{(?)}$]

$$\mathrm{Li}_{1,1,1}(x,y,z) \equiv \sum \mathrm{Li}_3\text{'s} \pmod{\textit{products}} \quad \Big(\text{including} \rightsquigarrow \mathrm{Li}_3\Big(\frac{(y-1)(1-xyz)}{(1-x)y(1-z)}\Big)\Big)\,.$$

What about weight 4? *Apparently not! (There is an obstruction)* Best one can find is something like [Zagier, Gangl]

$$\mathrm{Li}_{3,1}\Big(\frac{1-x}{y},y\Big)+\mathrm{Li}_{3,1}\Big(\frac{x}{y},y\Big) \equiv \sum \mathrm{Li}_4\text{'s} \pmod{\textit{products}} \quad \Big(\text{including} \rightsquigarrow \mathrm{Li}_4\Big(\frac{(1-x)y}{x(1-y)}\Big)\Big)\,.\,.$$

Question/aim: How to predict/understand/find/explain such reductions or obstructions?

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1. The depth conjecture and motivic MPL's

Write

$$\mathrm{Li}_{a_1,\ldots,a_d}(x_1,\ldots,x_d) = (-1)^d I(0; \tfrac{1}{x_1\cdots x_d}, \{0\}^{a_1-1}, \tfrac{1}{x_2\cdots x_d}, \{0\}^{a_2-1}, \ldots, \tfrac{1}{x_d}, \{0\}^{a_d-1}; 1)\,,$$

via iterated integral (along path γ)

$$I_{(\gamma)}(a; z_1, \ldots, z_n; b) = \int_{a<t_1<\cdots<t_n<b} \frac{\mathrm{d}t_1}{t_1 - x_1} \wedge \cdots \wedge \frac{\mathrm{d}t_n}{t_n - x_n}\,.$$

- Goncharov upgraded I to $I^{\mathfrak{u}}$ framed mixed Tate motives, in connected graded Hopf algebra $\mathcal{H}_n$.
- Define $\mathrm{Li}^{\mathfrak{u}}_{a_1,\ldots,a_d}$ via $I^{\mathfrak{u}}$.
- Coproduct gives us new tool to investigate the structure.

Introduce coproduct (cobracket) on the Lie coalgebra of irreducibles $\mathcal{L}_n = \mathcal{H}_n/\text{products}$.

$$\Delta I^{\mathfrak{L}}(x_0; x_1, \ldots, x_n; x_{n+1}) = \sum_{i<j} I^{\mathfrak{L}}(x_i; x_{i+1}, \ldots, x_{j-1}; x_j) \wedge I^{\mathfrak{L}}(x_0; x_1, \ldots, x_i, x_j, \ldots, x_n; x_{n+1})\,.$$

(Semicircular picture.)

Depth is motivic, $\mathcal{D}_d = \{$ depth $\leq d$ MPL's$\}$. As factor $I^{\mathcal{L}}(a; \overbrace{0, \ldots, 0}^{\geq 2}; b) = 0$ mod products,

$$\Delta(\mathcal{D}_d) \subset \mathcal{D}_d \wedge \mathcal{L}_1 \oplus \bigoplus_{i+j=d} \mathcal{D}_i \wedge \mathcal{D}_j\,,$$

E.g.

$$\begin{aligned} \Delta\,\mathrm{Li}^{\mathcal{L}}_4(x) &= \log^{\mathcal{L}}(x) \wedge \mathrm{Li}^{\mathcal{L}}_3(x) \\ \Delta\,\mathrm{Li}^{\mathcal{L}}_{3,1}(x) &= \underbrace{\log^{\mathcal{L}}(x) \wedge \mathrm{Li}^{\mathcal{L}}_{2,1}(x,y)}_{\text{symbol/derivative contribution}} + \mathrm{Li}_1 \wedge \mathrm{Li}_3\text{'s} - \mathrm{Li}^{\mathcal{L}}_2(x) \wedge \mathrm{Li}^{\mathcal{L}}_2(xy)\,. \end{aligned}$$

