Goncharov's programme, and depth reductions of multiple polylogarithms

Steven Charlton
MPIM

7 August 2024
Number Theory Lunch Seminar

Multiple polylogarithm definitions

Definition

Multiple polylogarithm (MPL) is

$$\mathrm{Li}_{k_1,k_2,\ldots,k_d}(x_1,x_2,\ldots,x_d) := \sum_{0<m_1<m_2<\cdots<m_d} \frac{x_1^{m_1}\cdots x_d^{m_d}}{m_1^{k_1}\cdots m_d^{k_d}}\,, \quad |x_i|<1\,.$$

Generalises

$$\mathrm{Li}_1(x) = -\log(1-x) = \sum_{m=1}^{\infty} \frac{x^m}{m}\,.$$

- Weight is $n = k_1 + \cdots + k_d$,
- Depth is d.

Goal: Fundamental challenge around MPL's is to understand functional equations, identities, and the filtration by depth.

Functional equations for polylogarithms

- Functional equations for Li_1

$$\log(x^{-1}) = -\log(x)\,, \qquad \log(xy) = \log(x) + \log(y)\,.$$

- Propagation to dilogarithm via $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Li}_2(x) = \frac{1}{x}\mathrm{Li}_1(x)$?

Proposition (Dilogarithm symmetries)

Dilogarithm satisfies two symmetries

$$\begin{aligned} \mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) &= \overbrace{\zeta(2)}^{=\mathrm{Li}_2(1)=\frac{\pi^2}{6}} - \log(1-x)\log(x)\,, \\ \mathrm{Li}_2(x) + \mathrm{Li}_2(x^{-1}) &= -\zeta(2) - \tfrac{1}{2}\log^2(-x)\,, \quad x \notin [0,1] \end{aligned}$$

Theorem (5-term relation, Abel, Spence, Kummer, ...)

For $|x| + |y| < 1$ we have 'fundamental identity'

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(y) - \mathrm{Li}_2\left(\frac{x}{1-y}\right) - \mathrm{Li}_2\left(\frac{y}{1-x}\right) + \mathrm{Li}_2\left(\frac{xy}{(1-x)(1-y)}\right)$$
$$= -\log(1-x)\log(1-y)$$

Relating different depths

Functions of different depths can be related:

$$\mathrm{Li}_1(x)\,\mathrm{Li}_1(y) = \mathrm{Li}_{1,1}(x,y) + \mathrm{Li}_{1,1}(y,x) + \mathrm{Li}_2(xy)$$

$$\leftrightarrow \quad \sum_{m=1}^{\infty}\sum_{\ell=1}^{\infty} \frac{x^m y^\ell}{m\cdot\ell} = \Big(\sum_{m<\ell} + \sum_{m>\ell} + \sum_{m=\ell}\Big)\frac{x^m y^\ell}{m\cdot\ell}$$

So-called stuffle product of MPL's $\rightsquigarrow$ this generalises easily

More surprising when the depth reduces

Proposition (Lewin$^{(?)}$, popularised by Goncharov, Zagier)

As a power-series identity, for $|xy|<1, |y|<1$,

$$\mathrm{Li}_{1,1}(x,y) = \mathrm{Li}_2\Big(\frac{y(x-1)}{1-y}\Big) - \mathrm{Li}_2\Big(\frac{-y}{1-y}\Big) - \mathrm{Li}_2(xy)\,.$$

Further depth reductions?

Proposition ($\mathrm{Li}_{1,1,1}$ is depth 1, Lewin$^{(?)}$, conceptionally by Goncharov, Zhao, ...)

$$\begin{aligned}\mathrm{Li}_{1,1,1}(x,y,z) \;=\; & -\mathrm{Li}_3\Big(\frac{1-xyz}{1-x}\Big) - \mathrm{Li}_3\Big(\frac{1-xyz}{xy(1-z)}\Big) + \mathrm{Li}_3\Big(\frac{(y-1)(1-xyz)}{(1-x)y(1-z)}\Big) + \mathrm{Li}_3(xy) \\ & -\mathrm{Li}_3\Big(\frac{y(1-x)}{y-1}\Big) + \mathrm{Li}_3(1-x) - \mathrm{Li}_3\Big(\frac{y(1-z)}{y-1}\Big) + \mathrm{Li}_3\Big(\frac{-y}{1-y}\Big) + \textit{products}\end{aligned}$$

