ALGEBRA 2 – MICHAELMAS 2015
TUTORIAL 3

2) Let $R = (\mathbb{Z}/2)[x]$, and let I be the ideal $(x^2+1)_R$ of R.

i) Let $f(x) \in R$. Use the division algorithm to prove that $f(x) - (ax+b) \in I$ for some $ax + b \in R$. Are a and b unique? Hence list the elements of $(\mathbb{Z}/2)[x]/I$.

Solution: Apply the division algorithm to divide $f(x)$ by x^2+1. It says we can write

$$f(x) = (x^2+1)q(x) + r(x)\,,$$

where $\deg(r) < \deg(x^2+1) = 2$. This means $r(x) = ax+b$ for some $a, b \in \mathbb{Z}/2$.
Now $f(x) - r(x) = (x^2+1)q(x) \in I$, which shows $f(x)$ is equivalent to $ax+b$. This means every equivalence class has a linear representative.
If also $f(x) - (cx - d) \in I$, then $(ax+b) - (cx+d) \in I$. This means $(a-c)x + (b-d)$ is divisible by x^2+1, so $a - c = b - d = 0$. So a and b are unique.
The elements of R/I are therefore

$$\overline{0} + I,\ \overline{1} + I,\ x + I,\ (x + \overline{1}) + I$$

ii) Check the following computations in $(\mathbb{Z}/2)[x]/I$

- $(\overline{1} + I) + ((x + \overline{1}) + I) = x + I$,
- $(x + I) \cdot (x + I) = \overline{1} + I$.

iii) Complete the addition and multiplication tables for $(\mathbb{Z}/2)[x]/I$.

Solution: Continuing on from the calculations above, the addition and multiplication tables are as follows. For notational ease, I write r, rather than $r + I$.

$+$	$\overline{0}$	$\overline{1}$	x	$x+\overline{1}$
$\overline{0}$	$\overline{0}$	$\overline{1}$	x	$x+\overline{1}$
$\overline{1}$	$\overline{1}$	$\overline{0}$	$x+\overline{1}$	x
x	x	$x+\overline{1}$	$\overline{0}$	$\overline{1}$
$x+\overline{1}$	$x+\overline{1}$	x	$\overline{1}$	$\overline{0}$

and

$\cdot$	$\overline{0}$	$\overline{1}$	x	$x+\overline{1}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	x	$x+\overline{1}$
x	$\overline{0}$	x	$\overline{1}$	$x+\overline{1}$
$x+\overline{1}$	$\overline{0}$	$x+\overline{1}$	$x+\overline{1}$	$\overline{0}$

iv) **Extra:** Is $(\mathbb{Z}/2)[x]/I$ a field?

Solution: $(\mathbb{Z}/2)[x]/I$ is not even an integral domain since $(x+\overline{1})+I$ is a zero-divisor. So this definitely cannot be a field.

Remark: Later, this will mean that $(x^2+\overline{1})_R$ is not a *maximal* ideal. Indeed

$$(x^2+\overline{1})_R \subsetneq (x+\overline{1})_R \subsetneq R$$