Ignore $\mathcal{L}_1$, and expressions simplify

$$\overline{\Delta}\,\mathrm{Li}^{\mathcal{L}}_4(x) = 0\,, \quad \overline{\Delta}\,\mathrm{Li}^{\mathcal{L}}_{3,1}(x,y) = -\mathrm{Li}^{\mathcal{L}}_2(x) \wedge \mathrm{Li}^{\mathcal{L}}_2(xy) \neq 0$$

So $\mathrm{Li}_{3,1}$ cannot be written via Li_4's alone. $\overline{\Delta}$ seems to distinguish depth $= 1$ and depth ≥ 2. [This explains previous examples.]

Can iterate $\overline{\Delta}$ on higher depth functions, to detect when depth 2 appears in $\overline{\Delta}$, i.e. original function has depth ≥ 3. $\overline{\Delta}^{[2]} = \overline{\Delta} \circ (\overline{\Delta}^{[1]} \otimes \mathrm{id})$ gives

$$\begin{aligned} \overline{\Delta}\,\mathrm{Li}_{4,1,1}(x,y,z) = {} & \mathrm{Li}^{\mathcal{L}}_2(y) \wedge \mathrm{Li}^{\mathcal{L}}_{3,1}(xy,z) - \mathrm{Li}^{\mathcal{L}}_{3,1}(yz,y^{-1}) \wedge \mathrm{Li}^{\mathcal{L}}_2(xyz) \\ & + \text{wt } 3 \wedge \text{wt } 3 + \mathrm{Li}^{\mathcal{L}}_4\text{'s} \wedge \text{wt } 2\,, \\ \overline{\Delta}^{[2]}\,\mathrm{Li}^{\mathcal{L}}_{4,1,1}(x,y,z) = {} & -\mathrm{Li}^{\mathcal{L}}_2(y) \otimes \mathrm{Li}^{\mathcal{L}}_2(z) \otimes \mathrm{Li}^{\mathcal{L}}_2(xyz) \in \mathrm{CoLie}_3 = \mathcal{L}^{\otimes 3}/\sqcup\!\sqcup \end{aligned}$$

Hence $\mathrm{Li}^{\mathcal{L}}_{4,1,1}(x,y,z) \neq \sum$ depth 2.

Conjecture 2 (Gonchrov Depth Conjecture)**.** *Write $\mathcal{B}_n = \{$classical polylogarithms, modulo products$\}$, for $k \geq 2$, the following map is an isomorphism*

$$\overline{\Delta}^{[k-1]} \colon \mathrm{gr}^{\mathcal{D}}_k \mathcal{L} \xrightarrow{\cong ?} \mathrm{CoLie}_k\Big(\bigoplus_{n\geq 2} \mathcal{B}_n\Big)$$

(Simplified: A linear combination of MPL's has depth $< k$ if and only if $(k-1)$-st iterated truncated cobracket $\overline{\Delta}^{[k-1]}$ vanishes.)

Implications:

- Volumes of hyperbolic manifolds are depth 1 MPL's (as Dehn invariant $\approx$ coproduct vanishes)
- Crucial part of Zagier's conjecture on Dedekind zeta $\zeta_F(k)$

2. Expectations, consequences and results

(I won't say much about the proofs, other than the general flavour. But feel free to ask later or afterwards.)

Observation: If $k > \frac{\text{weight}}{2}$, then $\overline{\Delta}^{[k-1]}\mathcal{L}_n = 0$. Because each factor of $\overline{\Delta}^{[k-1]}$ has weight ≥ 2.

Expectation: Every weight n MPL can be expressed via depth $< \lfloor \frac{n}{2} \rfloor$.

