The coproduct on multiple zeta values, and 'almost' identities

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The title is quite ambitious. I'm probably going to spend a lot of time talking first about the background, giving an introduction to MZVs and some of the standard results.

1 Introduction / recap of Multiple Zeta Values

1.1 Definition/Motivation

Firstly the definition

Definition 1.1. For $s_i \in \mathbb{Z}_{>0}$, the *multiple zeta value* is defined as

$$\zeta(s_1, s_2, \ldots, s_k) \coloneqq \sum_{0<n_1<n_2<\ldots<n_k} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}$$

For this sum to converge, we require $s_k > 1$. (Some conventions use $n_1 > n_2 > \cdots > n_k > 0$, but this just reverses the order of the arguments.)

We call k the *depth*, and $\sum_{i=0}^k s_i$ the *weight*.

View this as a multivariable generalisation of the Rieman zeta function. These are the sort of sums one gets if you look at products of RZV, and try to break them up into their basic blocks...

So why is it worth studying these things? We'll firstly, there they have a huge amount of structure hidden behind this definition. For example, at weight $k = 10$ there are a priori $2^{10-2} = 256$ different MZVs. But it turns out that there are, at most, 7 linearly independent ones. This means there is a huge number of relations. For example

$$\begin{aligned} \zeta(1,2) &= \zeta(3) \\ (2n+1)\zeta(\{1,3\}^n) &= \zeta(\{2\}^{2n}) = \frac{\pi^{2n}}{(2n+1)!} \\ 28\zeta(3,9) + 150\zeta(5,7) + 168\zeta(7,5) &= \frac{5197}{691}\zeta(12) \end{aligned}$$

First is Euler, Second was a conjecture of Zagier which Broadhusrt proved, and the last is Gangl-Kaneko-Zagier with a connection to non-trivial cusp forms of weight $2k$.

So how to find and understand these?

Secondly, studying them could be motivated by what we don't know about them, despite the apparent simplicity of the definition. Easy-sounding questions about MZVs can be incredibly difficult. Recall, Euler showed that

$$\zeta(2k) = \frac{(-1)^{k+1} B_{2k} (2\pi)^{2k}}{2(2k)!} \,,$$

in particularly all $\zeta(2k)$ are irrational (transcendental), and linearly independent. But the analogous equation for $\zeta(2k+1)$ is firmly unanswered. Arèry showed that $\zeta(3)$ is irrational, and it has been prove there are infinitely irrational $\zeta(odd)$, but we still don't know if $\zeta(5)$ is one of them. Similarly the three relations above are homogeneous in the weight. Are all relations weight graded? No one can prove this yet.

2 Standard results about MZVs

So let's now recall some facts about MZVs.

Integral Representation Kontsevich shows/observed that every multiple zeta value can be written as particular Chen iterated integrals. We have

$$\zeta(a_1, \ldots, a_k) = (-1)^k \int_0^1 \frac{\mathrm{d}t}{t-1} \left(\frac{\mathrm{d}t}{t-0} \right)^{a_1 - 1} \cdots \frac{\mathrm{d}t}{t-1} \left(\frac{\mathrm{d}t}{t-0} \right)^{a_k - 1} \,.$$

It is convenient to write this iterated integral using the following notation

$$(-1)^k I(0; 1, 0^{a_1-1}, \ldots, 1, 0^{a_k-1}; 1)$$

where 0; and ; 1 are endpoints of the integration, and the middle arguments encode the differential forms appearing in the integral

$$a \leftrightarrow \frac{\mathrm{d}t}{t-a}$$

This gives an association between MZVs of weight k, and binary words of length $k+2$ starting 01, and ending 01. Can also write this as $xy^{a_1-1} \cdots xy^{a_k-1}$, as an argument to ζ.

Duality: It was observed early on in the study of MZVs, that they shows a duality - pairs of unrelated MZVs have the same numerical value. For example, Euler showed $\zeta(3) = \zeta(2,1)$, but we also have things like $\zeta(3,4) = \zeta(1,1,2,1,2)$. The integral representation provides a very convenient way to describe, and prove, the duality of MZVs, which is otherwise very difficult to even formulate.

