2012 Q9)

9a) Define what is meant by the centre $Z(G)$ of a group G?

b) Show that $Z(G)$ is a subgroup of G, and that it is normal in G.

c) Determine the centre of $\mathrm{SL}_2(\mathbb{R})$

d) Show that the order of any conjugacy class in G divides the group order.

e) Let p be a prime number, and G a non-abelian group of order p^3. Show that the order of $Z(g)$ must be p.

f) Using the above, or otherwise, give the orders of $Z(D_4)$ and of $Z(Q_8)$.

Centre - Q9a)

Q9a) Define what is meant by the centre $Z(G)$ of a group G?

Definition (Week 12 - Q3, Centre)

The centre of a group G is the subset of elements $z \in G$ which commute with all elements in G, i.e.

$$Z(G) := \{ \, z \in G \mid gz = zg \text{ for all } g \in G \, \} \; .$$

Subgroup, Normal subgroup - Q9b)

9b) Show that $Z(G)$ is a subgroup of G, and that it is normal in G.

Proposition (Prop 2.7, Subgroup criterion)

A subset $H \subseteq G$ is a subgroup of $(G, \circ)$ if the following conditions holds

1. *H is non-empty (usually $e_G \in H$ suffices),*
2. *for all $h_1, h_2 \in H$, the product $h_1 \circ h_2 \in H$, and*
3. *for all $h \in H$, the inverse $h^{-1} \in H$.*

Definition (Def 2.17, Normal subgroup)

A subgroup $H < G$ is called normal if

- $gH = Hg$ for all $g \in G$.
- Equivalently $gHg^{-1} \subseteq H$ for all $g \in G$.
- Equivalently $ghg^{-1} \in H$ for all $g \in G$, and all $h \in H$.

9c) Determine the centre of $\mathrm{SL}_2(\mathbb{R})$.

Recall

The special linear group denotes matrices of determinant 1

$$\mathrm{SL}_2(\mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; ad - bc = 1 \text{ and } a, b, c, d \in \mathbb{R} \right\} .$$

9d) Show that the order of any conjugacy class in G divides the group order.

Recall

A group G acts on itself by conjugation. The orbits are conjugacy classes

$$\mathrm{ccl}_G(g) := \left\{ hgh^{-1} \;\middle|\; h \in G \right\} .$$

Theorem (Thm 8.18, Orbit-Stabiliser)

Suppose G acts on X. Then for any $x \in X$ there is a bijection

$$\begin{aligned} \beta \colon G(x) &\to \{ \text{left cosets of } G_x \text{ in } G \} \\ g(x) &\mapsto gG_x . \end{aligned}$$

Corollary (Cor 8.19)

Suppose G acts on a set X. Then for any $x \in X$

$$\#G(x) \cdot \#G_x = \#G .$$

Lagrange's Theorem - Q9e)

9e) Let p be a prime number and G a non-abelian group of order p^3. Show that the order of $Z(G)$ must be p.

Theorem (Thm 2.10, Lagrange)

Let H be a subgroup of the finite group G. Then

$$\#H \mid \#G\,.$$

Hint

- Compare with Corollary 11.3, dealing with groups of order p^2
- Use that

$$\#G = \underbrace{\sum_{\text{ccl size } 1} \#\mathrm{ccl}_G(g)}_{=\#Z(G)} + \sum_{\text{ccl size } > 1} \#\mathrm{ccl}_G(g)\,.$$

9f) Give the orders of $Z(D_4)$ and of $Z(Q_8)$.

Definition (Def 3.24, Dihedral Group)

The dihedral group D_n is defined by

$$D_n = \langle r, s \mid r^n = s^2 = e, sr = r^{-1}s \rangle \, .$$

Definition (Week 12 - Q3, Quaternion Group)

The quaternion group is defined by

$$Q_8 = \{ \pm 1, \pm i, \pm j, \pm k \} \, ,$$

with $i^2 = j^2 = k^2 = ijk = -1$.

2014 Q10)

10iii) For each of $n \in \{ 3, 4, 5, 6 \}$ decide whether

- A_n is isomorphic to a subgroup of D_n,
- D_n is isomorphic to a subgroup of A_n, or
- neither A_n is isomorphic to a subgroup of D_n, nor D_n is isomorphic to a subgroup of A_n.

iv) Show that $D_3 \times D_5$ cannot be isomorphic to A_5.

Distinguishing groups - Q10iii,iv)

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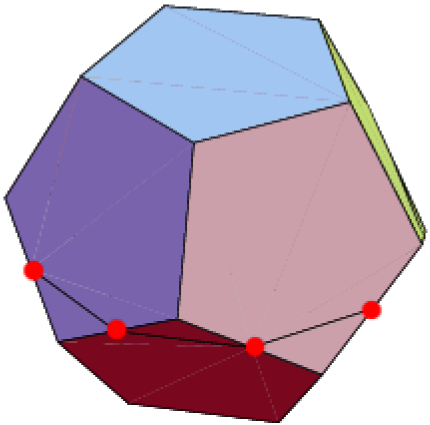
iv) Show that $D_3 \times D_5$ cannot be isomorphic to A_5.

