ALGEBRA 2 – EPIPHANY 2016
TUTORIAL 7

Are the groups $\mathbb{R}$ and $\mathbb{R} \times \mathbb{R}$ isomorphic?

Surprisingly, the answer is: *it depends*! (On whether you assume the Axiom of Choice.)

1 $\mathbb{Q}$-vector spaces

Remark 1. Let V be a vector space over a field F. Then $(V, +)$ is an abelian group, where $+$ is vector addition. This is really just restating the axioms of $+$ from a vector space.

Proposition 2. *Let V, W be vector spaces over $\mathbb{Q}$ and take some function $T\colon V \to W$. Then T is a $\mathbb{Q}$-linear map, if and only if it is a group homomorphism.*

Proof. **'⇒':** Let $v_1, v_2 \in V$. If T is $\mathbb{Q}$-linear, we have

$$T(v_1 + v_2) = T(1 \cdot v_1 + 1 \cdot v_2) = 1 \cdot T(v_1) + 1 \cdot T(v_2) = T(v_1) + T(v_2)\,.$$

This shows that T is a group homomorphism from $(V, +)$ to $(W, +)$.

'⇐': Now suppose $v_1 \in V$ and $\lambda = \frac{p}{q} \in \mathbb{Q}$. If T is a group homomorphism from $(V, +) \to (W, +)$ we have the following.

$$\begin{aligned} qT(\lambda v_1) &= \underbrace{T(\lambda v_1) + \cdots + T(\lambda v_1)}_{q \text{ times}} \\ &= T(\underbrace{\lambda v_1 + \cdots + \lambda v_1}_{q \text{ times}}) \\ &= T(q\lambda v_1) \\ &= T(pv_1) \\ &= T(\underbrace{v_1 + \cdots + v_1}_{p \text{ times}}) \\ &= \underbrace{T(v_1) + \cdots + T(v_1)}_{p \text{ times}} \\ &= pT(v_1)\,. \end{aligned}$$

So we have $qT(\lambda v_1) = pT(v_1)$, which implies $T(\lambda v_1) = \frac{p}{q}T(v_1) = \lambda T(v_1)$. This shows that T is a $\mathbb{Q}$-linear map. □

2 Group isomorphisms

Example 3. The groups $\mathbb{Q}$ and $\mathbb{Q} \times \mathbb{Q}$ are not isomorphic.

Proof. Suppose that $\phi\colon \mathbb{Q} \to \mathbb{Q} \times \mathbb{Q}$ is a group isomorphism. Then proposition 2 shows that ϕ is a $\mathbb{Q}$-linear map. This means that $\mathbb{Q}$ and $\mathbb{Q} \times \mathbb{Q}$ would be isomorphic as $\mathbb{Q}$-vector spaces. But $\dim_{\mathbb{Q}} \mathbb{Q} = 1$ and $\dim_{\mathbb{Q}} \mathbb{Q} \times \mathbb{Q} = 2$, so this is not possible. □

More generally…

Proposition 4. *Let V and W be $\mathbb{Q}$-vector spaces. Then the groups $(V, +)$ and $(W, +)$ are isomorphic if and only if* $\dim_{\mathbb{Q}} V = \dim_{\mathbb{Q}} W$.

Proof. **'⇒':** Like in example 3, a group isomorphism from $\phi\colon (V, +) \to (W, +)$ is already a $\mathbb{Q}$-linear map. This means V and W are isomorphic as $\mathbb{Q}$-vector spaces, so have the same dimension.

'⇐': Let $\mathcal{B}_V = \{ e_\lambda \}$ be a basis of V, and $\mathcal{B}_W = \{ f_\lambda \}$ be a basis of W. Since $\dim_{\mathbb{Q}} V = \dim_{\mathbb{Q}} W$, we can find a bijection $\phi\colon \mathcal{B}_V \to \mathcal{B}_W$. By relabelling, say $e_\lambda \leftrightarrow f_\lambda$. Then extend ϕ by $\mathbb{Q}$-linearity to a $\mathbb{Q}$-linear map $\phi\colon V \to W$.

The $\mathbb{Q}$-linear map ϕ is injective: If $\phi(\sum a_\lambda e_\lambda) = 0$, we find $\sum a_\lambda \phi(e_\lambda) = \sum a_\lambda f_\lambda = 0$. This means all $a_\lambda = 0$ since $\{ f_\lambda \}$ is a basis. Also ϕ is surjective: Given $\sum b_\lambda f_\lambda \in W$, we have $\phi(\sum b_\lambda e_\lambda) = \sum b_\lambda \phi(e_\lambda) = \sum b_\lambda f_\lambda$.

This shows ϕ is a bijective $\mathbb{Q}$-linear map, i.e. a vector space isomorphism. Then proposition 2 shows ϕ is a bijective group homomorphism, i.e. a group isomorphism. ☐

3 Axiom of Choice

The Axiom of Choice[1] is an additional set theory axiom which states that for every family of non-empty sets $(S_i)_{i \in I}$, there exists a family of elements $(x_i)_{i \in I}$ such that $x_i \in S_i$. It sounds quite obvious! But when there are infinitely many sets in the family, it can't be proven from the usual properties of sets! The difficulty lies in choosing an element from each set, when there is no rule to guide you.

One important use for the Axiom of Choice is to show that *every* vector space has a basis. Even when the dimension is *uncountably* infinite. Without the Axiom of Choice, some vector spaces don't have a basis…

Theorem 5. *If V is an infinite dimensional $\mathbb{Q}$-vector space, then* $\dim_{\mathbb{Q}} V = \#V$.

Proof. A proof requires some heavy set-theory machinery. For example, see `https://drexel28.wordpress.com/2010/10/22/the-dimension-of-r-over-q/` ☐

But finally, we have

Example 6. The groups $\mathbb{R}$ and $\mathbb{R} \times \mathbb{R}$ *are* isomorphic!

Proof. We can view $\mathbb{R}$ as a vector space over $\mathbb{Q}$. The Axiom of Choice shows that $\mathbb{R}$ has $\mathbb{Q}$-basis $\mathcal{B}_1 = \{ e_\lambda \}$, and $\mathbb{R} \times \mathbb{R}$ has a $\mathbb{Q}$-basis $\mathcal{B}_2 = \{ f_\lambda \}$.

But $\#\mathbb{R} = \#(\mathbb{R} \times \mathbb{R})$, for example by interleaving the decimal expansions

$$(0.a_1 a_2 a_3 \ldots, 0.b_1 b_2 b_3 \ldots) \mapsto 0.a_1 b_1 a_2 b_2 a_3 b_3 \ldots .$$

This means $\dim_{\mathbb{Q}} \mathbb{R} = \dim_{\mathbb{Q}} \mathbb{R} \times \mathbb{R}$, so proposition 4 shows the groups are isomorphic. ☐

[1]Read more at `https://en.wikipedia.org/wiki/Axiom_of_choice`