ALGEBRA 2 – MICHAELMAS 2015
TUTORIAL 3

Question: Suppose that $f\colon R \to S$ is a ring isomorphism (bijective ring homomorphism). Prove that $f^{-1}\colon S \to R$ is indeed a ring homomorphism.

Solution: Let $s_1, s_2 \in S$. We need to prove that $f^{-1}(s_1 + s_2) = f^{-1}(s_1) + f^{-1}(s_2)$. (And similarly for multiplication.)

Since f is bijective, we can find $r_1, r_2 \in R$ such that $f(r_1) = s_1$ and $f(r_2) = s_2$. Now consider

$$s_1 + s_2 = f(r_1) + f(r_2) = f(r_1 + r_2)\,.$$

Apply f^{-1} to both sides to get

$$f^{-1}(s_1 + s_2) = f^{-1}(f(r_1 + r_2)) = r_1 + r_2 = f^{-1}(s_1) + f^{-1}(s_2)\,.$$

This proves f^{-1} preserves addition.

A very similar proof works for multiplication. Lastly we have $f(1_R) = 1_S$, so that $f^{-1}(1_S) = 1_R$. This proves that $f^{-1}\colon R \to S$ is a ring homomorphism.