Theorem 3 ((Unobstructed case) Rudenko, 2022)**.** *This is expectation true.*

Proof idea. Explicit formula via quadrangular polylogarithms $\mathrm{QLi}_n(x_1, \ldots, x_{2n+2})$ of depth $\frac{n}{2}$. QLi is sum of correlators[1] $\mathrm{Cor}(x_{i_1}, \ldots, x_{i_n})$ so has a nice symbol. Then quadrangulation formula writes it via lower depth MPL's. □

Useful to introduce

$$\mathrm{Li}^{\mathfrak{L}}_{a_0\,;\,a_1,\ldots,a_d}(x_1, \ldots, x_d) = (-1)^d I^{\mathfrak{L}}(0; \{0\}^{a_0}, 1, \{0\}^{a_1-1}, x_1, \{0\}^{a_2-1}, x_1x_2, \{0\}^{a_2-1}, \ldots, x_1x_2\cdots x_{d-1}, \{0\}^{a_d-1}; x_1x_2\cdots x_d)\,.$$

Mainly consider

$$\mathrm{Li}^{\mathfrak{L}}_{k\,;\,1,\ldots,1}(x_1, \ldots, x_d) = (-1)^d I^{\mathfrak{L}}(0; \{0\}^k, 1, x_1, x_1x_2, \ldots, x_1x_2\cdots x_{d-1}; x_1x_2\cdots x_d)\,.$$

This is a "nice" variant of $\mathrm{Li}^{\mathfrak{L}}_{k+1,1,\ldots,1}(x_2, \ldots, x_d, (x_1x_2\cdots x_d)^{-1})$, modulo lower depth terms.
Then

$$\Delta^{[k-1]}\,\mathrm{Li}^{\mathfrak{L}}_{k\,;\,1,\ldots,1}(x_1, \ldots, x_k) = -\,\mathrm{Li}^{\mathfrak{L}}_2(x_1) \otimes \mathrm{Li}^{\mathfrak{L}}_2(x_2) \otimes \cdots \otimes \mathrm{Li}^{\mathfrak{L}}_2(x_k) \in \mathrm{CoLie}_k$$

So $\mathrm{Li}^{\mathfrak{L}}_{k\,;\,1,\ldots,1}(x_1, \ldots, x_k)$ behaves like Li_2 in each argument.

Expectations in even weight: One therefore conjectures reductions

(Nielsen-type) $$\mathrm{Li}_{k\,;\,1,\ldots,1}(1, x_2, \ldots, x_k) \equiv 0 \pmod{\text{depth} < k}$$

(Zagier-type) $$\mathrm{Li}_{k\,;\,1,\ldots,1}(x_1, x_2, \ldots, x_k) + \mathrm{Li}_{k\,;\,1,\ldots,1}(\underbrace{1-x_1}_{\text{or } x_1^{-1}}, x_2, \ldots, x_k) \equiv 0 \pmod{\text{depth} < k}$$

(Gangl-type) $$\mathrm{Li}_{k\,;\,1,\ldots,1}(\underbrace{V(y,z)}_{\text{dilogarithm 5-term relation: } -[y]+[z]-[\frac{z}{y}]+[\frac{1-z}{1-y}]-[\frac{y(1-z)}{(1-y)z}]}, x_2, \ldots, x_k) \equiv 0 \pmod{\text{depth} < k}$$

Here Nielsen refers to the reduction of Nielsen polylogarithm (c.f. Kölbig, Lewin, ...)

$$S^{\mathcal{L}}_{2,2}(x) = I^{\mathcal{L}}(0; 1, 1, 0, 0; x) = \mathrm{Li}^{\mathcal{L}}_{1,3}(1, x) \equiv -I^{\mathcal{L}}(0; 0, 0, x^{-1}, x^{-1}; 1) = -\,\mathrm{Li}^{\mathcal{L}}_{2\,;\,1,1}(1, x)$$
$$= -\,\mathrm{Li}_4(1-x) + \mathrm{Li}_4(x) + \mathrm{Li}_4\left(\frac{x}{x-1}\right).$$

Zagier refers to reduction given by Zagier for

$$\underbrace{\mathrm{Li}_{3,1}(\tfrac{1-x}{y}, y)}_{\mathrm{Li}_{2\,;\,1,1}(1-x,y)} + \underbrace{\mathrm{Li}_{3,1}(\tfrac{x}{y}, y)}_{\mathrm{Li}_{2\,;\,1,1}(x,y)} = \sum \mathrm{Li}_4\ .$$

Gangl refers to reduction given by Gangl for

$$\mathrm{Li}_{2\,;\,1,1}(V(x,y), z) = \sum_{i=1}^{122} \mathrm{Li}_4\ .$$

Expectations are known in weight 4. Now we have it in weight 6.

