Warmup Q2'ii)' and Q3i)

Let $R = (\mathbb{Z}/2)[x]$ and $I = (x^2 + \overline{1})_R$. In R/I, check

- $(\overline{1} + I) + ((x + \overline{1}) + I) = x + I$
- $(x + I) \cdot (x + I) = \overline{1} + I$

In $R = \mathbb{Z}[\sqrt{5}]$ show $(2, 3\sqrt{5})_R = (1)_R$.

Ideals

Ideal is a subset $I \subseteq R$ such that

1. I is closed under addition,
2. For all $x \in I$ and $r \in R$, we have $xr, rx \in I$,
3. I is *non-empty*

Can replace 1 and 3 by

- $(I, +)$ is a subgroup of $(R, +)$

Quotient rings

- Define $a \sim b$ iff $a - b \in I$
- Get equivalent classes

$$\overline{r} = r + I \coloneqq \{\, r + x \mid x \in I \,\}$$

Quotient ring is $R/I \coloneqq \{\, r + I \mid r \in R \,\}$

Operations

- $(a + I) + (b + I) \coloneqq (a + b) + I$
- $(a + I) \cdot (b + I) \coloneqq (a \cdot b) + I$

Q2'ii)'

Let $R = (\mathbb{Z}/2)[x]$ and $I = (x^2 + \overline{1})_R$. In R/I, check

$$(\overline{1} + I) + ((x + \overline{1}) + I) = x + I$$

- Use $(a + I) + (b + I) := (a + b) + I$
- $a = \overline{1}, b = x + \overline{1} \in (\mathbb{Z}/2)[x]$
- $a + b = \overline{1} + (x + \overline{1}) = x$

So calculation is correct

Q2'ii)'

Let $R = (\mathbb{Z}/2)[x]$ and $I = (x^2 + \overline{1})_R$. In R/I, check

$$(x + I) \cdot (x + I) = \overline{1} + I$$

- Use $(a + I) \cdot (b + I) \coloneqq (a \cdot b) + I$
- $a = x, b = x \in (\mathbb{Z}/2)[x]$
- $a \cdot b = x^2$

Is $x^2 + I = \overline{1} + I$?

- Yes, since $x^2 - \overline{1} = x^2 + \overline{1} \in I$

$\mathbb{C}$, a familiar quotient ring

$$\begin{array}{ccc} \mathbb{R}[x]/(x^2+1)_{\mathbb{R}[x]} & & \mathbb{C} \\ \Cup & \leftrightarrow & \Cup \\ (a+bx)+J & & a+b\mathrm{i} \end{array}$$

$$\begin{aligned} (2+3x)+J &= \big\{2+3x, 4+3x+2x^2, 1+3x-x^2, \\ &\qquad 3+4x+x^2+x^3, \dots\big\} \\ &= (4+3x+2x^2)+J = \cdots \end{aligned}$$

$$\begin{aligned} \leftrightarrow \quad 2+3\mathrm{i} &= 4+3\mathrm{i}+2\mathrm{i}^2 = 1+3\mathrm{i}-\mathrm{i}^2 \\ &= 3+4\mathrm{i}+\mathrm{i}^2+\mathrm{i}^3 = \cdots \end{aligned}$$

$\mathbb{C}$, a familiar quotient ring

$$
\begin{aligned}
&((1 + 2x) + J) + ((3 - x) + J) \\
&\qquad\qquad\qquad\qquad = (4 + x) + J
\end{aligned}
$$

$$\leftrightarrow \quad (1 + 2\mathrm{i}) + (3 - \mathrm{i}) = 4 + \mathrm{i}$$

$\mathbb{C}$, a familiar quotient ring

$$
\begin{aligned}
((1+2x)+J)\cdot((3-x)+J) \\
&= (3+5x-2x^2)+J \\
&= (5+5x)+J
\end{aligned}
$$

$$
\leftrightarrow \quad (1+2\mathrm{i})\cdot(3-\mathrm{i}) = 3+5\mathrm{i}-2\mathrm{i}^2 \\
= 5+5\mathrm{i}
$$

$\mathbb{C}$ is $\mathbb{R}[x]/(x^2+1)_{\mathbb{R}[x]}$ after writing $\mathrm{i} = x + J$.

Generating ideals

- Elements $r_1, \ldots, r_n \in R$ generate the ideal

$$(r_1, \ldots, r_n)_R$$

- Is the 'smallest' ideal containing $r_1, \ldots, r_n$:
  If $r_1, \ldots, r_n \in I$ then $(r_1, \ldots, r_n)_R \subseteq I$
- If R commutative

$$(r_1, \ldots, r_n)_R = \left\{ \sum_{i=1}^{n} a_i r_i \;\middle|\; a_i \in R \right\}$$

Equality of ideals

Ideals $I = (r_1, \ldots, r_n)_R$ and $J = (s_1, \ldots, s_m)_R$.

Want to show $I = J$. Show $I \subseteq J$ and $J \subseteq I$.

For $I \subseteq J$:

- Show $r_i \in J$, for all i

For $J \subseteq I$:

- Show $s_j \in I$, for all j

Conclude $I = J$.

Q3i)

In $R = \mathbb{Z}[\sqrt{5}]$, show $I = (2, 3\sqrt{5})_R$ equals $J = (1)_R$

For $I \subseteq J$:

- $2 = 2 \cdot 1 \in J$
- $3\sqrt{5} = 3\sqrt{5} \cdot 1 \in J$

For $J \subseteq I$:

- $1 = -7 \cdot 2 + \sqrt{5} \cdot 3\sqrt{5} \in I$

So $I = J$.