Lemma (Lem 6.1)

If two groups are isomorphic, then they

- *both have the same size,*
- *both have the same number of elements of a given order k,*
- *both have the same size centre,*
- *are both abelian, or are both non-abelian,*
- *both have the same number of (normal) subgroups of a given size ℓ,*
- *...*

Seeing D_5 a subgroup of A_5 geometrically



2012 Q4)

4iii) Find 5 non-isomorphic groups of order 8. In particular, prove that they are non-isomorphic.

Distinguishing groups - Q4iii)

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Finitely generated abelian groups - Q4iii)

4iii) Find 5 non-isomorphic groups of order 8. In particular, prove that they are non-isomorphic.

Theorem (Thm 12.13, FTFGAG)

Any finitely generated abelian group is isomorphic to a group of the form

$$\mathbb{Z}/d_1 \times \mathbb{Z}/d_2 \times \cdots \times \mathbb{Z}/d_k \times \mathbb{Z}^r \,,$$

where $r \geq 0$, $k \geq 0$, *and* $d_i \geq 1$.

Moreover, if we require $d_i \mid d_{i+1}$ *and* $d_1 > 1$ *then this form is unique.*

2012 Q5)

5i) Let $\phi\colon G \to H$ be a homomorphism of groups and let $H' < H$ be a subgroup. Show that $\phi^{-1}(H') := \{ g \in G \mid \phi(g) \in H' \}$ is a subgroup of G.

ii) Which of the following permutations, if any, are conjugate in S_6?

$$\sigma_1 = (1\,2\,3\,5)(5\,4\,3)(5\,6)\,; \quad \sigma_2 = (5\,6)(5\,4\,3)(1\,2\,3\,5)\,; \quad \sigma_3 = (2\,3\,5)(1\,4\,6)$$

Conjugacy in S_n - Q5ii)

ii) Which of the following permutations, if any, are conjugate in S_6?

$$\sigma_1 = (1\,2\,3\,5)(5\,4\,3)(5\,6)\,; \quad \sigma_2 = (5\,6)(5\,4\,3)(1\,2\,3\,5)\,; \quad \sigma_3 = (2\,3\,5)(1\,4\,6)$$

Theorem (Thm 10.5)

For $\sigma \in S_n$, the conjugacy class $\mathrm{ccl}_{S_n}(\sigma)$ consists of all permutations which have the same cycle shape as x.

2012 Q6)

6i) Write down the possible groups of order 218 and 289, up to isomorphism.

ii) Write down all the abelian groups of order 108, up to isomorphism. How many elements of order 6 does each of these groups contain?

6i) Write down the possible groups of order 218 and 289, up to isomorphism.

Theorem (Thm 9.2, Groups of order $2p$)

Let p be an odd prime. Then any group of order $2p$ is either isomorphic to the cyclic group $\mathbb{Z}/(2p)$, or isomorphic to the dihedral group D_p.

Theorem (Cor 11.4, Groups of order p^2)

Let p be a prime. Then any group of order p^2 is abelian. Moreover it is either isomorphic to $\mathbb{Z}/p^2$, or isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$.

6ii) Write down all the abelian groups of order 108, up to isomorphism. How many elements of order 6 does each of these groups contain?

Theorem (Thm 12.13, FTFGAG)

Any finitely generated abelian group is isomorphic to a group of the form

$$\mathbb{Z}/d_1 \times \mathbb{Z}/d_2 \times \cdots \times \mathbb{Z}/d_k \times \mathbb{Z}^r ,$$

where $r \geq 0$, $k \geq 0$, and $d_i \geq 1$.

Moreover, if we require $d_i \mid d_{i+1}$ and $d_1 > 1$ then this form is unique.

$A_n(G)$, $O_n(G)$, Counting elements of order n - Q6ii)

6ii) Write down all the abelian groups of order 108, up to isomorphism. How many elements of order 6 does each of these groups contain?

Definition (Def 12.17)

- $A_m(G) =$ number of elements of G with order dividing m
- $O_m(G) =$ number of elements of G with order exactly m

Fact (Lem 12.18, Prop 12.19)

- A_m is multiplicative, so $A_m(G \times H) = A_m(G)A_m(H)$
- $A_m(\mathbb{Z}/n) = \gcd(n, m)$
- Use inclusion/exclusion to relate O and A. E.g., for $m = pq$, with p, q prime:

$$O_{pq}(G) = A_{pq}(G) - A_p(G) - A_q(G) + A_1(G)$$