Theorem 4 (Matveiakin-Rudenko 2022)**.** *Gangl-type reduction holds in weight 6, modulo assuming the Zagier-type reductions*

Theorem 5 (C, 2023/24+)**.** *Nielsen-type and Zagier-type reductions hold in weight 6.*

[1]Essentially $I(\infty; x_{i_1}, \cdots ; x_{i_n})$

Hence: Goncharov's depth conjecture in weight 6 depth 3.

Proof idea. Degenerations and specialisations of the weight 6 quadrangular polylogarithm functional equation. (Zagier-type reduction is harder, as it involves combining many non-obvious specialisations, and finding already Nielsen type reductions. Final result for $\mathrm{Li}_{3\,;\,1,1,1}(x,y,z) + \mathrm{Li}_{3\,;\,1,1,1}(1-x,y,z)$ is aprpox. 20 thousand terms. Gangl type reduction is more tractable with linear algebra investigations, but still involves combining many specialisations.) ☐

Some roadmap to generalise to weight 8 and higher.

3. Results in odd weight

In odd weight, not as much progress

$$\Delta^{[k-1]} \mathrm{Li}^{\mathfrak{L}}_{k+1\,;\,1,\ldots,1}(x_1,\ldots,x_k) = \sum_{i_1+\cdots+i_k=1} -\mathrm{Li}^{\mathfrak{L}}_{2+i_1}(x_1) \otimes \mathrm{Li}^{\mathfrak{L}}_{2+i_2}(x_2) \otimes \cdots \otimes \mathrm{Li}^{\mathfrak{L}}_{2+i_k}(x_k) \in \mathrm{CoLie}_k \,.$$

So $\mathrm{Li}^{\mathfrak{L}}_{k\,;\,1,\ldots,1}(x_1,\ldots,x_k)$ has Li_2 and Li_3 component in each variable. Then e.g.

$$\Delta^{[2-1]}(\mathrm{Li}^{\mathfrak{L}}_{3\,;\,1,1}(x,z) + \mathrm{Li}^{\mathfrak{L}}_{4\,;\,1,1}(x,z^{-1})) = -2\,\mathrm{Li}^{\mathfrak{L}}_2(x) \otimes \otimes \mathrm{Li}^{\mathfrak{L}}_3(z) \in \mathrm{CoLie}_2$$
$$\Delta^{[3-1]}(\mathrm{Li}^{\mathfrak{L}}_{4\,;\,1,1,1}(x,y,z) + \mathrm{Li}^{\mathfrak{L}}_{4\,;\,1,1,1}(x,y,z^{-1})) = -2\,\mathrm{Li}^{\mathfrak{L}}_2(x) \otimes \mathrm{Li}^{\mathfrak{L}}_2(y) \otimes \mathrm{Li}^{\mathfrak{L}}_3(z) \in \mathrm{CoLie}_3$$

Should get reductions when x, y is dilogarithm identity, or when z is trilogarithm identity. Even in weight 5, only partial progress.

Expectation: Since Nielsen polylogarithm $S_{3,2}(z) = \mathrm{Li}_{1,4}(1,z) \equiv \mathrm{Li}_{3\,;\,1,1}(z,1)$, expect it satisfies 5-term relation

$$S_{3,2}(V(x,y)) = \sum \mathrm{Li}_5$$

Theorem 6 (C-Gangl-Radcheko, 2020)**.** $\sum_i (-1)^i S_{3,2}(cr(x_1,\ldots,x_i,\ldots,x_5))$ *is sum of 3-orbits of* Li_5 *of certain higher ratios*

$$r_1 = \cdots\,, r_2 = \cdots\,, r_3 = \frac{x_{12}^3 x_{15} x_{34}^2 x_{35}}{x_{13}^3 x_{14} x_{24} x_{25}^2}\,, \quad x_{ij} = x_i - x_j\,.$$