Change variables in the integral, so $t \mapsto 1-t$. Then $\mathrm{d}t/(t-1) \leftrightarrow \mathrm{d}t/t$, and the end points swap. So

$$I(0; 10^{a_1-1} \cdots 10^{a_k-1}; 1) = \pm I(1; 01^{a_1-1} \cdots 01^{a_k-1}; 0)$$

But then reversing the path of integration gives

$$= I(0; 1^{a_k-1}0 \cdots 1^{a_1-1}0; 1)$$

And this is the integral for another MZV. On the binary words: reverse and interchange $0 \mapsto 1$.

So

$$\zeta(1,2) = I(0; 110; 1) = I(0; 100; 1) = \zeta(3)$$

Shuffle product: There is a well known way to multiply Chen iterated integrals. By splitting up the integration simplex, one can show it is to take the shuffle product of the words defining the differential forms.

$$I(a; v; b)I(a; w; b) = I(a; v \sqcup\!\sqcup w; b)$$

Here $w \sqcup\!\sqcup v$ can be defined recursively by

- For any word w, $\sqcup\!\sqcup w = w \sqcup\!\sqcup = w$,
- For words v, w, and letters x, y, $(xv) \sqcup\!\sqcup (yw) = x(v \sqcup\!\sqcup yw) + y(xv \sqcup\!\sqcup w)$.

Idea: riffle shuffle the letters of the two words.

So

$$\zeta(2)\zeta(2) = I(0; 10 \sqcup\!\sqcup 10; 1) = I(0; 4 \cdot 1100 + 2 \cdot 1010; 1) = 4\zeta(1, 3) + 2\zeta(2, 2)$$

Stuffle product Instead of multiplying the integrals. Let's multiply the series representing the MZV. This leads to the stuffle product of MZVs, $\zeta(v)\zeta(w) = \zeta(v * w)$, where $*$ is defined recursively via:

- For any word w, $1 * w = w * 1 = w$,
- For any word w, and any integer $n \geq 1$:

$$x^n * w = w * x^n = wx^n$$

- For any words w_1, w_2, and integers $p, q \geq 0$:

$$yx^p w_1 * yx^q w_2 = yx^p(w_1 * yx^q w_2) + yx^q(yx^p w_1 * w_2) + yx^{p+q+1}(w_1 * w_2)$$

This has a much better interpretation as shuffling the arguments of the MZVs, and possibly *stuffing* two into one split.

$$\zeta(a)\zeta(b) = \zeta(a, b) + \zeta(b, a) + \zeta(a + b)$$
$$\zeta(2)\zeta(2) = 2\zeta(2, 2) + \zeta(4)$$

Double Shuffle With the two different ways of multiplying MZVs, we can compare the expressions and get linear relations between MZVs. This even works if we allow the divergent $\zeta(1)$ to appear formally, the divergences cancel out in a way which gives correct results.

$$2\zeta(2, 2) + \zeta(4) = 4\zeta(1, 3) + 2\zeta(2, 2) \implies \zeta(4) = 4\zeta(1, 3)$$

Conjecturally regularised doubles shuffle gives all relations, which in turn would imply they are weight graded.

3 Motivic MZVs and the Coproduct / coaction

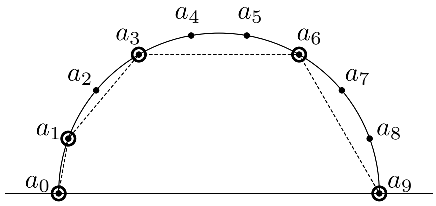
Many of the difficulties in proving results about MZVs is due to transcendence problems. If there were some way to replace the messy analytic object with some purely algebraic object, things would be easier.

Goncharov's motivic iterated integrals Goncharov (in *Galois symmetries of fundamental groupoids and non-commutative geometry*) showed how the ordinary iterated integrals $I(a_0; a_1, \ldots, a_n; a_{n+1})$ can be updated to frramed mixed Tate motives, so give the motivic iterated integrals $I^{\mathfrak{a}}(a_0; a_1, \ldots; a_n, a_{n+1})$, with a period map back down to $\mathbb{R}$.

