Multiple Zeta Values

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1 Motivation

1.1 Riemann Zeta Function

Rather than just throw the definition of an MZV up, I though I'd try to show how one could somewhat naturally arrive at the concept of a MZV. Recall first the Riemann zeta function:

$$\zeta(s) \coloneqq \sum_{n=1}^{\infty} \frac{1}{n^s}$$

This sum converges for any $s \in \mathbb{C}$, with $\Re(s) > 1$. And it can be analytically continued to a meromorphic function on all of $\mathbb{C}$, with a single simple pole at $s = 1$ – all the good stuff. From this point of view it's a very important function for analytic number theory. But we are going to look at another aspect: it's values when s is a positive integer, so we sum over reciprocals of squares, cubes, etc., have interested many mathematicians. These are the values we want to focus on.

Euler managed to evaluate $\zeta(2) = \pi^2/6$, and generalise his method to evaluate $\zeta(4) = \pi^4/90, \zeta(6) = \pi^6/945$ and all other $\zeta(\text{even})$. Generally:

$$\zeta(2k) = \frac{(-1)^{k+1} B_{2k} (2\pi)^{2k}}{2(2k)!}$$

where B_m are the Bernoulli numbers, defined as the coefficient of t^m in the Taylor series

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}$$

A question which immediately springs to mind might be, what is $\zeta(3)$? Or $\zeta(5)$? Nobody knows. We can say surprisingly little about the odd ζ's. We know (just) that $\zeta(3)$ is irrational, and that at least one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is also irrational. We also know that infinitely many (but maybe not all) of the odd ζ's are irrational. They are very mysterious.

1.2 Product of Riemann Zeta Values

It's not a unnatural thing to consider multiplying two Riemann Zeta Values, and to wonder what we get. So let's do this. If we try to find the product:

$$\zeta(a)\zeta(b)$$

by multiplying out the sums we get

$$\begin{aligned}\zeta(a)\zeta(b) &= \sum_{n=1}^{\infty}\frac{1}{n^a}\cdot\sum_{m=1}^{\infty}\frac{1}{m^b}\\ &= \sum_{n=1,m=1}^{\infty}\frac{1}{n^a m^b}\end{aligned}$$

This sum is over all lattice points with positive coordinates. We can break this up into a sum over the lower triangle, the diagonal and the upper triangle, to get:

$$\sum_{n,m=1}^{\infty}\frac{1}{n^a m^b} = (\sum_{0<n<m}^{\infty} + \sum_{n=m} + \sum_{n>m>0})\frac{1}{n^a m^b}$$

We can recognize one of the terms here as $\zeta(a+b)$. Let's call the other two terms $\zeta(a,b)$ and $\zeta(b,a)$.

2 Definitions

2.1 Multiple Zeta Values

This, at least, makes the full definition of a MZV seem somewhat less unnatural. The *multiple zeta value* $\zeta(a_1, a_2, \dots, a_k)$, with a_i positive integers, is defined by

$$\zeta(a_1, a_2, \dots, a_k) := \sum_{0<n_1<n_2<\cdots<n_k} \frac{1}{n_1^{a_1} n_2^{a_2} \cdots n_k^{a_k}}$$

For this sum to converge, we need $a_k > 1$.

The sum $\sum_{i=1}^{k} a_i$ of the a_i is called *weight* of the multiple zeta value. This is a commonly used notion, and is makes an appearance in various conjetures and open questions surrounding MZVs.

2.2 Integral Representation

Playing around with iterated integrals, one can write a mutiple zeta value $\zeta(a_1, \dots, a_k)$ as a Chen iterated integral

$$\begin{aligned}\zeta(a_1, \dots, a_k) &= \int_{[0,1]} \frac{\mathrm{d}x}{x-1}\left(\frac{\mathrm{d}x}{x}\right)^{a_1-1} \cdots \frac{\mathrm{d}x}{x-1}\left(\frac{\mathrm{d}x}{x}\right)^{a_k-1}\\ &=: I(0; 10^{a_1-1} \cdots 10^{a_k-1}; 1)\end{aligned}$$

The proof of this is just a case of expanding out the iterated integral as a geometric series, and integrating term by term. Doing this should give the sum defining this MZV.

A compact notation for this iterated integral is given in the next line. The 0; and ;1 denote the end points of the integral. The digits in the middle are read off from the differential forms appearing:

$$a \leftrightarrow \frac{\mathrm{d}x}{x-a}$$

An the powers just mean that digit repeated so many times.

