An very brief overview of Surreal Numbers
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1 History and Introduction

Surreal numbers were created by John Horton Conway (of Game of Life fame), as a greatly simplified construction of an earlier object (Alling's ordered field associated to the class of all ordinals, as constructed via modified Hahn series). The name surreal numbers was coined by Donald Knuth (of TEX and the Art of Computer Programming fame) in his novel '*Surreal Numbers*' [2], where the idea was first presented.

Surreal numbers form an ordered Field (Field with a capital F since surreal numbers aren't a set but a class), and are in some sense the largest possible ordered Field. All other ordered fields, rationals, reals, rational functions, Levi-Civita field, Laurent series, superreals, hyperreals, ..., can be found as subfields of the surreals. The definition/construction of surreal numbers leads to a system where we can talk about and deal with infinite and infinitesimal numbers as naturally and consistently as any 'ordinary' number. In fact it let's can deal with even more 'wonderful' expressions

$$\infty - 1, \frac{1}{2}\infty, \sqrt{\infty}, \frac{1}{\infty}, \ldots$$

in exactly the same way[1].

One large area where surreal numbers (or a slight generalisation of them) finds application is in the study and analysis of combinatorial *games*, and game theory. Conway discusses this in detail in his book '*On Numbers and Games*' [1].

2 Basic Definitions

All surreal numbers are constructed iteratively out of two basic definitions. This is an wonderful illustration on how a huge amount of structure can arise from very simple origins. To start with you should forget everything you know about numbers, inequalities, addition, et cetera, and make the following definition.

Definition 2.1. A *surreal number* x is a pair of sets, the left set X_L and the right set X_R, of previously created surreal numbers, such that no element $r \in X_R$ of the right set is to be $\leq$ any element $\ell \in X_L$ of the left set, i.e. $\neg\, \exists \ell \in X_L\, \exists r \in X_R\, (r \leq \ell)$. (This requirement means that x is well-formed.)

Write $x \equiv \{\, X_L \mid X_R \,\}$ to mean x is given by this particular presentation, for many different presentations can lead to the same surreal number value.

And to make sense of this we need the definition of $\leq$, which is as follows

Definition 2.2. For two surreal numbers $x \equiv \{\, X_L \mid X_R \,\}$ and $y \equiv \{\, Y_L \mid Y_R \,\}$, we say $x \leq y$ iff

[1] You should write ω instead of ∞, but that spoils the fun.

- y is not $\leq$ any element of X_L, i.e. $\neg\,\exists x_\ell \in X_L\,(y \leq x_i)$, and
- no element of Y_R is $\leq x$, i.e. $\neg\,\exists y_r \in Y_R\,(y_r \leq x)$.

Example 2.3. For for example, forgetting that you've forgotten everything, and pretending that some of the ordinary numbers we know are surreal numbers, then following is another surreal number

$$\{\,1,2,7 \mid 9,10,12\,\}\ ,$$

since nothing in X_R is $\leq$ anything in X_L.

But this is not a surreal number

$$\{\,1,2,7 \mid 6,8,10\,\}\ ,$$

since $x_r = 6 \in X_R$, $x_\ell = 7 \in X_L$ and $x_r = 6 \leq x_\ell = 7$; nor is

$$\{\,1,2,7 \mid 7,8,10\,\}\ ,$$

since $x_r = 7 \in X_R$, $x_\ell = 7 \in X_L$ and $x_r = 7 \leq x_\ell = 7$.

And for $\leq$:

$$3 = \{\,1,2 \mid 5,6\,\} \leq 7 = \{\,4,5,6 \mid 8\,\}$$

since:

- 7 is not $\leq$ than any element of $\{\,1,2\,\}$ and
- no element of $\{\,8\,\}$ is ≤ 3.

These definitions are very circular, but each time we apply them, we always move to a 'simpler' condition to check as we move to previously created surreal numbers. This means eventually we get to statements we can check.

3 The first surreal number

So how can we start making surreal numbers? Both of these definitions are very circular, so where do we start?

The first definition says that a new surreal number must be a pair of sets of previously created surreal numbers. At this point we don't know any surreal numbers, but the empty set is a perfectly good set, so the first surreal number we can possibly create is

$$\{\ \mid\ \}\ .$$

But before we claim victory at creating a surreal number, we need to check that this is well formed.

Proposition 3.1. *The surreal number $X \equiv \{\ \mid\ \}$ is well formed.*

Proof. To show X is well formed, we need to show that no element of X_R is $\leq$ any element of X_L. This only fails if there is some element of X_R which is $\leq$ some element of X_L. But as $X_R = \emptyset$ contains no elements, this doesn't fail. Hence $\{\ \mid\ \}$ is well formed. $\square$

For the moment we'll write $0 \equiv \{\ \mid\ \}$ for this. This name can be justified later. To keep track of the order in such we have created the various surreal numbers, we say this surreal number is created on day *zero*.