These form a graded Hopf algebra structure $\mathcal{A}$, with the period map being a homomorphism of rings. (Restrict to $a_i = 0, 1$. Goncharov works more generally, but we're interested in MZVs, so this is fine.) The grading is n. He deduces the following expression for the coproduct:

$$\Delta I^{\mathfrak{a}}(a_0; a_1, \ldots; a_n, a_{n+1}) =$$
$$\sum_{0=i_0<i_1<\cdots<i_k<i_{k+1}=n+1} I^{\mathfrak{a}}(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1}) \otimes \prod_{p=0}^{k} I^{\mathfrak{a}}(a_{i_p}; a_{i_p+1}, \ldots; a_{i_{p+1}-1}, a_{i_{p+1}})$$

The way to remember this using the semicircle polygon pictorial representation



This is the term

$$I^{\mathfrak{a}}(a_0; a_1, a_3, a_6; a_9) \otimes I^{\mathfrak{a}}(a_0; a_1) I^{\mathfrak{a}}(a_1; a_2; a_3) I^{\mathfrak{a}}(a_3; a_4, a_5; a_6) I^{\mathfrak{a}}(a_6; a_7, a_8; a_9)$$

So this for all possible polygons.

By using the Kontsevich integral representation of MZVs, we get their motivic version.

$$\zeta^{\mathfrak{a}}(a_1, \ldots, a_n) = (-1)^n I^{\mathfrak{a}}(0; 10^{a_1} \cdots 10^{a_n}; 1) .$$

For free this gives us some results that have so far been impossible to prove for usual MZVs. The elements $\zeta^{\mathfrak{m}}(2k+1)$ lie in different componenets A_{2k+1}, so must be linearly independent! Goncharov has proved that any relations between the motivic DZVs follows from the motivic double shuffle relations. Similarly Goncharov has shown that $\zeta^{\mathfrak{a}}(3,5)$ is irreducible, i.e. not a product of classical motivic ζ's.

Brown's motivic MZVs Goncharov's motivic MZVs aren't quite good enough. For him, $\zeta^{\mathfrak{a}}(2) = 0$. Francis Brown (*Mixed Tate motives over* $\mathbb{Z}$ and *On the decomposition of motivic multiple zeta values*) shows how to lift these even further in such a way that $\zeta^{\mathfrak{m}}(2) \neq 0$. This is done with a graded algebra comodule $\mathcal{H} \cong \mathcal{A} \otimes_{\mathbb{Q}} \mathbb{Q}[\zeta^{\mathfrak{m}}(2)]$ over $\mathcal{A}$, and Goncharov's coproduct lifts to a coaction $\Delta\colon \mathcal{H} \to \mathcal{A} \otimes_{\mathbb{Q}} \mathcal{H}$, defined by the same formula as before (up to swapping factors)

$$\Delta I^{\mathfrak{m}}(a_0; a_1, \ldots; a_n, a_{n+1}) =$$
$$\sum_{0=i_0<i_1<\cdots<i_k<i_{k+1}=n+1} \prod_{p=0}^{k} I^{\mathfrak{a}}(a_{i_p}; a_{i_p+1}, \ldots; a_{i_{p+1}-1}, a_{i_{p+1}}) \otimes I^{\mathfrak{m}}(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1})$$

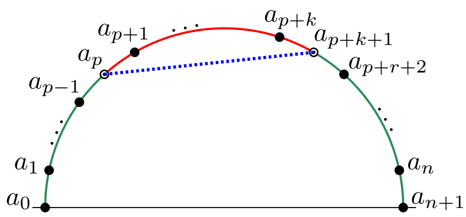
To make this coaction easier to work with, Brown introduces an infinitesimal version of it via the operators he calls D_r, as follows. Take $\mathcal{L} = \mathcal{A}_{>0}/\mathcal{A}_{>0}\mathcal{A}_{>0}$, and π is the projection. Then

$$D_r\colon \mathcal{H}_N \xrightarrow{\Delta_{r,N-r}} \mathcal{A}_r \otimes_{\mathbb{Q}} \mathcal{H}_{N-r} \xrightarrow{\pi\otimes\mathrm{id}} \mathcal{L}_r \otimes_{\mathbb{Q}} \mathcal{H}_{N-r}$$

The action of this on a motivic iterated integral can be explicitly computed as

$$D_r I^{\mathfrak{m}}(a_0; a_1, \ldots, a_n; a_{n+1}) =$$
$$\sum_{p=0}^{n-r} I^{\mathfrak{L}}(a_p; a_{p+1}, \ldots, a_{p+r}; a_{p+r+1}) \otimes I^{\mathfrak{m}}(a_0; a_1, \ldots, a_p, a_{p+1}, \ldots, a_n; a_{n+1})$$

So in the picture from before, we're just cutting off one segment with r interior points each time.