This gives a correspondence between MZVs and a certain subset of binary words (those starting in 1 and ending in 0), (if we ignore the end points of integration.) For the most part, don't worry about the exact details, with this notation the properties of iterated integrals come through as simple combinatorial manipulation of binary words. This way of encoding MZVs had some very nice side effects, it allows statements of theorems to be given very elegantly.

3 Algebraic Properties

3.1 Relations

I now want to convince you that multiple zeta values satisfy a huge number of relations. To start with here are some examples:

$$\begin{aligned} \zeta(1,2) &= \zeta(3) \\ \zeta(\{1,3\}^n) &= \zeta(\{2\}^{2n}) = \frac{\pi^{2n}}{(2n+1)!} \\ 28\zeta(3,9) + 150\zeta(5,7) + 168\zeta(7,5) &= \frac{5197}{691}\zeta(12) \end{aligned}$$

The last relation is an exception relation between double zeta values, and comes from a connection to modular forms. Such relations for weight $2k$ exist when there is a non-trivial cusp form of weight $2k$ on $\Gamma_0(N)$? Or so?

One of the important questions concerning MZVs has to do with studying and understanding all the relations they satisfy. A few sharp-eyed people may have spotted that in all these relations, the weight of both sides is the same. Does this always happen? Well actually nobody knows. We conjecture that space of MZVs is weight graded (break up into distinct pieces for each weight), so there are no relations between MZVs of different weights, but this is still only a conjecture.

3.2 Duality

The duality between MZVs was first noticed numerically before the integral representation was known. It seemed that certain pairs of MZVs were equal, for mysterious reasons. Even writing out the statement of duality was quite tricky then, and a proof impossible. Once the integral representation was discovered, and the notation for MZVs as binary words, the statement becomes very elegant and the proof immediate.

If we substitute $1-x$ for x in the integral representation, and work thing through, we find a different iterated integral which equals our MZV.

$$\begin{aligned} &\zeta(a_1, \ldots, a_k) \\ &= I(0; 10^{a_1-1} \cdots 10^{a_k-1}; 1) \\ &= I(0; 1^{a_k-1}0 \cdots 1^{a_1-1}0; 1) \end{aligned}$$

This iterated integral also corresponds to a MZV, so we get an equality between two different MZV.

On binary words the effect is to reverse and interchange $0 \leftrightarrow 1$. So for example, the proof that $\zeta(1,2) = \zeta(3)$ is completely trivial now.

3.3 Shuffle Product

One property of Chen iterated integrals is how they multiply. The product of two integrals can be written as a sum of other iterated integrals. This is done by splitting up the product of the integration simplicies into new simplices and recognizing the integrals arising other iterated integrals.

This leads to the shuffle product admissible words and on MZVs. Given words, their the shuffle product is defined recursively by:

- For any word w, $1 \sqcup\!\sqcup\, w = w \sqcup\!\sqcup\, 1 = w$, where 1 is the empty word.
- For any words w_1, w_2, and symbols $a, b \in \{x, y\}$:

  $$aw_1 \sqcup\!\sqcup\, bw_2 = a(w_1 \sqcup\!\sqcup\, bw_2) + b(aw_1 \sqcup\!\sqcup\, w_2)$$

  The idea here is to riffle shuffle the letters of the two words. I.e. to look at all permutations of all the letters, but making sure the letters from each word keep the original order.

3.4 Stuffle Product

We've already seen how the product of two RZVs can be written as a combination of multiple zeta values by multiplying the defining sums. More generally this procedure works. If we multiply two MZVs by multiplying their series representations we get the stuffle product of the MZVs. This is defined by:

- For any word w, $1 * w = w * 1 = w$,
- For any word w, and any integer $n \geq 1$:

  $$x^n * w = w * x^n = wx^n$$

- For any words w_1, w_2, and integers $p, q \geq 0$:

  $$yx^p w_1 * yx^q w_2 = yx^p(w_1 * yx^q w_2) + yx^q(yx^p w_1 * w_2) + yx^{p+q+1}(w_1 * w_2)$$

  This has a better interpretation on the arguments of the MZVs, than on the binary words. Here we shuffle the arguments of the two MZVs (this is from the first two terms above), but we may also *stuff* (hence the name) two arguments into the same slot (this is the third term above).

