The coproduct on multiple zeta values, and 'almost' identities

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1 Introduction / recap of Multiple Zeta Values

1.1 Definition/Motivation

Before I start talking about coproducts on multiple zeta values, I ought to recap some of the basic definitions and facts about MZVs, and give some motivation.

Firstly the definition

Definition 1.1. For $a_i \in \mathbb{Z}_{>0}$, the *multiple zeta value* is defined as

$$\zeta(a_1, a_2, \ldots, a_k) := \sum_{0<n_1<n_2<\ldots<n_k} \frac{1}{a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k}}$$

For this sum to converge, we require $a_k > 1$. (Some conventions use $n_1 > n_2 > \cdots > n_k > 0$, but this just reverses the order of the arguments.)

We call k the *depth*, and $\sum_{i=0}^{k} a_i$ the *weight*.

So why is it worth studying these things? We'll firstly, there they have a huge amount of structure hidden behind this definition. For example, at weight $k = 10$ there are a priori $2^{10-2} = 256$ different MZVs. But it turns out that there are, at most, 7 linearly independent ones. This means there is a huge number of relations. For example

$$\begin{aligned} \zeta(1,2) &= \zeta(3) \\ \zeta(\{1,3\}^n) = \zeta(\{2\}^{2n}) &= \frac{\pi^{2n}}{(2n+1)!} \\ 28\zeta(3,9) + 150\zeta(5,7) + 168\zeta(7,5) &= \frac{5197}{691}\zeta(12) \end{aligned}$$

First is Euler, Second was a conjecture of Zagier which Broadhusrt proved, and the last is Gangl-Kaneko-Zagier with a connection to non-trivial cusp forms of weight $2k$.

So how to find and understand these?

Secondly, studying them could be motivated by what we don't know about them, despite the apparent simplicity of the definition. Easy-sounding questions about MZVs can be incredibly difficult. Recall, Euler showed that

$$\zeta(2k) = \frac{(-1)^{k+1} B_{2k} (2\pi)^{2k}}{2(2k)!},$$

in particularly all $\zeta(2k)$ are irrational (transcendental), and linearly independent. But the analogous equation for $\zeta(2k+1)$ is firmly unanswered. Arèry showed that $\zeta(3)$ is irrational, and it has been prove there are infinitely irrational $\zeta(odd)$, but we still don't know if $\zeta(5)$ is one of them. Similarly the three relations above are homogeneous in the weight. Are all relations weight graded? No one can prove this yet.

2 Facts about MZVs

So let's now recall some facts about MZVs.

Integral Representation Kontsevich shows/observed that every multiple zeta value can be written as particular Chen iterated integrals. We have

$$\zeta(a_1,\ldots,a_k) = (-1)^k \int_0^1 \frac{\mathrm{d}t}{t-1}\left(\frac{\mathrm{d}t}{t-0}\right)^{a_1-1}\cdots\frac{\mathrm{d}t}{t-1}\left(\frac{\mathrm{d}t}{t-0}\right)^{a_k-1}.$$

It is convenient to write this iterated integral using the following notation

$$(-1)^k I(0;1,0^{a_1-1},\ldots,1,0^{a_k-1};1)$$

where 0; and ; 1 are endpoints of the integration, and the middle arguments encode the differential forms appearing in the integral

$$a \leftrightarrow \frac{\mathrm{d}t}{t-a}$$

This gives an association between MZVs of weight k, and binary words of length $k+2$ starting 01, and ending 01. Can also write this as $xy^{a_1-1}\cdots xy^{a_k-1}$, as an argument to ζ.

Duality: It was observed early on in the study of MZVs, that they shows a duality - pairs of unrelated MZVs have the same numerical value. For example, Euler showed $\zeta(3)=\zeta(2,1)$, but we also have things like $\zeta(3,4)=\zeta(1,1,2,1,2)$. The integral representation provides a very convenient way to describe, and prove, the duality of MZVs, which is otherwise very difficult to even formulate.

Change variables in the integral, so $t\mapsto 1-t$. Then $\mathrm{d}t/(t-1)\leftrightarrow \mathrm{d}t/t$, and the end points swap. So

$$I(0;10^{a_1-1}\cdots 10^{a_k-1};1) = \pm I(1;01^{a_1-1}\cdots 01^{a_k-1};0)$$

But then reversing the path of integration gives

$$= I(0;1^{a_k-1}0\cdots 1^{a_1-1}0;1)$$

And this is the integral for another MZV. On the binary words: reverse and interchange $0\mapsto 1$.