The real upshot of this comes from the following Theorem of Brown

Theorem 3.1. *The kernel of* $D_{<N} \coloneqq \bigoplus_{3 \leq 2k+1 < N} D_{2k+1}$ *is* $\zeta^{\mathfrak{m}}(N)\mathbb{Q}$ *in weight* N*.*

Brown uses this and the D_r operators to provide an exactly-numerical algorithm to decompose motivic multiple zeta values into a chosen basis. This provides a combinatorial method to find/prove certain identities on the level of real MZVs using the period map.

Some simple examples of this include

Example 3.2.

$$t = \zeta(\underbrace{2, 2, \ldots, 2}_{n}) \in \pi^{2n}\mathbb{Q}$$

because if we compute $D_r I^{\mathfrak{m}}(0; (10)^n; 1)$, then (draw picture) cut off segment always starts and ends with the same symbol. So $D_{<N} t^{\mathfrak{m}} = 0$, which implies $t^{\mathfrak{m}} \in \zeta^{\mathfrak{m}}(2n)\mathbb{Q}$, and gives the above result on taking the period map.

Sadly, Brown's decomposition method cannot find the coefficient exactly in this case, so we'd have to resort to numerical evaluation write an explicit version of the 'almost' identity

$$\zeta(\underbrace{2, 2, \ldots, 2}_{n}) = \alpha \pi^{2n} \, .$$

4 'Almost' identities

Now I'd like to use this setup to explore some (more elaborate) 'almost-identities' that I have found, and set them in context of conjectured results.

For convenience, let's introduce the notation

$$Z(a_0, a_1, \ldots, a_{2m}) \coloneqq \zeta(\{2\}^{a_0}, 1, \{2\}^{a_1}, 3, \ldots, 1, \{2\}^{a_{2m-1}}, 3, \{2\}^{a_{2m}}) \, ,$$

so denote the MZV obtained by inserting 2^{a_i} in the i-th gap between consecutive terms of $1, 3, 1, 3, \ldots, 1, 3$.

The cyclic insertion conjecture (Borwein, Bradley, Broadhurst, Lisonek in *Combinatorial Aspects of MZVs*) evaluates the sum obtained by inserting some fixed blocks of 2s into $\zeta(1, 3, 1, 3, \ldots)$, and taking all cyclic shifts.

Conjecture 4.1 (BBBL, Cyclic insertion conjecture)**.** More precisely

$$\sum_{r\in C_{2n+1}} Z(a_{r(0)},\ldots,a_{r(2n)}) \stackrel{?}{=} \frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}\,.$$

Where wt is the weight of the MZVs involved, so is $4n+2\sum a_i$.

Numerically this checks out easily enough in every case, but it hasn't been proven.

So far, the best that can be proven exactly is the Bowman-Bradley theorem,

Theorem 4.2 (Bowman Bradley, *Algebra and Combinatorics of Shuffles and MZVs*)**.**

$$\sum_{j_0+\ldots+j_{2n}=m} Z(a_{j_0},\ldots,a_{j_{2n}}) = \frac{1}{2n+1}\binom{m+2n}{m}\frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$$

This is the sum over all (weak) compositions of m into $2n+1$ parts.

The proof of this is not too complicated, it relies on establishing some properties of the shuffle algebra (shuffle product on MZVs). But structurally the shuffle algebra makes it difficult to isolate particular size blocks of 2.

If you're happy to look just at 'almost' identities I can take another little steps towards the cyclic insertion conjecture. If I instead sum over all possible permutations (not just cyclic ones), then the result is a rational multiple of π^{wt}.

Theorem 4.3 (Symmetric Insertion, C)**.**

$$\sum_{\sigma\in S_{2n+1}} Z(a_{\sigma(0)},\ldots,a_{\sigma(2n)}) \in \pi^{\mathrm{wt}}\mathbb{Q}\,.$$

Proof. The proof of this is to set up a pairwise cancellation in each D_r on Brown's motivic MZVs. The binary string corresponding to $Z(a_1,a_2,\ldots,a_{2n})$ is

$$(01)^{a_1+1} \mid (10)^{a_2+1} \mid \cdots \mid (01)^{a_2n+1}$$

Then computing D_r involves marking out subsequences of length $r+2$ to get the left hand factor, and chopping this out to get the right hand factor.

