Multiple Zeta Values

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Outline

Riemann Zeta Function

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$$\zeta(2k) = \frac{(-1)^{k+1} B_{2k} (2\pi)^{2k}}{2(2k)!}$$

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$$\zeta(a)\zeta(b)$$

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$$= \sum_{n,m=1}^{\infty} \frac{1}{n^a m^b}$$

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$$\begin{aligned}\zeta(a)\zeta(b) &= \sum_{n=1}^{\infty}\frac{1}{n^a}\cdot\sum_{m=1}^{\infty}\frac{1}{m^b}\\ &= \sum_{n,m=1}^{\infty}\frac{1}{n^a m^b}\\ &= \sum_{0<n<m}\frac{1}{n^a m^b}+\sum_{0<n=m}\frac{1}{n^a m^b}+\sum_{n>m>0}\frac{1}{n^a m^b}\end{aligned}$$

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Multiple Zeta Values

Definition

The multiple zeta value (MZV) $\zeta(a_1, a_2, \ldots, a_k)$ is defined by

$$\zeta(a_1, a_2, \ldots, a_k) \coloneqq \sum_{0<n_1<n_2<\cdots<n_k} \frac{1}{n_1^{a_1} n_2^{a_2} \cdots n_k^{a_k}} .$$

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- For convergence $a_k > 1$
- Sum $a_1 + \cdots + a_k$ is the weight

Integral Representation

Can write MZV as a Chen iterated integral

$$\zeta(a_1, \ldots, a_k) = \int_{[0,1]} \frac{\mathrm{d}x}{1-x} \left(\frac{\mathrm{d}x}{x}\right)^{a_1 - 1} \cdots \frac{\mathrm{d}x}{1-x} \left(\frac{\mathrm{d}x}{x}\right)^{a_k - 1}$$
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- Encode $\zeta(a_1, \ldots, a_k)$ as binary word $10^{a_1-1} \cdots 10^{a_k-1}$
- Also write $\zeta(yx^{a_1-1} \cdots yx^{a_k-1})$

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- Can we describe all relations?
- Are they weight graded?

Duality

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Substitute $x \mapsto 1 - x$ in integral representation

$$\begin{aligned} &\zeta(a_1, \ldots, a_k) \\ &= I(0; 10^{a_1-1} \cdots 10^{a_k-1}; 1) \\ &= I(0; 1^{a_k-1}0 \cdots 1^{a_1-1}0; 1) \end{aligned}$$

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- Effect – reverse word and interchange $0 \leftrightarrow 1$

$$\zeta(yx^{a_1-1} \cdots yx^{a_k-1}) = \zeta(y^{a_k-1}x \cdots y^{a_1-1}x)$$

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Example

$$\zeta(1, 2) = I(0; 110; 1) = I(0; 100; 1) = \zeta(3)$$

Shuffle Product

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Definition

The shuffle product $ш$ is defined by recursively by:

- For any words w_1, w_2, and letters $a, b \in \{x, y\}$

$$aw_1 ш bw_2 = a(w_1 ш bw_2) + b(aw_1 ш w_2)$$

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Example

$$\begin{aligned}\zeta(2)\zeta(2) &= \zeta(yx)\zeta(yx) = \zeta(yx \sqcup\!\sqcup\, yx) \\ &= \zeta(2 \cdot yxyx + 4 \cdot yyxx) = 2\zeta(2,2) + 4\zeta(1,3)\end{aligned}$$

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The stuffle product $*$ is defined by recursively by:

- For any words w_1, w_2, and integers $p, q \geq 0$:

$$yx^p w_1 * yx^q w_2 = yx^p(w_1 * yx^q w_2) + yx^q(yx^p w_1 * w_2) + yx^{p+q+1}(w_1 * w_2)$$

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$$\begin{aligned}\zeta(a)\zeta(b) &= \zeta(yx^{a-1})\zeta(yx^{b-1}) = \zeta(yx^{a-1} * yx^{b-1}) \\ &= \zeta(yx^{a-1}yx^{b-1} + yx^{b-1}yx^{a-1} + yx^{a+b-1}) \\ &= \zeta(a,b) + \zeta(b,a) + \zeta(a+b)\end{aligned}$$

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$$\zeta(2)\zeta(2) = \zeta(yx)\zeta(yx) = \zeta(yx * yx) = \zeta(2 \cdot yxyx + yxxx) = 2\zeta(2,2) + \zeta(2+2)$$

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Compare these two products

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$$\zeta(w_1 \sqcup\!\sqcup w_2) - \zeta(w_1 * w_2) = 0$$

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- Does this give all relations?