3.5 Double Shuffle

Since MZVs are just real numbers, it doesn't matter how we multiply them, stuffle and shuffle must give the same thing. So the different is zero. Comparing them like this we get *linear* relations between MZVs.

We've found the product $\zeta(2)\zeta(2)$ in two different ways above, so when we compare them they must be equal. The shuffle product must equal the stuffle product. With the expressions we found above, we deduce that $4\zeta(1, 3) = \zeta(4)$.

By itself, comparing shuffle and stuffle doesn't generate all relations between MZVs. If we formally allow $\zeta(1)$, divergent, to appear in the product it happens that all divergent terms cancel out when comparing stuffle and shuffle. Moreover they cancel out in a way which gives correct results. This gives the regularised double shuffle relations, and conjecturally these do generate all relations between MZVs.

4 Motivic MZVs

4.1 Motivic MZVs

Recently Francis Brown, building on results of Alexander Goncharov has found a purely algebraic lifting/analogue of the multiple zeta values. By getting rid of the analytic aspect - the defining of MZVs in terms of infinite sums - many of the mysterious aspects of MZVs are eliminated. Also have motivic iterated integrals.

The exact definition of these objects is quite involved, and not something I can explain in this talk other than to say it involves some very scary sounding words: Tanakian category, mixed Tate motives, group scheme. Don't worry about it, like with the integral representation stuff in number of applications the relevant properties show through as simple combinatorial manipulations.

By construction these objects lie in a weight graded vector space - a vector space along with the a decompmosition into the direct sum of subspace $Z_1, \ldots, Z_n$. So automatically we the weight grading of relations. For example since $\zeta^{\mathfrak{m}}(3)$ and $\zeta^{\mathfrak{m}}(5)$ lie in different subspaces Z_3 and Z_5 respectively, they cannot be linearly dependent.

In fact, they form more than just a weight graded vector space. They have much more algebraic structure. They form what is called a Hopf algebra there is something called a coproduct $\Delta\colon \mathcal{Z} \to \mathcal{Z} \otimes \mathcal{Z}$. It's always a good thing when we find that an object has more structure, it makes it more rigid, and so easier to study and discover results about.

We can use these motivic MZVs to study real MZVs. There is a so-called period map, which takes this algebraic object $\zeta^{\mathfrak{m}}(a_1, \ldots, a_k)$ and returns the real number $\zeta(a_1, \ldots, a_k)$. This is a map with very good structure - it's a ring homomorphism between the motivic and classical MZVs. So the sum or product of two motivic MZVs maps to the sum or product of their classical counterparts. Overall this means that relations between motivic MZVs will descend to classical MZVs. So if we can us the extra structure of motivic MZVs to discover some relation or fact, this will automatically hold for the real MZVs as well.

4.2 Transcendental Galois Theory

I want to give a more concrete idea of how this fairly abstract motivic MZV stuff can be used to answer some questions about real MZVs. The result is a kind of analogue of galois theory for transcendental numbers, for MZVs. Maybe recall that in Galois theory the behaviour of numbers under symmetries of a field extension forces various properties on the number. A number fixed under all Galois automorphisms must lie in the base field. For example, a complex number invariant under $i \mapsto -i$ (complex conjugation) must be real.

The analogue here comes in the form of a family of operators defined on the motivic MZVs. They are related to the coproduct Δ, they are an 'infinitesimal' version of this coproduct. Write the motivic MZV as a motivic iterated integral using the integral representation from before. Then we define D_k on this motivic iterated integral by looking at all subwords of length $k+2$, i.e. k letters inside plus the start and end letter, of the word defining the integral. We take this as one new iterated integral. We then chop out the interior (inside / without boundary) from the original word, we look at what remains without this subword, and take that as the word of another interated integral.

The mnemonic picture below gives a good way to remember/interpret this formula. We arrange the letters of the word w around the edge of a semicircle. We then look at all segments of the semicircle of length $k+2$, take this segment as a new semicircle/iterated integral, and then throw this away and look at what remains for the other integral.

The pay off of this comes with the following theorem. If we take some combination of motivic multiple zeta values of weight N, and compute $D_3, D_5, \ldots$, and all the other odd D_k upto and including N. If these all vanish identically then the combination we started with is a rational multiple of the Riemann zeta value $\zeta^{\mathfrak{m}}(N)$.

Once we are happy to work with motivic iterated integral, and know their various properties we have some very combinatorial tools at our disposal to study and deduce things about multiple zeta values.