Can weight 4 be reduced to depth 1? Apparently not (e.g. Wojtkowiak), but:

Proposition ($\mathrm{Li}_{3,1}$ relations, mod depth 1, Zagier, Gangl)

$$\begin{aligned}\mathrm{Li}_{3,1}\Big(\frac{1-x}{y}, y\Big) + \mathrm{Li}_{3,1}\Big(\frac{x}{y}, y\Big) \;=\; & -\tfrac{1}{2}\,\mathrm{Li}_4\Big(\frac{(1-x)y}{x(1-y)}\Big) - \tfrac{1}{2}\,\mathrm{Li}_4\Big(\frac{xy}{(1-x)(1-y)}\Big) + \tfrac{1}{2}\,\mathrm{Li}_4\Big(\frac{(1-y)y}{(1-x)x}\Big) \\ & -\mathrm{Li}_4\Big(\frac{1-y}{1-x}\Big) - \mathrm{Li}_4\Big(\frac{1-y}{x}\Big) + \mathrm{Li}_4\Big(-\frac{y}{1-y}\Big) \\ & -\mathrm{Li}_4(1-x) - \mathrm{Li}_4(x) + 2\,\mathrm{Li}_4(y) + \textit{products}\end{aligned}$$

Question: How to predict or understand, such reductions and obstructions?

Motivations & connections elsewhere

Values of zeta functions:

- Dedekind zeta function of a number field F is $\zeta_F(s) = \sum_{I \neq 0 \subset \mathcal{O}_F} N(I)^{-s}$.
- Analytic class number formula: $\mathrm{Res}_{s=1}\,\zeta_F(s)$ via $\underbrace{\text{regulator } R_F}_{\log\,/\,\mathrm{Li}_1 \text{ of units}}$ (& arithmetic data).
- Zagier: conjecture for $\zeta_F(n)$ via Li_n (& 'higher units' in K-theory).

Example: $$\zeta_{\mathbb{Q}(\sqrt{-7})}(2) = \frac{4\pi^2}{21\sqrt{7}}\left(2\underbrace{D_2}_{\substack{\text{single-valued}\\ \text{version of } \mathrm{Li}_2}}\left(\frac{1+\sqrt{-7}}{2}\right) + D_2\left(\frac{-1+\sqrt{-7}}{4}\right)\right)$$

- Goncharov relates $\zeta_F(n)$ to high-depth "Grassmannian polylogarithm".

Task: How to reduce to depth 1?

Also: hyperbolic geometry, cluster algebras, high-energy physics, . . .

Motivic multiple polylogarithms

Recall: Iterated integral

$$I_{(\gamma)}(a; x_1, \ldots, x_n; b) \coloneqq \int_{a<t_1<\cdots<t_n<b} \frac{\mathrm{d}t_1}{t_1 - x_1} \wedge \frac{\mathrm{d}t_2}{t_2 - x_2} \wedge \cdots \wedge \frac{\mathrm{d}t_n}{t_n - x_n} .$$

By term-wise integration of geometric series (with $X_j \coloneqq \prod_{i=1}^{j} x_i$),

$$\mathrm{Li}_{k_1,k_2,\ldots,k_d}(x_1, \ldots, x_d) = (-1)^d I(0; 1, \underbrace{0, \ldots, 0, X_1}_{k_1}, \underbrace{0, \ldots, 0, X_2}_{k_2}, \ldots, X_{d-1}, \underbrace{0, \ldots, 0; X_d}_{k_d}) .$$

- Goncharov: upgrade of I to framed mixed Tate motives $I^{\mathfrak{u}}(a; x_1, \ldots, x_n; b)$, $I^{\mathfrak{u}}$ lives in connected, weight graded Hopf algebra $(\mathcal{A}_\bullet(\overline{\mathbb{Q}}), \Delta)$