So

$$\zeta(1,2) = I(0;110;1) = I(0;100;1) = \zeta(3)$$

Shuffle product: There is a well known way to multiply Chen iterated integrals. By splitting up the integration simplex, one can show it is to take the shuffle product of the words defining the differential forms.

$$I(a;v;b)I(a;w;b) = I(a;v ⧢ w;b)$$

Here $w ⧢ v$ can be defined recursively by

- For any word w, $⧢ w = w ⧢ = w$,
- For words v, w, and letters x, y, $(xv) ⧢ (yw) = x(v ⧢ yw) + y(xv ⧢ w)$.

Idea: riffle shuffle the letters of the two words.

So

$$\zeta(2)\zeta(2) = I(0; 10 ⧢ 10; 1) = I(0; 4 \cdot 1100 + 2 \cdot 1010; 1) = 4\zeta(1,3) + 2\zeta(2,2)$$

Stuffle product Instead of multiplying the integrals. Let's multiply the series representing the MZV. This leads to the stuffle product of MZVs, $\zeta(v)\zeta(w) = \zeta(v * w)$, where $*$ is defined recursively via:

- For any word w, $1 * w = w * 1 = w$,
- For any word w, and any integer $n \geq 1$:

$$x^n * w = w * x^n = wx^n$$

- For any words w_1, w_2, and integers $p, q \geq 0$:

$$yx^p w_1 * yx^q w_2 = yx^p(w_1 * yx^q w_2) + yx^q(yx^p w_1 * w_2) + yx^{p+q+1}(w_1 * w_2)$$

This has a much better interpretation as shuffling the arguments of the MZVs, and possibly *stuffing* two into one split.

$$\zeta(a)\zeta(b) = \zeta(a,b) + \zeta(b,a) + \zeta(a+b)$$
$$\zeta(2)\zeta(2) = 2\zeta(2,2) + \zeta(4)$$

Double Shuffle With the two different ways of multiplying MZVs, we can compare the expressions and get linear relations between MZVs. This even works if we allow the divergent $\zeta(1)$ to appear formally, the divergences cancel out in a way which gives correct results.

$$2\zeta(2,2) + \zeta(4) = 4\zeta(1,3) + 2\zeta(2,2) \implies \zeta(4) = 4\zeta(1,3)$$

Conjecturally regularised doubles shuffle gives all relations, which in turn would imply they are weight graded.

3 Motivic MZVs and the Coproduct / coaction

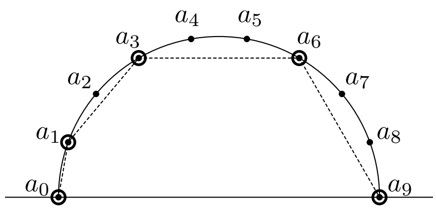
Many of the difficulties in proving results about MZVs is due to transcendence problems. If there were some way to replace the messy analytic object with some purely algebraic object, things would be easier.

Goncharov's motivic iterated integrals Goncharov (in Galois symmetries of fundamental groupoids and non-commutative geometry) showed how the ordinary iterated integrals $I(a_0; a_1, \ldots, a_n; a_{n+1})$ can be updated to frramed mixed Tate motives, so give the motivic iterated integrals $I^{\mathfrak{a}}(a_0; a_1, \ldots; a_n, a_{n+1})$, with a period map back down to $\mathbb{R}$.

These form a graded Hopf algebra structure $\mathcal{A}$, with the period map being a homomorphism of rings. (Restrict to $a_i = 0, 1$. Goncharov works more generally, but we're interested in MZVs, so this is fine.) The grading is n. He deduces the following expression for the coproduct:

$$\Delta I^{\mathfrak{a}}(a_0; a_1, \ldots; a_n, a_{n+1}) =$$
$$\sum_{0=i_0<i_1<\cdots<i_k<i_{k+1}=n+1} I^{\mathfrak{a}}(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1}) \otimes \prod_{p=0}^{k} I^{\mathfrak{a}}(a_{i_p}; a_{i_p+1}, \ldots; a_{i_{p+1}-1}, a_{i_{p+1}})$$

The way to remember this using the semicircle polygon pictorial representation



This is the term

$$I^{\mathfrak{a}}(a_0; a_1, a_3, a_6; a_9) \otimes I^{\mathfrak{a}}(a_0; a_1) I^{\mathfrak{a}}(a_1; a_2; a_3) I^{\mathfrak{a}}(a_3; a_4, a_5; a_6) I^{\mathfrak{a}}(a_6; a_7, a_8; a_9)$$

So this for all possible polygons.

