Warmup Q2i)

Find the orbits and stabilisers for

- $G = S_3$ acting on itself (that is on $X = S_3$)

by conjugation.

Recall

If G acts on $X = G$ by conjugation, then for $g \in G$, and $x \in X$

$$g(x) = gxg^{-1}$$

Group actions

Definition (Group Action, Definition 8.1)

An action of group G on set X is a homomorphism

$$\varphi\colon G \to S_X$$

Useful shorthand $g(x) := \varphi(g)(x)$.

In reality this means we have to check

- $\varphi(g)$ is a bijection $X \to X$,
- $\varphi(e) = \mathrm{id}_X$, meaning $e(x) = x$, and (Identity)
- $\varphi(g) \circ \varphi(h) = \varphi(gh)$ meaning (Compatibility)

$$g(h(x)) = (gh)(x)$$

Orbits, stabilisers

For action of G on X

Definition (Orbit)

The (G-)orbit of $x \in X$ is

$$G(x) := \{\, g(x) \mid g \in G \,\} \subset X$$

Definition (Stabiliser)

The stabiliser of $x \in X$ is

$$G_x := \{\, g \in G \mid g(x) = x \,\} \subset G$$

Orbit-Stabiliser Theorem

Theorem (Corollary 8.19)

If G acts on X, then

$$\#G(x) \cdot \#G_x = \#G$$

Size of the orbit tells you the size of the stabiliser.

Useful: Finding enough elements let's you determine G_x exactly.

Q2i) - Orbits

Orbits and stabilisers of $G = S_3$ acting on itself by conjugation?

- These orbits are conjugacy classes
- Conjugate elements have the same order (week 13, Q5i)

Good idea for conjugacy classes: gather elements by their orders

$$\overbrace{\{\, e = (1)\,\}}^{\text{order 1 elements}}, \quad \overbrace{\{\,(1\,2), (1\,3), (2\,3)\,\}}^{\text{order 2 elements}} \quad \text{and} \quad \overbrace{\{\,(1\,2\,3), (1\,3\,2)\,\}}^{\text{order 3 elements}}$$

Observe $(1\,3)(1\,2)(1\,3)^{-1} = (2\,3)$, and $(2\,3)(1\,2)(2\,3)^{-1} = (1\,3)$.

Also $(1\,2)(1\,2\,3)(1\,2)^{-1} = (1\,3\,2)$.

So these sets are already the orbits. (Why. . . ?)

Q2i) - Stabilisers

$G = S_3$, so $\#G = 6$. From Orbit-Stabiliser Theorem get $\#G_x$

Element x	Orbit $G(x)$	$\#G(x)$	$\#G_x$
e	$\{ e \}$	1	6
$(1\,2)$, $(1\,3)$, or $(2\,3)$	$\{ (1\,2), (1\,3), (2\,3) \}$	3	2
$(1\,2\,3)$ or $(1\,3\,2)$	$\{ (1\,2\,3), (1\,3\,2) \}$	2	3

Warning: Same orbit means same *size* stabiliser. Maybe not equal.

- Already get $G_e = S_3$, for size reasons.

Easy elements in G_x? For conjugation, have $\langle g \rangle < G_g$. In the remaining cases $\langle g \rangle$ is already big enough. So

- $G_{(1\,2)} = \langle (1\,2) \rangle$, $G_{(1\,3)} = \langle (1\,3) \rangle$, $G_{(2\,3)} = \langle (2\,3) \rangle$, and
- $G_{(1\,2\,3)} = G_{(1\,3\,2)} = \langle (1\,2\,3) \rangle$