Goncharov's programme,
and depth reductions of multiple polylogarithms

Steven Charlton
Max Planck Institute for Mathematics, Bonn

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🏳️‍🌈 $\mathrm{Spec}(\overline{\mathbb{Q}}(2\pi i))$, Fields Institute, Toronto

Multiple polylogarithm definitions

Definition

Multiple polylogarithm (MPL) is

$$\mathrm{Li}_{k_1, k_2, \ldots, k_d}(x_1, x_2, \ldots, x_d) := \sum_{0 < m_1 < m_2 < \cdots < m_d} \frac{x_1^{m_1} \cdots x_d^{m_d}}{m_1^{k_1} \cdots m_d^{k_d}}, \quad |x_i| < 1 \, .$$

Generalises

$$\mathrm{Li}_1(x) = -\log(1 - x) \, .$$

- Weight is $n = k_1 + \cdots + k_d$,

- Depth is d.

Goal: Fundamental challenge around MPL's is to understand the filtration by depth.

Relating different depths

Functions of different depths can be related:

$$\mathrm{Li}_1(x)\,\mathrm{Li}_1(y) = \mathrm{Li}_{1,1}(x,y) + \mathrm{Li}_{1,1}(y,x) + \mathrm{Li}_2(xy)$$

$$\leftrightarrow \quad \sum_{m=1}^{\infty}\sum_{\ell=1}^{\infty} \frac{x^m y^\ell}{m\cdot\ell} = \left(\sum_{m<\ell} + \sum_{m>\ell} + \sum_{m=\ell}\right)\frac{x^m y^\ell}{m\cdot\ell}$$

So-called stuffle product of MPL's ⤳ this generalises easily

More surprising when the depth reduces

> **Proposition (Goncharov, Zagier, Lewin[(?)])**
>
> *As a power-series identity, for $|xy| < 1, |y| < 1$,*
>
> $$\mathrm{Li}_{1,1}(x,y) = \mathrm{Li}_2\left(\frac{y(x-1)}{1-y}\right) - \mathrm{Li}_2\left(\frac{-y}{1-y}\right) - \mathrm{Li}_2(xy)\,.$$

Further depth reductions?

Proposition ($\mathrm{Li}_{1,1,1}$ is depth 1,   Goncharov, Zhao, Lewin[(?)], ...)

$$\mathrm{Li}_{1,1,1}(x,y,z) \;=\; -\mathrm{Li}_3\Big(\tfrac{1-xyz}{1-x}\Big) - \mathrm{Li}_3\Big(\tfrac{1-xyz}{xy(1-z)}\Big) + \mathrm{Li}_3\Big(\tfrac{(y-1)(1-xyz)}{(1-x)y(1-z)}\Big) + \mathrm{Li}_3(xy)$$
$$-\mathrm{Li}_3\Big(\tfrac{y(1-x)}{y-1}\Big) + \mathrm{Li}_3(1-x) - \mathrm{Li}_3\Big(\tfrac{y(1-z)}{y-1}\Big) + \mathrm{Li}_3\Big(\tfrac{-y}{1-y}\Big) + \textit{products}$$

Can weight 4 be reduced to depth 1? Apparently not, but:

Proposition ($\mathrm{Li}_{3,1}$ relations, mod depth 1,   Zagier, Gangl)

$$\mathrm{Li}_{3,1}\Big(\tfrac{1-x}{y},y\Big) + \mathrm{Li}_{3,1}\Big(\tfrac{x}{y},y\Big) \;=\; -\tfrac{1}{2}\mathrm{Li}_4\Big(\tfrac{(1-x)y}{x(1-y)}\Big) - \tfrac{1}{2}\mathrm{Li}_4\Big(\tfrac{xy}{(1-x)(1-y)}\Big) + \tfrac{1}{2}\mathrm{Li}_4\Big(\tfrac{(1-y)y}{(1-x)x}\Big)$$
$$-\mathrm{Li}_4\Big(\tfrac{1-y}{1-x}\Big) - \mathrm{Li}_4\Big(\tfrac{1-y}{x}\Big) + \mathrm{Li}_4\Big(-\tfrac{y}{1-y}\Big)$$
$$-\mathrm{Li}_4(1-x) - \mathrm{Li}_4(x) + 2\mathrm{Li}_4(y) + \textit{products}$$

Question: How to predict or understand, such reductions and obstructions?