By using the Kontsevich integral representation of MZVs, we get their motivic version.

$$\zeta^{\mathfrak{a}}(a_1, \ldots, a_n) = (-1)^n I^{\mathfrak{a}}(0; 10^{a_1} \cdots 10^{a_n}; 1) .$$

For free this gives us some results that have so far been impossible to prove for usual MZVs. The elements $\zeta^{\mathfrak{m}}(2k+1)$ lie in different componenets A_{2k+1}, so must be linearly independent! Goncharov has proved that any relations between the motivic DZVs follows from the motivic double shuffle relations. Similarly Goncharov has shown that $\zeta^{\mathfrak{a}}(3, 5)$ is irreducible, i.e. not a product of classical motivic ζ's.

Brown's motivic MZVs Goncharov's motivic MZVs aren't quite good enough. For him, $\zeta^{\mathfrak{a}}(2) = 0$. Francis Brown (Mixed tate motives over Z and on the decomposition of motivic multiple zeta values) shows how to lift these even further in such a way that $\zeta^{\mathfrak{m}}(2) \neq 0$. This is done with a graded algebra comodule $\mathcal{H} \cong \mathcal{A} \otimes_{\mathbb{Q}} \mathbb{Q}[\zeta^{\mathfrak{m}}(2)]$ over $\mathcal{A}$, and Goncharov's coproduct lifts to a coaction $\Delta\colon \mathcal{H} \to \mathcal{A} \otimes_{\mathbb{Q}} \mathcal{H}$, defined by the same formula as before (up to swapping factors)

$$\Delta I^{\mathfrak{a}}(a_0; a_1, \ldots; a_n, a_{n+1}) =$$
$$\sum_{0=i_0<i_1<\cdots<i_k<i_{k+1}=n+1} \prod_{p=0}^{k} I^{\mathfrak{a}}(a_{i_p}; a_{i_p+1}, \ldots; a_{i_{p+1}-1}, a_{i_{p+1}}) \otimes I^{\mathfrak{m}}(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1})$$

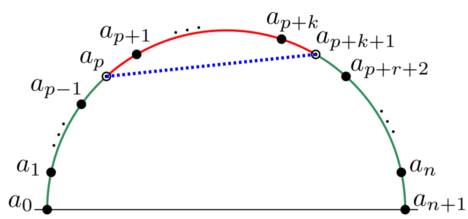
To make this coaction easier to work with, Brown introduces an infinitesimal version of it via the operators he calls D_r, as follows. Take $\mathcal{L} = \mathcal{A}_{>0}/\mathcal{A}_{>0}\mathcal{A}_{>0}$, and π is the projection. Then

$$D_r\colon \mathcal{H}_N \xrightarrow{\Delta_{r,N-r}} \mathcal{A}_r \otimes_{\mathbb{Q}} \mathcal{H}_{N-r} \xrightarrow{\pi\otimes\mathrm{id}} \mathcal{L}_r \otimes_{\mathbb{Q}} \mathcal{H}_{N-r}$$

The action of this on a motivic iterated integral can be explicitly computed as

$$D_r I^{\mathfrak{m}}(a_0; a_1, \ldots, a_n; a_{n+1}) =$$
$$\sum_{p=0}^{n-r} I^{\mathfrak{L}}(a_p; a_{p+1}, \ldots, a_{p+r}; a_{p+r+1}) \otimes I^{\mathfrak{m}}(a_0; a_1, \ldots, a_p, a_{p+1}, \ldots, a_n; a_{n+1})$$

So in the picture from before, we're just cutting off one segment with r interior points each time.



The real upshot of this comes from the following Theorem of Brown

Theorem 3.1. *The kernel of* $D_{<N} := \bigoplus_{3 \leq 2k+1 < N} D_{2k+1}$ *is* $\zeta^{\mathfrak{m}}(N)\mathbb{Q}$ *in weight* N*.*

Brown uses this and the D_r operators to provide an exactly-numerical algorithm to decompose motivic multiple zeta values into a chosen basis. This provides a combinatorial method to find/prove certain identities on the level of real MZVs using the period map.