We can prove the following fact about 0

Theorem 3.2. $\underbrace{0}_{x} \le \underbrace{0}_{y}$

Proof. We need to show that $y = 0$ is not $\le$ any element of $X_L = \emptyset$, and that no element of Y_R is $\le x = 0$.

The first fails only if $y = 0$ is $\le$ some $x_\ell \in X_L$, but X_L is empty, so there is no such x_ℓ and it doesn't fail.

Similarly, the second only fails some $y_r \in Y_R$ is $\le x = 0$, but Y_R is empty, so there is no such y_r and it doesn't fail. □

4 More surreal numbers

Now we have the following new pairs of surreal numbers

$$\{\, 0 \mid \,\}\,, \{\, \mid 0 \,\}\,, \{\, 0 \mid 0 \,\}\ .$$

Since $0 \le 0$, the last one isn't well formed, so isn't a surreal number. But as before, the others are surreal numbers.

So on day *one* we have created

$$\begin{aligned} 1 &\equiv \{\, 0 \mid \,\} \\ -1 &\equiv \{\, \mid 0 \,\} \end{aligned}$$

Again these names can be justified later. We can prove

Theorem 4.1. $0 \equiv \{\, \mid \,\} \le 1 \equiv \{\, 0 \mid \,\}$.

Proof. We need to prove that 1 is not $\le$ any element of $X_L = \emptyset$, and no element of $Y_R = \emptyset$ is $\le$ 0. Both relevant sets are empty, so this is true. □

And similarly $-1 \le 1$ and $-1 \le 0$. We can also prove

Theorem 4.2. $\neg\, 1 \le 0$

Proof. $1 \le 0$ would mean 0 is not $\le$ any element of $X_L = 0$, and no element of $Y_R = \emptyset$ is ≤ 1. But take $x_\ell = 0 \in X_L$, then $0 \le x_\ell$, so the first fails, giving $1 \not\le 0$. □

We can introduce some shorthand notation for convenience. Say $x = y$ iff $x \le y$ and $y \le x$. Say $x \ne y$ iff $\neg\, x = y$. Say $x < y$ iff $x \le y$ and $x \ne y$. Then from the above, we've got

$$0 = 0,\ 0 < 1,\ 0 \ne 1,\ \text{and so on.}$$

Exactly what you would expect from $-1, 0, 1$.

On the second day we get the a lot more numbers, the new ones are

$$\begin{aligned} 2 &\equiv \{\, 1 \mid \,\} \\ \tfrac{1}{2} &\equiv \{\, 0 \mid 1 \,\} \\ \tfrac{1}{2} &\equiv \{\, -1 \mid 1 \,\} \\ -2 &\equiv \{\, \mid -1 \,\}\ . \end{aligned}$$

(Still names which need justifying!)

We get a couple of different representations for numbers we already know, such as

$$0 = \{ -1 \mid 1 \} = \{ -1 \mid \ \} = \{ \ \mid 1 \} ,$$

these equalities follow from the definitions, but finally require a non-vacuous statement to be proven.

Proving $0 \equiv \{ \ \mid \ \} \leq \{ -1 \mid 1 \}$ finally involves proving a non-vacuous statement. We need to prove that $\{ -1 \mid 1 \}$ is not $\leq$ any member of $X_L = \emptyset$, and that no element of $Y_R = \{ 1 \}$ is ≤ 0. The first is vacuous, but the second is true since $\neg\, 1 \leq 0$ was shown above.

5 Some general theorems about surreal numbers

We can see the beginnings of a pattern in which new numbers are created on each day. The pattern would become more obvious if we looked at day 3, but for time I'll just tell you:

Theorem 5.1. *On day n, we create $n = \{ n-1 \mid \ \}$, and $-n = \{ \ \mid -(n-1) \}$, and all midpoints between previously existing surreal numbers.*

This means on finite days we create only integers, and dyadic fractions.

We can also see some structure on the values a surreal number represent. The number $x \equiv \{ X_L \mid X_R \}$ is always between all values in X_L and X_R. More precisely

Theorem 5.2. *The value of $x \equiv \{ X_L \mid X_R \}$ is the earliest-created surreal y such that $y <$ every element of X_R, and every element of X_L is $< y$.*

For example

$$\{ -2 \mid 1 \} = 0, \{ -2, -1, 0 \mid 2 \} = 1 ,$$

Once we move to allowing infinite sets, on day ω, we can write down

$$\frac{1}{3} = \left\{ 0, \frac{1}{4}, \frac{5}{16}, \frac{21}{64} \dots \,\middle|\, 1, \frac{1}{2}, \frac{3}{8}, \frac{11}{32}, \dots \right\}$$

The left hand set is all dyadic fractions $< \frac{1}{3}$, and the right hand set is all dyadic fractions $> \frac{1}{3}$. (This is very similar to the Dedekind cut definition of the real numbers, especially if we look for the surreal number with value π.)