Connections elsewhere

Values of zeta functions: ⟦see Kairi Black's lightning talk, Tuesday⟧

- Dedekind zeta function of a number field F is $\zeta_F(s) = \sum_{I \neq 0 \subset \mathcal{O}_F} N(I)^{-s}$.

- Analytic class number formula: $\mathrm{Res}_{s=1} \zeta_F(s)$ via $\underbrace{\text{regulator } R_F}_{\log / \mathrm{Li}_1 \text{ of units}}$ (& arithmetic data).

- Zagier: conjecture for $\zeta_F(n)$ via Li_n (& 'higher units' in K-theory).

- Goncharov relates $\zeta_F(n)$ to high-depth "Grassmannian polylogarithm".

Task: How to reduce to depth 1?

Also: hyperbolic geometry, cluster algebras, high-energy physics, . . .

Motivic multiple polylogarithms

Recall: Iterated integral ⟦special case from Rosa Preiß's talk, Thursday⟧

$$I_{(\gamma)}(a; x_1, \ldots, x_n; b) := \int_{a < t_1 < \cdots < t_n < b} \frac{dt_1}{t_1 - x_1} \wedge \frac{dt_2}{t_2 - x_2} \wedge \cdots \wedge \frac{dt_n}{t_n - x_n}.$$

By term-wise integration of geometric series (with $X_j := \prod_{i=1}^{j} x_i$),

$$\mathrm{Li}_{k_1, k_2, \ldots, k_d}(x_1, \ldots, x_d) = (-1)^d I(0; 1, \underbrace{0, \ldots, 0, X_1}_{k_1}, \underbrace{0, \ldots, 0, X_2}_{k_2}, \ldots, X_{d-1}, \underbrace{0, \ldots, 0; X_d}_{k_d}).$$

- Goncharov: upgrade of I to framed mixed Tate motives $I^{\mathfrak{u}}(a; x_1, \ldots, x_n; b)$,
  $I^{\mathfrak{u}}$ lives in connected, weight graded Hopf algebra $(\mathcal{A}_\bullet(\overline{\mathbb{Q}}), \Delta)$
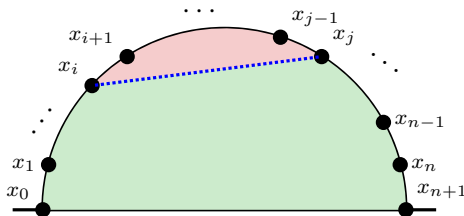
Idea: Can keep track of differential form and integration chain separately

- Define $\mathrm{Li}^{\mathfrak{u}}_{k_1, k_2, \ldots, k_d}(x_1, \ldots, x_d)$ via $I^{\mathfrak{u}}$

Lie coalgebra, cobracket, depth filtration

- $\mathcal{L}_\bullet = \mathcal{A}_\bullet / \mathcal{A}_{>0}^2 = $ motivic MPL's modulo products, obtain $\Delta \colon \mathcal{L}_\bullet(F) \to \bigwedge^2 \mathcal{L}_\bullet(F)$

$$\Delta \operatorname{I}^\mathcal{L}(x_0; x_1, \ldots, x_n; x_{n+1}) = \sum_{i<j} \operatorname{I}^\mathcal{L}(x_i; x_{i+1}, \ldots, x_{j-1}; x_j) \wedge \operatorname{I}^\mathcal{L}(x_0; x_1, \ldots, x_i, x_j, \ldots, x_n; x_{n+1})$$

- Depth filtration $\mathcal{D}_d = \{$depth $\leq d$ MPL's in $\mathcal{L}_\bullet\}$ has

$$\Delta(\mathcal{D}_d) \quad \subset \quad \mathcal{D}_d \wedge \mathcal{L}_1 \oplus \bigoplus_{\substack{i+j=d \\ i,j \geq 1}} \mathcal{D}_i \wedge \mathcal{D}_j \,.$$

