Warmup Q1 parts (1) and (8)

Are the following $(S, \circ)$ groups?

(1) $S = \{ \frac{a}{2} : a \in \mathbb{Z} \}$, and $\circ = +$?

(8) $S = \{ z \in \mathbb{C} : |z| = 2^{2a+1}$ with $a \in \mathbb{Z} \}$ and $\circ = \cdot$?

Motivation for group definition

Capture notion of symmetry mathematically.

Symmetry

Something you can do to an object, which leaves it unchanged.

- Have the do nothing symmetry (Identity)
- Can do one symmetry followed by another (Closure)
- Can undo a symmetry (Inverse)
- Doing three symmetries makes sense (Associativity)

Definition of a group

Definition (Group)

A group $(G, \circ)$ is a set G with binary operation $\circ \colon G \times G \to G$ such that

- (Identity) There is an identity $e \in G$, which satisfies $e \circ g = g \circ e = g$ for every $g \in G$.
- (Inverse) Each element $g \in G$ has an inverse $h \in G$, which satisfies $g \circ h = h \circ g = e$.
- (Associativity) For every $g, h, k \in G$, we have $g \circ (h \circ k) = (g \circ h) \circ k$.

Note: Closure is part of the definition of a binary operation.

Q1 part (1)

Is $S = \left\{ \frac{a}{2} : a \in \mathbb{Z} \right\}$, under $\circ = +$ a group?

Start checking the conditions. Helpful to note these are rational numbers, with the usual addition.

- (Closure) Let $\frac{a}{2}, \frac{b}{2} \in S$. Then $\frac{a}{2} + \frac{b}{2} = \frac{a+b}{2} \in S$. ✓
- (Identity) The identity is $0 = \frac{0}{2} \in S$. ✓
- (Inverse) Let $\frac{a}{2} \in S$. The inverse is $\frac{-a}{2} \in S$. ✓
- (Associativity) Addition of rationals is associative. ✓

So $(S, \circ)$ is a group.

Q1 part (8)

Is $S = \{ z \in \mathbb{C} : |z| = 2^{2a+1} \text{ with } a \in \mathbb{Z} \}$ under $\circ = \cdot$ a group?

Start checking the conditions.

- (Closure) Let $w, z \in S$. Say $|z| = 2^{2a+1}$ and $|w| = 2^{2b+1}$. Then $|zw| = 2^{2a+2b+2}$. But this won't be in S. ✗

Better to give *explicit* example of failure.

Take $w = z = 2 \in S$, with $|z| = |w| = 2^1$. Then $wz = 4$ has $|wz| = 4 = 2^2 \notin S$.

So $(S, \circ)$ is not a group because closure fails. (Identity also fails.)

Checking associativity

Generally difficult. Can't read from Cayley table. Have to check every single possibility.

$\circ$	a	b	c
a	a	b	c
b	b	a	c
c	c	c	a

Is $(\{ a, b, c \}, \circ)$ a group?

No. But only failure is associativity in two (out of 27) cases

$$(b \circ c) \circ c \neq b \circ (c \circ c)$$
$$(c \circ c) \circ b \neq c \circ (c \circ b)$$

(Note: there are better ways to see $(\{ a, b, c \}, \circ)$ is not a group.)