By reflecting the blocks containing the subsequence, we can pair this term of D_r up with another term in such a way that they cancel. For example

For $\zeta(2,2,1,3)$, we have the binary string 0101|10|01. If we mark out

01**0101**|**10**|01

Then reflecting the first two blocks gives

0**1**|**1010**10|01

on $\zeta 1,2,2,3$. Since the subsequence is reversed, we pick up a minus sign $(-1)^r=-1$, this means those terms in D_r cancel.

Taking the sum over all permutations guarantees all terms are paired up, and so all terms cancel. ☐

Unfortunately doing this for cyclic insertion doesn't work as well. There is no obvious pairwise cancellation, so any attempt to prove it would have to use much more of the structure of motivic MZVs.

But still we can now write down 'almost' identities like

$$\zeta(2,2,1,2,3)+\zeta(2,2,1,3,2)+\zeta(1,2,2,3,2)+\zeta(2,1,2,2,3)+\zeta(1,2,3,2,2)+\zeta(2,1,3,2,2) = \beta\pi^{10}$$

with confidence.

With this result we have the corollary

Corollary 4.4. *The MZV*

$$\zeta(\{\{2\}^m, 1, \{2\}^m, 3\}^n, \{2\}^m) \in \pi^{\mathrm{wt}}\mathbb{Q}\,,$$

that is, this MZV is evaluable*.*

An explicit conjecture for the value for this value is given by BBBL (Special Values of Multiple Polylogs *in Eg 6.5 pg 18), but also appears as a special case of the cyclic insertion conjecture where all blocks have the same length.*

4.1 Generalisation of cyclic/symmetric insertion

There is another conjectural family of identities, which has much the same spirit as the BBBL cyclic insertion conjecture. Zeta's with blocks of 2's...

Conjecture 4.5 (Hoffman, MZV Info page)**.**

$$\zeta(3,3,2^n) - \zeta(3,2^n,1,2) + \zeta(2^n,1,2,1,2) \stackrel{?}{=} -\zeta(2^{n+3})\,,$$

although Hoffman writes it with the first and third term combined via duality...

This has been checked up to weight 22, $a = 8$, by J. Vermaseren using tables of relations. But no general proof has been given yet.

I can also prove this identity, up to a rational, using Brown's motivic MZVs.

Even more than this is true. Hoffman's identity, and the BBBL cyclic insertion identity are both special cases of an even more general family of identities.

Take any MZV (any iterated integral), and write down the binary word encoding it. Split this into blocks whenever 00, or 11 occurs. As in the cyclic insertion case. In Hoffman's examplm, we have

$$I(010 \mid 010 \mid 0(10)^n 1) + I(010 \mid 0(10)^n 1 \mid 101) + I(0(10)^n 1 \mid 101 \mid 101)$$

Notice that the block-sizes are the same across the three integrals. A similar thing happens in the cyclic insertion conjecture. From any permutation of the blocks, we can reconstruct the corresponding iterated integral, because we know the first block starts with a 0, and blocks always join with 00 or 11.

Theorem 4.6 (C)**.** *If we do this with any iterated integral, and take all permutations of the block sizes, we can set up a similar pairwise cancellation in D_r to deduce that the sum is $\in \pi^{\mathrm{wt}}\mathbb{Q}$.*

(Note, that when moving around these block, you might end up with integrals starting 00, or ending 11. These don't directly correspond to MZVsk, but these integrals can always be *regularised* to get sums of MZVs, so things are okay.)

What's more, Hoffman and the Cyclic insertion conjecture work by taking *cyclic* shifts of the blocks.

Conjecture 4.7 (C)**.** More generally it appears that (for iterated integrals not containing 0000, or 1111), taking cyclic shifts alone is sufficient to get $\pi^{\mathrm{wt}}\mathbb{Q}$. Amazingly, the value is apparently (up to sign) $\frac{\pi^{\mathrm{wt}}}{(\mathrm{wt}+1)!}$. For odd weight cyclic, shifts sum to 0 for integrals with no 000 or 111.