Why? Follows since $\operatorname{I}^\mathcal{L}(a; \overbrace{0, \ldots, 0}^{\ell \geq 2}; b) = \frac{1}{\ell!} \operatorname{I}^\mathcal{L}(a; 0, b)^\ell \equiv 0 \pmod{\text{products}}$

Cobrackets and obstructions to depth 1

$$\Delta \operatorname{Li}_4^{\mathcal{L}}(x) = \log^{\mathcal{L}}(x) \wedge \operatorname{Li}_3^{\mathcal{L}}(x) \qquad\qquad\qquad \in \mathcal{L}_1 \wedge \mathcal{L}_3\,,$$

$$\left.\begin{aligned}\Delta \operatorname{Li}_{3,1}^{\mathcal{L}}(x,y) = {}& -\log^{\mathcal{L}}(x) \wedge \operatorname{Li}_{2,1}^{\mathcal{L}}(x,y) + \operatorname{Li}_1^{\mathcal{L}}(y) \wedge \operatorname{Li}_3^{\mathcal{L}}(x) - \operatorname{Li}_1^{\mathcal{L}}(y) \wedge \operatorname{Li}_3^{\mathcal{L}}(xy)\\ & - \operatorname{Li}_1^{\mathcal{L}}(xy) \wedge \operatorname{Li}_3^{\mathcal{L}}(x) + \operatorname{Li}_1^{\mathcal{L}}(xy) \wedge \operatorname{Li}_3^{\mathcal{L}}(y)\end{aligned}\right\} \in \mathcal{L}_1 \wedge \mathcal{L}_3$$

$$+ \operatorname{Li}_2^{\mathcal{L}}(y) \wedge \operatorname{Li}_2^{\mathcal{L}}(xy) \qquad\qquad\qquad \in \mathcal{L}_2 \wedge \mathcal{L}_2$$

- Write $\overline{\Delta}$ to mean ignore $\mathcal{L}_1$, then we obtain

$$\overline{\Delta} \operatorname{Li}_4^{\mathcal{L}}(x) = 0\,, \qquad \overline{\Delta} \operatorname{Li}_{3,1}^{\mathcal{L}}(x,y) = \operatorname{Li}_2^{\mathcal{L}}(y) \wedge \operatorname{Li}_2^{\mathcal{L}}(xy)\,.$$

Hence $\operatorname{Li}_{3,1}^{\mathcal{L}}(x,y) \neq \sum \operatorname{Li}_4^{\mathcal{L}}$'s.

Question: Is this the only obstruction?

Goncharov Depth Conjecture (depth 1, simplified version)

A linear combination of MPL's has depth 1 if and only if $\overline{\Delta}$ vanishes.

Expectations and evidence

Weight 2, 3: No space for non-trivial contribution, so

$$\overline{\Delta}(\mathcal{L}_2) = 0 \,, \quad \overline{\Delta}(\mathcal{L}_3) = 0 \,.$$

Expect $\mathrm{Li}_{1,1}^{\mathcal{L}}(x,y) = \sum \mathrm{Li}_2^{\mathcal{L}}$'s, and $\mathrm{Li}_{1,1,1}^{\mathcal{L}}(x,y,z) = \sum \mathrm{Li}_3^{\mathcal{L}}$'s. $\qquad \checkmark$ (Intro)

Weight 4: After tweaking variables $\overline{\Delta}\,\mathrm{Li}_{3,1}^{\mathcal{L}}(\frac{x}{y},y) = -\,\mathrm{Li}_2^{\mathcal{L}}(x) \wedge \mathrm{Li}_3^{\mathcal{L}}(y)$.

Vanishing of $\overline{\Delta}\big(\sum_i \lambda_i \,\mathrm{Li}_{3,1}^{\mathcal{L}}(\frac{x_i}{y},y)\big)$ under dilogarithm identities $\sum_i \lambda_i \,\mathrm{Li}_2(x_i) = 0$.