Idea: Can keep track of differential form and integration chain separately

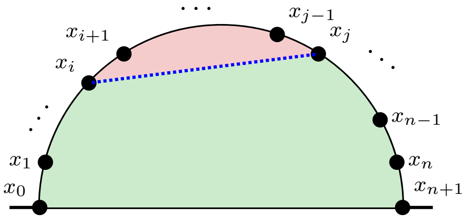
- Define $\mathrm{Li}^{\mathfrak{u}}_{k_1,k_2,\ldots,k_d}(x_1, \ldots, x_d)$ via $I^{\mathfrak{u}}$

Lie coalgebra, cobracket, depth filtration

Write: $\mathcal{L}_\bullet = \mathcal{A}_\bullet / \mathcal{A}_{>0}^2$ = motivic MPL's modulo products

- Image of $I^{\mathfrak{u}}, \mathrm{Li}^{\mathfrak{u}}_{k_1,\ldots,k_d}$ in $\mathcal{L}_\bullet$ is $\mathrm{I}^{\mathcal{L}}, \mathrm{Li}^{\mathcal{L}}_{k_1,\ldots,k_d}$
- Obtain $\Delta\colon \mathcal{L}_\bullet(F) \to \bigwedge^2 \mathcal{L}_\bullet(F)$

$$\Delta\, \mathrm{I}^{\mathcal{L}}(x_0; x_1, \ldots, x_n; x_{n+1}) = \sum_{i<j} \mathrm{I}^{\mathcal{L}}(x_i; x_{i+1}, \ldots, x_{j-1}; x_j) \wedge \mathrm{I}^{\mathcal{L}}(x_0; x_1, \ldots, x_i, x_j, \ldots, x_n; x_{n+1})$$



- Depth filtration $\mathcal{D}_d = \{$depth $\leq d$ MPL's in $\mathcal{L}_\bullet\}$ has

$$\Delta(\mathcal{D}_d) \quad \subset \quad \mathcal{D}_d \wedge \mathcal{L}_1 \oplus \bigoplus_{\substack{i+j=d \\ i,j\geq 1}} \mathcal{D}_i \wedge \mathcal{D}_j \, .$$

Why? Follows since $\mathrm{I}^{\mathcal{L}}(a; \overbrace{0, \ldots, 0}^{\ell \geq 2}; b) = \frac{1}{\ell!}\, \mathrm{I}^{\mathcal{L}}(a; 0, b)^\ell \equiv 0 \pmod{\text{products}}$

Cobrackets and obstructions to depth 1

$$\Delta \operatorname{Li}_4^{\mathcal{L}}(x) = \log^{\mathcal{L}}(x) \wedge \operatorname{Li}_3^{\mathcal{L}}(x) \qquad \in \mathcal{L}_1 \wedge \mathcal{L}_3\,,$$

$$\begin{aligned}\Delta \operatorname{Li}_{3,1}^{\mathcal{L}}(x,y) = &-\log^{\mathcal{L}}(x) \wedge \operatorname{Li}_{2,1}^{\mathcal{L}}(x,y) + \operatorname{Li}_1^{\mathcal{L}}(y) \wedge \operatorname{Li}_3^{\mathcal{L}}(x) - \operatorname{Li}_1^{\mathcal{L}}(y) \wedge \operatorname{Li}_3^{\mathcal{L}}(xy) \\ &- \operatorname{Li}_1^{\mathcal{L}}(xy) \wedge \operatorname{Li}_3^{\mathcal{L}}(x) + \operatorname{Li}_1^{\mathcal{L}}(xy) \wedge \operatorname{Li}_3^{\mathcal{L}}(y) \Bigg\} \in \mathcal{L}_1 \wedge \mathcal{L}_3 \\ &+ \operatorname{Li}_2^{\mathcal{L}}(y) \wedge \operatorname{Li}_2^{\mathcal{L}}(xy) \qquad \in \mathcal{L}_2 \wedge \mathcal{L}_2\end{aligned}$$

- Write $\overline{\Delta}$ to mean ignore $\mathcal{L}_1$, then we obtain

$$\overline{\Delta} \operatorname{Li}_4^{\mathcal{L}}(x) = 0\,, \qquad \overline{\Delta} \operatorname{Li}_{3,1}^{\mathcal{L}}(x,y) = \operatorname{Li}_2^{\mathcal{L}}(y) \wedge \operatorname{Li}_2^{\mathcal{L}}(xy)\,.$$

  Hence $\operatorname{Li}_{3,1}^{\mathcal{L}}(x,y) \neq \sum \operatorname{Li}_4^{\mathcal{L}}$'s.

