Warmup Q4

Are the groups

- $\mathbb{Z} \times \mathbb{Z}$, and $\mathbb{Z}$

isomorphic? (Why?)

Recall

Direct product of $(G, \circ)$ and $(H, \bullet)$ is

$$G \times H = \{ (g, h) \mid g \in G, h \in H \} ,$$

with operation $(g_1, h_1) \cdot (g_2, h_2) = (g_1 \circ g_2, h_1 \bullet h_2)$.

Distinguishing groups

Isomorphic groups share the same 'algebraic' properties

(Lemma 6.1) If $G \cong H$, then G and H

- Are both Abelian, or both non-Abelian
- Both have same size: $\#G = \#H$
- Both have same number of elements of order n
- Are both cyclic, or both non-cyclic
- ...

Warning: Can't say $G \cong H$ even if these all agree! Need to give an isomorphism (explicitly, or by abstract theory).

Q4

Are the groups $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z}$ isomorphic?

- Are both abelian ✓ Both same size ✓ Same orders of elements ✓ What now?
- $\mathbb{Z}$ is cyclic. Is $\mathbb{Z} \times \mathbb{Z}$ cyclic? I claim no.

Suppose $\mathbb{Z} \times \mathbb{Z}$ is generated by $(a, b) \in \mathbb{Z} \times \mathbb{Z}$. Then $\langle (a, b) \rangle = \{ (ka, kb) \mid k \in \mathbb{Z} \}$.

- Case $b = 0$: Then $(a, 1) \notin \langle (a, b) \rangle$.
- Case $b \neq 0$: Then $(a, 2b) \notin \langle (a, b) \rangle$.

So $\mathbb{Z} \times \mathbb{Z}$ is not cyclic. And $\mathbb{Z} \times \mathbb{Z}$ is not isomorphic to $\mathbb{Z}$.