- Symmetries $\mathrm{Li}_2^{\mathcal{L}}(x) + \mathrm{Li}_2^{\mathcal{L}}(1-x) = 0$ and $\mathrm{Li}_2^{\mathcal{L}}(x) + \mathrm{Li}_2^{\mathcal{L}}(x^{-1}) = 0$ $\qquad \checkmark$ (Intro $+\,\varepsilon$)

- 5-term relation $\sum_{i=0}^{4}(-1)^i \,\mathrm{Li}_2^{\mathcal{L}}([z_0,\ldots,\widehat{z_i},\ldots,z_4]) = 0, \quad [z_1,\ldots,z_4] = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$

Five-term & cluster algebras [[see José Simental's talk, Thursday]]

Cluster algebra magic: $\sum_{i=1}^{5} \mathrm{Li}_2^{\mathcal{L}}(-x_i) = 0$

Expectations and evidence

Weight 2, 3: No space for non-trivial contribution, so

$$\overline{\Delta}(\mathcal{L}_2) = 0\,, \quad \overline{\Delta}(\mathcal{L}_3) = 0\,.$$

Expect $\mathrm{Li}_{1,1}^{\mathcal{L}}(x,y) = \sum \mathrm{Li}_2^{\mathcal{L}}$'s, and $\mathrm{Li}_{1,1,1}^{\mathcal{L}}(x,y,z) = \sum \mathrm{Li}_3^{\mathcal{L}}$'s. $\qquad \checkmark$ (Intro)

Weight 4: After tweaking variables $\overline{\Delta}\,\mathrm{Li}_{3,1}^{\mathcal{L}}(\frac{x}{y},y) = -\,\mathrm{Li}_2^{\mathcal{L}}(x) \wedge \mathrm{Li}_3^{\mathcal{L}}(y)\,.$
Vanishing of $\overline{\Delta}\big(\sum_i \lambda_i \,\mathrm{Li}_{3,1}^{\mathcal{L}}(\frac{x_i}{y},y)\big)$ under dilogarithm identities $\sum_i \lambda_i \,\mathrm{Li}_2(x_i) = 0$.

- Symmetries $\mathrm{Li}_2^{\mathcal{L}}(x) + \mathrm{Li}_2^{\mathcal{L}}(1-x) = 0$ and $\mathrm{Li}_2^{\mathcal{L}}(x) + \mathrm{Li}_2^{\mathcal{L}}(x^{-1}) = 0$ $\qquad \checkmark$ (Intro $+\,\varepsilon$)

- 5-term relation $\sum_{i=0}^{4}(-1)^i \,\mathrm{Li}_2^{\mathcal{L}}([z_0,\ldots,\widehat{z_i},\ldots,z_4]) = 0, \quad [z_1,\ldots,z_4] = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$

Theorem (5-term reduction, Gangl 2012; Goncharov-Rudenko, Matveiakin-Rudenko)

$$\sum_{i=0}^{4}(-1)^i \,\mathrm{Li}_{3,1}^{\mathcal{L}}\left(\frac{\mathrm{cr}(z_0,\ldots,\widehat{z_i},\ldots,z_4)}{y}, y\right) = \sum_{i=1}^{122+\varepsilon} \mathrm{Li}_4^{\mathcal{L}}(g_i(z_0,\ldots,z_4,y))\,, \quad g_i \text{ rational}$$

Remark: Crucial ingredient to Goncharov-Rudenko proof of Zagier's Conjecture on $\zeta_F(4)$.

Iterating to understand higher depth

- Computing $\overline{\Delta}$ gets complicated:

$$\overline{\Delta}\,\mathrm{Li}_{4,1,1}^{\mathcal{L}}(x,y,z) = \mathrm{Li}_2^{\mathcal{L}}(y) \wedge \mathrm{Li}_{3,1}^{\mathcal{L}}(xy,z) - \mathrm{Li}_{3,1}^{\mathcal{L}}\Big(yz,\tfrac{1}{y}\Big) \wedge \mathrm{Li}_2^{\mathcal{L}}(xyz) - \mathrm{Li}_3^{\mathcal{L}}(y) \wedge \mathrm{Li}_{2,1}^{\mathcal{L}}(xy,z)$$
$$- \mathrm{Li}_{2,1}^{\mathcal{L}}\Big(yz,\tfrac{1}{y}\Big) \wedge \mathrm{Li}_3^{\mathcal{L}}(xyz) + \mathrm{Li}_4^{\mathcal{L}}\text{'s} \wedge \text{weight } 2 + \mathrm{Li}_n^{\mathcal{L}}\text{'s} \wedge \mathrm{Li}_m^{\mathcal{L}}\text{'s}.$$