Proof idea. Direct (structured) calculation, using representation theory, once explicit identity was found on computer. (Goal: revisit via quadrangular polylogarithm identities.) ☐

Theorem 7 (C, 2019–???)**.** *Expression for*

$$\mathrm{Li}^{\mathfrak{L}}_{3\,;\,1,1}(x, \textit{22-term}) = \sum_i \mathrm{Li}^{\mathfrak{L}}_{3\,;\,1,1}(V(p_i,q_i), r_i) \pmod{\textit{depth } 1}\,.$$

4. Other predictions of Goncharov's depth conjecture

It is straightforward to see $\overline{\Delta}^{[k-1]}\big(\mathrm{gr}^{\mathcal{D}}_k \mathcal{L}\big)$ is expressed by depth 1 (i.e. $\bigoplus \mathcal{B}_n$). So map in Goncharov's depth conjecture is well-defined. Surjectivity is *not* clear, but if the field is quadratically closed, there is a cute proof.

Theorem 8 (CGRaRu, 2024)**.** *If F quadratically closed* $\overline{\Delta}^{[k-1]}$ *surjective.*

Proof idea.

$$\overline{\Delta}^{[k-1]} \mathrm{Li}^{\mathfrak{L}}_{n-k\,;\,1,\ldots,1}(a_1,\ldots,a_k) = \sum_{n_1+\cdots+n_k=n, n_i\geq 2} \mathrm{Li}^{\mathfrak{L}}_{n_1}(a_1) \otimes \cdots \otimes \mathrm{Li}^{\mathfrak{L}}_{n_k}(a_k)\,.$$

Then use distribution relation $\mathrm{Li}^{\mathfrak{L}}_n(a^r) = r^{n-1} \sum_{\zeta^r=1} \mathrm{Li}^{\mathfrak{L}}_n(\zeta a)$, and properties of Vandermonde determinant to isolate individual terms. ☐

[**Corollary:** The case $k = 1$ of the depth conjecture implies the full depth conjecture.]
[*Question:* Can this be done over arbitrary fields? I.e. with only rational functions of arguments?]

This surjectivity leads to a *surprising* prediction. As $\mathrm{Li}_{n-d\,;\,1,\ldots,1}$ generates image of $\overline{\Delta}^{[d-1]}$, then modulo depth $< d$, every depth d MPL can be expressed via

$$\mathrm{Li}_{n-d\,;\,1,\ldots,1} \approx \mathrm{Li}_{0\,;\,n-d+1,1,\ldots,1}\ .$$

In depth 2, there is a cute proof

Theorem 9 (CGRaRu, 2024)**.** $\mathrm{Li}_{k,n-k}(x,y)$ *can be expressed via* $\mathrm{Li}_{n-1,1}(\sqrt[N]{x}^r\sqrt[N]{y}^s, \sqrt[N]{x}^t\sqrt[N]{y}^u)$, *and products of depth 1 MPL's, for some N.*

Proof idea. Partial fractions decomposition to show a certain identity for sum of $\mathrm{Li}_{n-1,1}(x/y,y) + \mathrm{Li}_{n-1,1}(y/x,x) + \mathrm{Li}_{n-1,1}(y,x)$'s expressed via $\sum \mathrm{Li}_{k,l}(y,x)$, then Vandermonde matrix inversion to isolate a single $\mathrm{Li}_{k,\ell}$. □

[**Remark:** More recent work (in progress) C-Ra-Ru, where we try to generalise this to higher depth.]

Conclusions

Lots of progress on/around Goncharov's depth conjecture. Main focus for research is odd weight (weight 5!), and weight $2k$ depth k. Also try to understand other MPL relations (with algebraic arguments), like the reduction to $\mathrm{Li}_{n+1-d,1,\ldots,1}$.