Consider the example of $\zeta^{\mathfrak{m}}(2, \ldots, 2)$. It is known explicitly that this is equal to $\pi^{2n}/(2n+1)!$, but this is really not obvious. One proof involves comparing coefficients of power series to extract this result. Using motivic MZVs we can see part of the result very straightforwardly. If we try to compute D_{odd}, we start by cutting out a subword of length 5 to compute D_3. Look at the word in the iterated integral, it's just an alternating sequence of 0's and 1's. So no matter where to start, the first and last letters of the subword will be equal. An integral where the start and end points are the same is just 0, and this holds on the motivic level too, so all the terms in D_k are identically 0. Hence the above theorem tells us that $\zeta^{(\mathfrak{m})}(2, \ldots, 2) \in \zeta^{(\mathfrak{m})}(2n)\mathbb{Q}$. And remember back to Euler's evaluation of the Riemann zeta function at even integers. It's a rational multiple of $\pi^2 n$, so we get that $\zeta(2, \ldots, 2) \in \pi^{2n}\mathbb{Q}$.

5 Cyclic Insertion

Lastly I'd like to give an example of what I've done with the motivic MZV stuff. A lot of numerical evidence has lead to a conjecture by Borwein, Bradley, Broadhurst and Lisonek which states that cyclically inserting blocks of 2 into the multiple zeta value $\zeta(1, 3)$ should give some explicit rational multiple of a power of π.

Full the conjecture states that given n and given $2n + 1$ non-negative integers $a_0, a_1, \ldots, a_{2n}$ with sum n. The following sum, taken over all cyclic shifts of the a_i should equal this rational multiple of π^*. In this sum I write 2^b to mean the argument 2 is repeated b times. For example with $a_0 = 1$ and $a_1 = 0$ and $a_2 = 0$, we expect sum $\pi^6/7!$, and numerically this does work. With $a_0 = 2$, $a_1 = 0$, $a_2 = 1$, $a_3 = a_4 = 0$ cyclically inserting these the sum should be $\pi^1 4/15!$, again this works numerically.

5.1 Symmetric Insertion

My result is that if we sum up over not just the cyclic shifts, but over all possible permutations of these a_i, then the sum is indeed a rational multiple of π^{4n+2m}. So if we add in the other 3 terms for the first example above, or the rest of 120 terms in the second. Those definitely are rational multiples of π^{4n+2m}.

Sketch of proof: Firstly we want to lift the result to the motivic MZV level. Then the key idea is to write the associated iterated integral in a particularly nice way. With a bit of staring at it, it ends up being alternating blocks of 01s or 10s. Start of with the lower bound of integration 0, then we repeat 10 some number of times and then hit the 1. Then we follow this by 10 and then 100. This 100 is 10 followed by 0 so ends this block and starts the next.

Once we've done this, compute D_{2k+1}. Since the word in the iterated integral is very symmetrical and all possible permutations of these a_i appear in the rest of the sum, there is enough symmetry around to set up a nice pairwise cancellation. The idea is so take a subword and reflect the blocks is spans to get another term in the result. This flipping introduces a minus sign, so they cancel.

Once the dust clears, all terms cancel and so D_{2k+1} is 0. So the result follows.

6 Summary

In this talk I've introduced you to multiple zeta values and their algebraic structure, and some of the open questions surrounding them.

We've seen the definition of a multiple zeta value, as something of a generalisation of the Riemann zeta function. We've mentioned that little is known about the irrationality of $\zeta(odd)$, other than for $\zeta(3)$, and still have no idea what $\zeta(5)$ is like. We've been the shuffle and stuffle products as a way of expressing the product of two MZVs as a linear combination of other MZVs, and how comparing these to products give a source of relations between MZVs. Whether this double shuffle comparison generates all relations is still unknown.

I've then given a brief run through of motivic MZVs, particularly the combinatorial tools on offer, these operators D_k, which allow a kind of transcendental Galois theory to be developed. The vanishing of a motivic MZV under all D_{odd} tell us that is is a rational multiple of a motivic Riemann zeta value.

Lastly I've shown you how I used these tools to prove a symmetric insertion result about MZVs, inserting all permutations of blocks of 2's into $\zeta(1, 3, 1, 3)$ gives a rational multiple of a power of pi. This result is maybe a small step towards the cyclic insertion conjecture.