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- Map back to MZVs $\zeta^{\mathfrak{m}}(a_1, \ldots, a_k) \mapsto \zeta(a_1, \ldots, a_k)$

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- Motivic MZV relations imply classical MZV relations

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$$D_k I^{\mathfrak{m}}(w) := \sum_{\substack{S \text{ subword} \\ \text{of length } k+2}} I^{\mathfrak{L}}(S) \otimes I^{\mathfrak{m}}(w/\operatorname{int} S)$$

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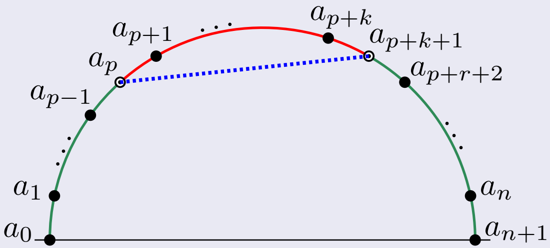
Mnemonic



- $w = a_0 a_1 \ldots a_n a_{n+1}$
- $S = a_p \cdots a_{p+k+1}$, $\quad w/\operatorname{int} S = a_0 \cdots a_p \, a_{p+k+1} \cdots a_{n+1}$

‘Transcendental’ Galois Theory

Theorem (Brown, 2011)

If $Z^{\mathfrak{m}}$ *has weight* N *and* $D_k Z^{\mathfrak{m}} = 0$ *for* $3 \leq k \leq N$ *odd, then*

$$Z^{\mathfrak{m}} \in \zeta^{\mathfrak{m}}(N)\mathbb{Q}$$

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Example

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- Can show all $D_k \zeta^{\mathfrak{m}}(2, \ldots, 2) = 0$
- So $\zeta(2, \ldots, 2) \in \zeta(2n)\mathbb{Q} = \pi^{2n}\mathbb{Q}$

Cyclic Insertion

Conjecture (BBBL)

Given $a_0, a_1, \ldots, a_{2n}$ *with sum* m *we get*

$$\sum_{\substack{\text{cyclic shifts} \\ \text{of } a_i}} \zeta(2^{a_0}, 1, 2^{a_1}, 3, \ldots, 1, 2^{a_{2n-1}}, 3, 2^{a_{2n}}) \stackrel{?}{=} \frac{\pi^{4n+2m}}{(4m+2n+1)!}$$

Examples

$$\zeta(2,1,3) + \zeta(1,2,3) + \zeta(1,3,2) \stackrel{?}{=} \frac{\pi^6}{7!}$$

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$$\zeta(2,2,1,3,2,1,3) + \zeta(1,2,2,3,1,2,3) + \zeta(1,3,2,2,1,3,2) + \zeta(2,1,3,1,2,2,3) + \zeta(2,1,3,1,3,2,2) \stackrel{?}{=} \frac{\pi^{14}}{15!}$$

Symmetric Insertion

Theorem

Given $a_0, a_1, \ldots, a_{2n}$ *with sum* m *we get*

$$\sum_{\substack{\text{all permutations} \\ \text{of } a_i}} \zeta(2^{a_0}, 1, 2^{a_1}, 3, \ldots, 1, 2^{a_{2n-1}}, 3, 2^{a_{2n}}) \in \pi^{4n+2m}\mathbb{Q}$$

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Sketch of Proof

- Lift to motivic MZVs
- Write the iterated integral nicely as

$$I^{\mathfrak{m}}((01)^{a_0+1}\,(10)^{a_1+1}\,\cdots\,(01)^{a_{2n+1}+1})$$

- Compute D_{2k+1}
- Pairwise cancel using symmetry □

Summary

- Definition of multiple zeta values
- Irrationality and transcendence?
- Relations between MZVs
  - Shuffle and stuffle product
  - Double shuffle generates all?
- Motivic MZVs
  - Combinatorial tools
  - Transcendental Galois theory
- Symmetric insertion