- Use $\overline{\Delta}$ to pick "genuine" depth 2 terms, so iterate[1] $\overline{\Delta}^{[2]} := \overline{\Delta} \circ (\overline{\Delta} \otimes \mathrm{id})$

$$\overline{\Delta}^{[2]}\,\mathrm{Li}_{4,1,1}^{\mathcal{L}}(x,y,z) = \mathrm{Li}_2^{\mathcal{L}}(y) \otimes \mathrm{Li}_2^{\mathcal{L}}(z) \otimes \mathrm{Li}_2^{\mathcal{L}}(xyz) \quad (\text{in } \mathrm{CoLie}_3).$$

Since $\overline{\Delta}^{[2]}\,\mathrm{Li}_{a,b}^{\mathcal{L}}(x,y) = 0$, we deduce $\mathrm{Li}_{4,1,1}^{\mathcal{L}}(x,y,z) \neq \sum \text{depth 2}$.

Question: Is this the only obstruction?

Goncharov Depth Conjecture (simplified version)

A linear combination of MPL's has depth $< k$ if and only if $\overline{\Delta}^{[k-1]}$ vanishes.

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[1]needs to be formalised: in a coLie algebra, the coJacobi identity would imply naïve $\Delta^2 = 0$

Expectations and evidence (II)

Observe: If $k > \frac{n}{2}$, then $\overline{\Delta}^{[k-1]}(\mathcal{L}_n) = 0$ (No space for k-many weight 2 pieces.)

Theorem (Rudenko, 2022)

Every weight n multiple polylogarithm can be expressed via depth $\leq \lfloor \frac{n}{2} \rfloor$ functions.

Idea: Explicit formula via "quadrangular" polylogs with depth $\frac{n}{2}$

Write:
$$f(x, y, z) := \mathrm{Li}_{4,1,1}^{\mathcal{L}}((xyz)^{-1}, y, z)$$
$$\overline{\Delta}^{[2]} f(x, y, z) = -\mathrm{Li}_2^{\mathcal{L}}(x) \otimes \mathrm{Li}_2^{\mathcal{L}}(y) \otimes \mathrm{Li}_2^{\mathcal{L}}(z) \quad (\text{in } \mathrm{CoLie}_k)$$

Then $f(x, y, z)$ behaves like $\mathrm{Li}_2^{\mathcal{L}}$ in each argument. Expect the following reductions:

- $f(x, y, z) + f(\underbrace{1 - x}_{\text{or } x^{-1}}, y, z) \overset{?}{\equiv} 0 \pmod{\mathsf{dp} \leq 2}$ (Zagier-type)

- $f(\text{dilog 5-term}, y, z) \overset{?}{\equiv} 0 \pmod{\mathsf{dp} \leq 2}$ (Gangl-type)

Weight 6 results

Saw similar expectations hold in weight 4. They are also recently known in weight 6. Write $f(x, y, z) \coloneqq \mathrm{Li}_{4,1,1}^{\mathcal{L}}((xyz)^{-1}, y, z)$, as before

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Theorem (Gangl-type reduction,   Matveiakin-Rudenko 2022)

$$f(\text{dilog 5-term}, y, z) \equiv 0 \quad (\text{mod depth} \leq 2 \ \& \ \text{the Zagier reductions})$$

Theorem (Zagier-type reductions,   C 2024,   arXiv:2405.13853)

$$f(x, y, z) + f(1 - x, y, z) \equiv 0 \quad (\text{mod depth} \leq 2)$$
$$f(x, y, z) + f(x^{-1}, y, z) \equiv 0 \quad (\text{mod depth} \leq 2)$$

Consequence: Goncharov's Depth Conjecture holds for weight 6 depth 3 ($k = 3$).

(Zagier-reductions seem more difficult than the Gangl-reduction in general!)

Remark: Similar conjectural expectations in all even weights.

