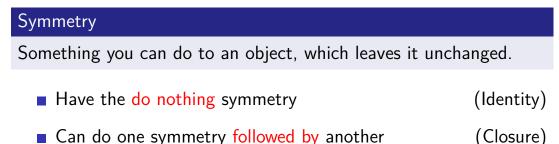
Are the following (S, \circ) groups? (1) $S = \left\{ \frac{a}{2} : a \in \mathbb{Z} \right\}$, and $\circ = +$? (8) $S = \left\{ z \in \mathbb{C} : |z| = 2^{2a+1} \text{ with } a \in \mathbb{Z} \right\}$ and $\circ = \cdot$?

Motivation for group definition

Capture notion of symmetry mathematically.



- (Inverse)
- Can undo a symmetry
- Doing three symmetries makes sense

(Associativity)

Definition of a group

Definition (Group)

A group (G,\circ) is a set G with binary operation $\circ\colon G\times G\to G$ such that

- (Identity) There is an identity $e \in G$, which satisfies $e \circ g = g \circ e = g$ for every $g \in G$.
- Inverse) Each element g ∈ G has an inverse h ∈ G, which satisfies g ∘ h = h ∘ g = e.
- (Associativity) For every $g, h, k \in G$, we have $g \circ (h \circ k) = (g \circ h) \circ k$.

Note: Closure is part of the definition of a binary operation.

Q1 part (1)

Is
$$S = \left\{ \frac{a}{2} : a \in \mathbb{Z} \right\}$$
, under $\circ = +$ a group?

Start checking the conditions. Helpful to note these are rational numbers, with the usual addition.

• (Closure) Let
$$\frac{a}{2}, \frac{b}{2} \in S$$
. Then $\frac{a}{2} + \frac{b}{2} = \frac{a+b}{2} \in S$. \checkmark

• (Identity) The identity is $0 = \frac{0}{2} \in S$.

- (Inverse) Let $\frac{a}{2} \in S$. The inverse is $\frac{-a}{2} \in S$.
- (Associativity) Addition of rationals is associative. \checkmark

So (S, \circ) is a group.

Q1 part (8)

Is $S = \{ z \in \mathbb{C} : |z| = 2^{2a+1} \text{ with } a \in \mathbb{Z} \}$ under $\circ = \cdot$ a group?

Start checking the conditions.

• (Closure) Let
$$w, z \in S$$
. Say $|z| = 2^{2a+1}$ and $|w| = 2^{2b+1}$.
Then $|zw| = 2^{2a+2b+2}$. But this won't be in S .

Better to give *explicit* example of failure.

Take $w = z = 2 \in S$, with $|z| = |w| = 2^1$. Then wz = 4 has $|wz| = 4 = 2^2 \notin S$.

So (S, \circ) is not a group because closure fails. (Identity also fails.)

Checking associativity

Generally difficult. Can't read from Cayley table. Have to check every single possibility.

0	a	b	c
a	a	b	С
b	b	a	c
c	c	c	a

Is $(\left\{ \, a,b,c \, \right\},\circ)$ a group?

No. But only failure is associativity in two (out of 27) cases

$$(b \circ c) \circ c \neq b \circ (c \circ c)$$
$$(c \circ c) \circ b \neq c \circ (c \circ b)$$

(Note: there are better ways to see $(\{a, b, c\}, \circ)$ is not a group.)