Question: Is this the only obstruction?

Goncharov Depth Conjecture (depth 1, simplified version)

A linear combination of MPL's has depth 1 if and only if $\overline{\Delta}$ vanishes.

Expectations and evidence

Weight 2, 3: No space for non-trivial contribution, so

$$\overline{\Delta}(\mathcal{L}_2) = 0\,, \quad \overline{\Delta}(\mathcal{L}_3) = 0\,.$$

Expect $\mathrm{Li}^{\mathcal{L}}_{1,1}(x,y) = \sum \mathrm{Li}^{\mathcal{L}}_2$'s, and $\mathrm{Li}^{\mathcal{L}}_{1,1,1}(x,y,z) = \sum \mathrm{Li}^{\mathcal{L}}_3$'s. ✓ (Intro)

Weight 4: After tweaking variables $\overline{\Delta}\,\mathrm{Li}^{\mathcal{L}}_{3,1}(\frac{x}{y}, y) = -\,\mathrm{Li}^{\mathcal{L}}_2(x) \wedge \mathrm{Li}^{\mathcal{L}}_3(y)\,.$
Vanishing of $\overline{\Delta}\big(\sum_i \lambda_i \,\mathrm{Li}^{\mathcal{L}}_{3,1}(\frac{x_i}{y}, y)\big)$ under dilogarithm identities $\sum_i \lambda_i \,\mathrm{Li}_2(x_i) = 0$.

- Symmetries $\mathrm{Li}^{\mathcal{L}}_2(x) + \mathrm{Li}^{\mathcal{L}}_2(1-x) = 0$ and $\mathrm{Li}^{\mathcal{L}}_2(x) + \mathrm{Li}^{\mathcal{L}}_2(x^{-1}) = 0$ ✓ (Intro $+\,\varepsilon$)
- 5-term relation $\sum_{i=0}^4 (-1)^i \,\mathrm{Li}^{\mathcal{L}}_2([z_0, \ldots, \widehat{z_i}, \ldots, z_4]) = 0$, $\quad [z_1, \ldots, z_4] = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$

Theorem (5-term reduction, Gangl 2012; Goncharov-Rudenko, Matveiakin-Rudenko)

$$\sum_{i=0}^{4} (-1)^i \,\mathrm{Li}^{\mathcal{L}}_{3,1}\Big(\frac{\mathrm{cr}(z_0, \ldots, \widehat{z_i}, \ldots, z_4)}{y}, y\Big) = \sum_{i=1}^{122+\varepsilon} \mathrm{Li}^{\mathcal{L}}_4(g_i(z_0, \ldots, z_4, y))\,, \quad g_i \text{ rational}$$

Remark: Crucial ingredient to Goncharov-Rudenko proof of Zagier's Conjecture on $\zeta_F(4)$.

Iterating to understand higher depth

- Computing $\overline{\Delta}$ gets complicated:

$$\overline{\Delta}\,\mathrm{Li}^{\mathcal{L}}_{4,1,1}(x,y,z) = \mathrm{Li}^{\mathcal{L}}_{2}(y)\wedge\mathrm{Li}^{\mathcal{L}}_{3,1}(xy,z) - \mathrm{Li}^{\mathcal{L}}_{3,1}\Big(yz,\tfrac{1}{y}\Big)\wedge\mathrm{Li}^{\mathcal{L}}_{2}(xyz) - \mathrm{Li}^{\mathcal{L}}_{3}(y)\wedge\mathrm{Li}^{\mathcal{L}}_{2,1}(xy,z)$$
$$- \mathrm{Li}^{\mathcal{L}}_{2,1}\Big(yz,\tfrac{1}{y}\Big)\wedge\mathrm{Li}^{\mathcal{L}}_{3}(xyz) + \mathrm{Li}^{\mathcal{L}}_{4}\text{'s}\wedge\text{weight } 2 + \mathrm{Li}^{\mathcal{L}}_{n}\text{'s}\wedge\mathrm{Li}^{\mathcal{L}}_{m}\text{'s}\,.$$