I'm not going to prove these theorems, but I'll give an indication of how things can be proven for all surreal numbers. The idea is a version of induction. Prove the result for $\{ \ \mid \ \}$. Then assuming the result is true for the parents of $x = \{ X_L \mid X_R \}$ (that is the elements of X_L and X_R), show that the result is true for x.

6 Addition, and justifying the names

Finally, once we introduce the notion of addition of surreal numbers, we can justify the names we have given to these surreal numbers.

Definition 6.1. For surreal numbers $x \equiv \{ X_L \mid X_R \}$ and $y \equiv \{ Y_L \mid Y_R \}$, we define

$$x + y \equiv \{ X_L + y, x + Y_L \mid X_R + y, x + Y_R \} ,$$

where number + set means add number to each element of set.

We should check that this gives a well-formed surreal number. It does. Moreover, does it make sense as addition? Certainly we can justify it somewhat by noting that since x lies between X_L, X_R, and y between Y_L, Y_R, the sum lies between the left hand elements and the right hand elements.

Theorem 6.2. $1 + 1 = 2$

Proof. $\{\, 0 \mid \,\} + \{\, 0 \mid \,\} = \{\, 0+1, 1+0 \mid \emptyset + 1, 1 + \emptyset \,\} = \{\, 1 \mid \,\}$, where we've made use of a simpler result that we should have proven beforehand $0 + 1 = 1 + 0 = 1$. □

So what we labelled two is justified. Similarly we can show the name $\frac{1}{2}$ is justified:

Theorem 6.3. $\frac{1}{2} + \frac{1}{2} = 1$

Proof. $\{\, 0 \mid 1 \,\} + \{\, 0 \mid 1 \,\} = \left\{\, 0 + \frac{1}{2}, \frac{1}{2} + 0 \;\middle|\; \frac{1}{2} + 1, 1 + \frac{1}{2} \,\right\}$. Both sets here contain simpler elements (the day sum is ≤ 3 compared to the original 4), so unspooling the definition further gives $= \left\{\, \frac{1}{2} \;\middle|\; 1\frac{1}{2} \,\right\}$. Now the earliest created surreal number between the left and right sets is 1. □

7 To infinity and beyond...

I want to finish this with some examples of the fun we can have by using infinite sets as the left and right pairs of surreal numbers. We've already seen that to get $\frac{1}{3}$ we need infinite left and right sets, and we can similarly get all other irrational numbers.

But what if we use the following

$$\{\, 0, 1, 2, 3, \ldots \mid \,\}$$

This is a surreal number greater than every positive integer, so shall we call it ∞? The infinity it describes is the ordinal ω.

But we also have

$$\epsilon = \left\{\, 0 \;\middle|\; 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \,\right\}$$

This is greater than 0, but smaller than every real number. It's an infinitesimal! If we have defined multiplication one could prove that $\epsilon\omega = 1$.

But what about

$$\{\, 0, 1, 2, 3, \ldots \mid \omega \,\}\,?$$

This should be greater than every positive integer, but less than infinity. It turns out this is accurately described as $\omega - 1$, since

$$\begin{aligned}(\omega - 1) + 1 &= \{\, \omega - 1 + \{0\}, \{0, 1, 2, \ldots\} + 1 \mid \omega - 1 + \emptyset, \{\omega\} + 1 \,\} \\ &= \{\, \omega - 1, 1, 2, 3, \ldots \mid \omega + 1 \,\} \\ &= \omega\end{aligned}$$

It gets weirder, but I'll leave you with some final examples

$$\frac{\omega}{2} = \{\, 0, 1, 2, 3, \ldots \mid \omega, \omega - 1, \omega - 2, \omega - 3, \ldots \,\}\ ,$$

since you can show $\frac{\omega}{2} + \frac{\omega}{2} = \omega$, and

$$\sqrt{\omega} = \left\{\, 0, 1, 2, 3, \ldots \;\middle|\; \omega, \frac{\omega}{2}, \frac{\omega}{3}, \frac{\omega}{4}, \ldots \,\right\}$$

since the definition of multiplication leads to $\sqrt{\omega} \times \sqrt{\omega} = \omega$.

This is to say nothing of *pseudo-numbers*, the things we get by dropping the requirement for well-formedness, and how they fit into the picture.

References

[1] J.H. Conway. *On Numbers and Games*. Ak Peters Series. Taylor & Francis, 2000. ISBN: 9781568811277.

[2] D.E. Knuth. *Surreal Numbers. How Two Ex-students Turned on to Pure Mathematics and Found Total Happiness : a Mathematical Novelette*. Addison-Wesley Publishing Company, 1974. ISBN: 9780201038125.

[3] Claus Tøndering. *Surreal Numbers – An Introduction*. URL: `http://www.tondering.dk/claus/surreal.html`.