Some simple examples of this include

Example 3.2.
$$t = \zeta(\underbrace{2, 2, \ldots, 2}_{n}) \in \pi^{2n}\mathbb{Q}$$

because if we compute $D_r I^{\mathfrak{m}}(0; (10)^n; 1)$, then (draw picture) cut off segment always starts and ends with the same symbol. So $D_{<N} t^{\mathfrak{m}} = 0$, which implies $t^{\mathfrak{m}} \in \zeta^{\mathfrak{m}}(2n)\mathbb{Q}$, and gives the above result on taking the period map.

Sadly, Brown's decomposition method cannot find the coefficient exactly in this case, so we'd have to resort to numerical evaluation write an explicit version of the 'almost' identity

$$\zeta(\underbrace{2, 2, \ldots, 2}_{n}) = \alpha \pi^{2n} \,.$$

4 'Almost' identities

Now I'd like to use this setup to explore some (more elaborate) 'almost-identities' that I have found, and set them in context of conjectured results.

For convenience, let's introduce the notation

$$Z(a_0, a_1, \ldots, a_{2m}) := \zeta(\{2\}^{a_0}, 1, \{2\}^{a_1}, 3, \ldots, 1, \{2\}^{a_{2m-1}}, 3, \{2\}^{a_{2m}}) \,,$$

so denote the MZV obtained by inserting 2^{a_i} in the i-th gap between consecutive terms of $1, 3, 1, 3, \ldots, 1, 3$.

The cyclic insertion conjecture (Borwein, Bradley, Broadhurst, Lisonek) evaluates the sum obtained by inserting some fixed blocks of 2s into $\zeta(1, 3, 1, 3, \ldots)$, and taking all cyclic shifts.

Conjecture 4.1. More precisely

$$\sum_{r \in C_{2n+1}} Z(a_{r(0)}, \dots, a_{r(2n)}) \stackrel{?}{=} \frac{\pi^{\text{wt}}}{(\text{wt}+1)!} .$$

Where wt is the weight of the MZVs involved, so is $4n + 2\sum a_i$.

Numerically this checks out easily enough in every case, but it hasn't been proven.

So far, the best that can be proven exactly is the Bowman-Bradley theorem,

Theorem 4.2.

$$\sum_{j_0+\ldots+j_{2n}=m} Z(a_{j_0}, \dots, a_{j_{2n}}) = \frac{1}{2n+1} \binom{m+2n}{m} \frac{\pi^{\text{wt}}}{(\text{wt}+1)!}$$

This is the sum over all (weak) compositions of m into $2n+1$ parts.

The proof of this is not too complicated, it relies on establishing some properties of the shuffle algebra (shuffle product on MZVs). But structurally the shuffle algebra makes it difficult to isolate particular size blocks of 2.

If you're happy to look just at 'almost' identities I can take another little steps towards the cyclic insertion conjecture. If I instead sum over all possible permutations (not just cyclic ones), then the result is a rational multiple of π^{wt}.

Theorem 4.3 (Symmetric Insertion)**.**

$$\sum_{\sigma \in S_{2n+1}} Z(a_{\sigma(0)}, \dots, a_{\sigma(2n)}) \in \pi^{\text{wt}}\mathbb{Q} .$$

Proof. The proof of this is to set up a pairwise cancellation in each D_r on Brown's motivic MZVs. The binary string corresponding to $Z(a_1, a_2, \dots, a_{2n})$ is

$$(01)^{a_1+1} \mid (10)^{a_2+1} \mid \cdots \mid (01)^{a_2n+1}$$

Then computing D_r involves marking out subsequences of length $r+2$ to get the left hand factor, and chopping this out to get the right hand factor.

By reflecting the blocks containing the subsequence, we can pair this term of D_r up with another term in such a way that they cancel. For example

For $\zeta(2,2,1,3)$, we have the binary string 0101|10|01. If we mark out

01**0101**|**10**|01

Then reflecting the first two blocks gives

0**1**|**1010**10|01

on $\zeta 1, 2, 2, 3$. Since the subsequence is reversed, we pick up a minus sign $(-1)^r = -1$, this means those terms in D_r cancel.

Taking the sum over all permutations guarantees all terms are paired up, and so all terms cancel. □

Unfortunately doing this for cyclic insertion doesn't work as well. There is no obvious pairwise cancellation, so any attempt to prove it would have to use much more of the structure of motivic MZVs.

But still we can now write down 'almost' identities like

$$\zeta(2,2,1,2,3)+\zeta(2,2,1,3,2)+\zeta(1,2,2,3,2)+\zeta(2,1,2,2,3)+\zeta(1,2,3,2,2)+\zeta(2,1,3,2,2) = \beta\pi^{10}$$

with confidence.