- Use $\overline{\Delta}$ to pick "genuine" depth 2 terms, so iterate[1] $\overline{\Delta}^{[2]} := \overline{\Delta}\circ(\overline{\Delta}\otimes\mathrm{id})$

$$\overline{\Delta}^{[2]}\,\mathrm{Li}^{\mathcal{L}}_{4,1,1}(x,y,z) = \mathrm{Li}^{\mathcal{L}}_{2}(y)\otimes\mathrm{Li}^{\mathcal{L}}_{2}(z)\otimes\mathrm{Li}^{\mathcal{L}}_{2}(xyz) \quad (\text{in } \mathrm{CoLie}_3)\,.$$

Since $\overline{\Delta}^{[2]}\,\mathrm{Li}^{\mathcal{L}}_{a,b}(x,y) = 0$, we deduce $\mathrm{Li}^{\mathcal{L}}_{4,1,1}(x,y,z) \neq \sum$ depth 2.

Question: Is this the only obstruction?

Goncharov Depth Conjecture (simplified version)

A linear combination of MPL's has depth $< k$ if and only if $\overline{\Delta}^{[k-1]}$ vanishes.

[1] needs to be formalised: in a coLie algebra, the coJacobi identity would imply naïve $\Delta^2 = 0$

Expectations and evidence (II)

Observe: If $k > \frac{n}{2}$, then $\overline{\Delta}^{[k-1]}(\mathcal{L}_n) = 0$ (No space for k-many weight 2 pieces.)

Theorem (Rudenko, 2022)

Every weight n multiple polylogarithm can be expressed via depth $\leq \lfloor \frac{n}{2} \rfloor$ functions.

Idea: Explicit formula via "quadrangular" polylogs with depth $\frac{n}{2}$

Write:

$$f(x, y, z) := \mathrm{Li}^{\mathcal{L}}_{4,1,1}((xyz)^{-1}, y, z)$$

$$\overline{\Delta}^{[2]} f(x, y, z) = -\,\mathrm{Li}^{\mathcal{L}}_2(x) \otimes \mathrm{Li}^{\mathcal{L}}_2(y) \otimes \mathrm{Li}^{\mathcal{L}}_2(z) \quad (\text{in } \mathrm{CoLie}_k)$$

Then $f(x, y, z)$ behaves like $\mathrm{Li}^{\mathcal{L}}_2$ in each argument. Expect the following reductions:

- $f(x, y, z) + f(\underbrace{1 - x}_{\text{or } x^{-1}}, y, z) \overset{?}{\equiv} 0 \pmod{\text{dp} \leq 2}$ (Zagier-type)
- $f(\text{dilog 5-term}, y, z) \overset{?}{\equiv} 0 \pmod{\text{dp} \leq 2}$ (Gangl-type)

Weight 6 results

Saw similar expectations hold in weight 4. They are also recently known in weight 6.
Write $f(x,y,z) \coloneqq \mathrm{Li}^{\mathcal{L}}_{4,1,1}((xyz)^{-1}, y, z)$, as before

Theorem (Gangl-type reduction, Matveiakin-Rudenko 2022)

$$f(\text{dilog 5-term}, y, z) \equiv 0 \quad (\text{mod depth} \leq 2 \;\&\; \text{the Zagier reductions})$$

Theorem (Zagier-type reductions, C 2024, arXiv:2405.13853)

$$f(x,y,z) + f(1-x,y,z) \equiv 0 \quad (\text{mod depth} \leq 2)$$
$$f(x,y,z) + f(x^{-1},y,z) \equiv 0 \quad (\text{mod depth} \leq 2)$$

Consequence: Goncharov's Depth Conjecture holds for weight 6 depth 3 $(k = 3)$.

(Zagier-reductions seem more difficult than the Gangl-reduction in general!)

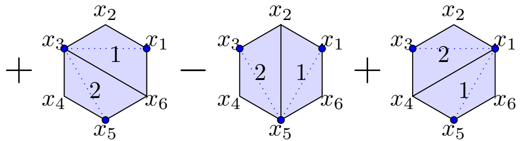
Remark: Similar conjectural expectations in all even weights.