Ingredients for the proof – Quadrangular polylogarithms

Rudenko's quadrangular polylogarithms $\mathrm{QLi}_{n+k}(x_0, \ldots, x_{2n+1})$

- "Nice" MPL functions on $\mathfrak{M}_{0,2n+2}$, expressible via "quadrangulations" of polygons
- Weight-independent depth n expression via cross-ratio $[x_a, x_b, x_c, x_d] = \frac{(x_a - x_b)(x_c - x_d)}{(x_b - x_c)(x_d - x_a)}$

$$\mathrm{QLi}_{k+2}(x_1, \ldots, x_6) = \mathrm{Li}^{\mathcal{L}}_{k\,;\,1,1}\left(+ \raisebox{-1.5em}{} - \raisebox{-1.5em}{} + \raisebox{-1.5em}{} \right)$$

$$= \mathrm{Li}^{\mathcal{L}}_{k\,;\,1,1}\left([x_1, x_2, x_3, x_6], [x_3, x_4, x_5, x_6]\right) - \mathrm{Li}^{\mathcal{L}}_{k\,;\,1,1}\left([x_1, x_2, x_5, x_6], [x_3, x_4, x_5, x_2]\right)$$
$$+ \mathrm{Li}^{\mathcal{L}}_{k\,;\,1,1}\left([x_1, x_4, x_5, x_6], [x_1, x_2, x_3, x_4]\right)$$
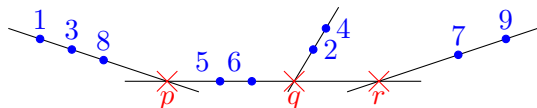
General functional equation

$$\mathbf{Q}_{2k}: \qquad \sum_{i=1}^{2k+3}(-1)^i \mathrm{QLi}_{2k}(x_1, \ldots, \widehat{x_i}, \ldots, x_{2k+3}) \equiv 0 \pmod{\text{depth} < k}$$

Ingredients (II) – Stable genus 0 curves

Part of boundary strata of Deligne-Mumford compactification $\overline{\mathfrak{M}}_{0,n}$
⟦From Hannah Larson's talk, Monday⟧

- Components are isomorphic to $\mathbb{P}^1$
- Only singular points are simple double points
- Number of marked & singular points per components ≥ 3

Gadget to encode limits of projective configurations
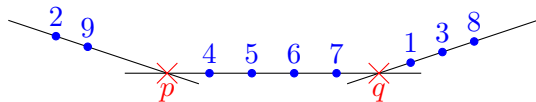
Idea: The points x_1, x_3, x_8 all go to p

But: There is a always projective transformation moving (x_1, x_3, x_8, p) to $(\infty, 0, 1, z)$

So: Points x_1, x_3, x_8 split off as a separate $\mathbb{P}^1$

Cross-ratios well-defined: $[x_1, x_3, x_8, x_i] = [x_1, x_3, x_8, p]$ and $[x_1, x_3, x_2, x_4] = [x_1, x_3, p, p] = 0$

Ingredients (III) – Build to symmetries step-by-step

Specialise $\mathbf{Q}_6$ to

and use $\lim_{x_i \to 0 \text{ or } \infty} \mathrm{Li}^{\mathcal{L}}_{k_1, \ldots, k_d}(x_1, \ldots, x_d) = 0 \pmod{\mathsf{dp} < d}$

Write $f(x, y, z) := \mathrm{Li}^{\mathcal{L}}_{4,1,1}((xyz)^{-1}, y, z)$, as before

Lemma (Full Symmetry 1)

$$f(A, B, C) \equiv -f\left(1 - A,\, \frac{B}{B-1},\, 1 - C\right) \pmod{\text{depth } 2}$$

Remark: First needed to show $f(1, x, y) \equiv 0 \pmod{\text{depth } 2}$, already non-trivial(!)

- Find a second *different* symmetry, and a four-term relation
- Play the symmetries against each other
- Extract (eventually. . .) the Zagier-type reduction

Summary

- Depth reductions of MPL's
  - Appearances in Number Theory, Hyperbolic Geometry, High-Energy Physics

- Goncharov's Depth Conjecture
  - Cobracket on (motivic) MPL's

  - Truncation and iteration of the cobracket

  - Cobracket conjecturally detects depth

- Zagier-, & Gangl-type reductions in weight 4, and weight 6

- Ideas from the proof
  - Quadrangular polylogarithm functional equations

  - Degenerations to stable genus 0 curves

  - Discover and exploit many symmetries/short identities