ALGEBRA 2 – EPIPHANY 2016
TUTORIAL 5

Investigating redundancy in the group axioms

Typically mathematicians want to eliminate any unnecessary axioms from their definitions,and any unnecessary hypothesis from their theorems. But, surprisingly, there are some redundancies in the group axioms!

Historically a weaker set of axioms was used in defining a group, but the 'benefit' of having fewer axioms was outweighed by the increased tediousness when using them.

1 Usual group axioms

For the sake of being self-contained, recall the axioms of a group are as follows. A group is a set G together with a binary operation $\circ\colon G \times G \to G$, such that

- (Associativity) The operation $\circ$ is associative,
- (Identity) There exists an element $e \in G$ with $e \circ x = x \circ e = x$ for any $x \in G$, and
- (Inverse) For any $x \in G$ there exists $x' \in G$ such that $x \circ x' = x' \circ x = e$.

Warning: In the exam you *must* use the above version of the group axioms, since this is the definition given in lectures. Using any of the versions below will cost you marks!

2 Right sided axioms

One can weaken the axioms to the 'right-sided' version

- (Associativity) The operation $\circ$ is associative,
- (Right Identity) There exists an element $e \in G$ with $x \circ e = x$ for any $x \in G$, and
- (Right Inverse) For any $x \in G$ there exists $x' \in G$ such that $x \circ x' = e$.

You are going to show how (Identity) and (Inverse) can be derived from (Right Identity), (Right Inverse) and (Associativity).

Do this by solving the following questions. You may only use the 'right-sided' version of the axioms.

Q1) Suppose $a \circ a = a$. Prove that $a = e$.

Q2) Use Q1) to show that $a' \circ a = e$, and so deduce that a' is also a left inverse of a.

Q3) Use Q2) to show that for all a, we have $e \circ a = a$, so that e is also a left identity.

Now we can conclude that the 'right-sided' version of the axioms still define a group.

3 Left sided axioms

Similarly, we can weaken the axioms to the 'left-sided' version

- (Associativity) The operation $\circ$ is associative,
- (Left Identity) There exists an element $e \in G$ with $e \circ x = x$ for all $x \in G$, and
- (Left Inverse) For any $x \in G$ there exists $x' \in G$ such that $x' \circ x = e$.

But we can still recover the usual group axioms from these. So the 'left-sided' version of the axioms still define a group.

Q4) Show how to do this. (Modify the approach above appropriately.)

4 Mixed axioms

Suppose now we weaken the axioms to a 'mixed' version

- (Associativity) The operation $\circ$ is associative,
- (Left Identity) There exists an element $e \in G$ with $e \circ x = x$ for all $x \in G$, and
- (Right Inverse) For any $x \in G$ there exists $x' \in G$ such that $x \circ x' = e$.

Obviously the proofs above do not go through to derive the usual group axioms from the 'mixed' weak version.

Q5) Is it possible to recover the usual group axioms from the 'mixed' weak version? Give a proof, or a counter-example.