Ingredients for the proof – Quadrangular polylogarithms

Rudenko's quadrangular polylogarithms $\mathrm{QLi}_{n+k}(x_0, \ldots, x_{2n+1})$

- "Nice" MPL functions on $\mathfrak{M}_{0,2n+2}$, expressible via "quadrangulations" of polygons
- Weight-independent depth n expression via cross-ratio $[x_a, x_b, x_c, x_d] = \frac{(x_a - x_b)(x_c - x_d)}{(x_b - x_c)(x_d - x_a)}$

$\mathrm{QLi}_{k+2}(x_1, \ldots, x_6) = \mathrm{Li}^{\mathcal{L}}_{k\,;\,1,1} \Bigg($



$\Bigg)$

$$= \mathrm{Li}^{\mathcal{L}}_{k\,;\,1,1}\left([x_1, x_2, x_3, x_6], [x_3, x_4, x_5, x_6]\right) - \mathrm{Li}^{\mathcal{L}}_{k\,;\,1,1}\left([x_1, x_2, x_5, x_6], [x_3, x_4, x_5, x_2]\right)$$
$$+ \mathrm{Li}^{\mathcal{L}}_{k\,;\,1,1}\left([x_1, x_4, x_5, x_6], [x_1, x_2, x_3, x_4]\right)$$

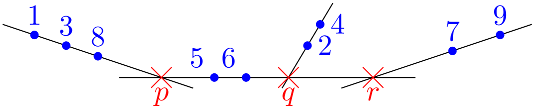
General functional equation

$$\mathbf{Q}_{2k}: \qquad \sum_{i=1}^{2k+3} (-1)^i \, \mathrm{QLi}_{2k}(x_1, \ldots, \widehat{x_i}, \ldots, x_{2k+3}) \equiv 0 \quad (\text{mod depth} < k)$$

Ingredients (II) – Stable genus 0 curves

Part of boundary strata of Deligne-Mumford compactification $\overline{\mathfrak{M}}_{0,n}$

- Components are isomorphic to $\mathbb{P}^1$
- Only singular points are simple double points
- Number of marked & singular points per components ≥ 3



Gadget to encode limits of projective configurations

Idea: The points x_1, x_3, x_8 all go to p

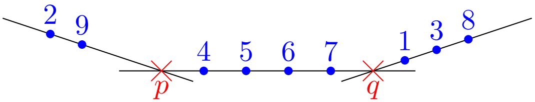
But: There is a always projective transformation moving (x_1, x_3, x_8, p) to $(\infty, 0, 1, z)$

So: Points x_1, x_3, x_8 split off as a separate $\mathbb{P}^1$

Cross-ratios well-defined: $[x_1,x_3,x_8,x_i] = [x_1,x_3,x_8,p]$ and $[x_1,x_3,x_2,x_4] = [x_1,x_3,p,p] = 0$

Ingredients (III) – Build to symmetries step-by-step

Specialise $\mathbf{Q}_6$ to



and use $\lim_{x_i \to 0 \text{ or } \infty} \mathrm{Li}^{\mathcal{L}}_{k_1,\ldots,k_d}(x_1, \ldots, x_d) = 0 \pmod{\text{dp } < d}$

Write $f(x,y,z) \coloneqq \mathrm{Li}^{\mathcal{L}}_{4,1,1}((xyz)^{-1}, y, z)$, as before

Lemma (Full Symmetry 1)

$$f(A,B,C) \equiv -f\Big(1-A\,,\,\frac{B}{B-1}\,,\,1-C\Big) \pmod{\text{depth } 2}$$

Remark: First needed to show $f(1,x,y) \equiv 0 \pmod{\text{depth } 2}$, already non-trivial(!)

- Find a second *different* symmetry, and a four-term relation
- Play the symmetries against each other
- Extract (eventually. . .) the Zagier-type reduction

Summary

- Depth reductions of MPL's
  - Appearances in Number Theory, Hyperbolic Geometry, High-Energy Physics
- Goncharov's Depth Conjecture
  - Cobracket on (motivic) MPL's
  - Truncation and iteration of the cobracket
  - Cobracket conjecturally detects depth
- Zagier-, & Gangl-type reductions in weight 4, and weight 6
- Ideas from the proof
  - Quadrangular polylogarithm functional equations
  - Degenerations to stable genus 0 curves
  - Discover and exploit many symmetries/short identities